STATIC REPLICATION OF EUROPEAN STANDARD DISPERSION OPTIONS

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Abstract. The replication of any European contingent claim by a static portfolio of calls and puts with strikes forming a continuum, formally proven by Carr and Madan (1998), extends to "standard dispersion" options written on the Euclidean norm of a vector of n asset performances. With the help of integral equation techniques we derive replicating portfolios for calls, puts and indeed any claim contingent on standard dispersion using vanilla basket calls whose basket weights span an n-dimensional continuum. Consequently multi-asset standard dispersion options admit a model-free price enforced by arbitrage, just as single-asset European claims do.

1. Introduction

Over the past few decades, an array of derivative instruments and trading strategies have appeared where the payoff is based on some measure of statistical dispersion of one or more underlying assets. In the single-asset category, realized volatility and variance swaps appeared in the 1990s, then VIX futures and options in the 2000s as well as other volatility-related exotic options. In the multi-asset category, examples include vanilla price dispersion trades, realized variance dispersion trades, correlation swaps, or call and put options written on cross-sectional price dispersion[1] as illustrated in figure 1. Significant market activity for dispersion instruments can be observed in annual reports of many large quantitative hedge funds[2]. Accurate pricing and hedging of these instruments is notoriously more complex compared to other multi-asset options such as basket options (e.g. Brigo et al., 2004) or worst-of and best-of options.

In our preceding publication (2021) we considered the inverse problem of replicating a single-asset European option with cash, the asset and a "continuous portfolio" of arbitrary "replicant" options indexed by a single real variable such as a strike price. In this paper we extend our framework to the multi-asset class of "standard dispersion" options written on the Euclidean norm of a vector of n asset performances, which we seek to replicate with cash and a continuous portfolio of replicant basket calls indexed along n real variables corresponding to basket levers or weights.

Specifically, given a target payoff function $F(s)$ written on standard dispersion $s := \sqrt{\sum_i x_i^2}$ of $n \geq 2$ asset performances $x_1, x_2, \ldots, x_n$, we wish to find quantities $\varphi(y_1, y_2, \cdots, y_n)$ of vanilla basket

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[1] In the financial industry, price dispersion is more commonly defined as mean absolute deviation corresponding to the "taxicab" ℓ_1-norm, whereas our approach is based on the Euclidean ℓ_2-norm for ease of mathematics. We do not discuss to what extent our ℓ_2 approach may approximate ℓ_1 instruments because an exact replication of the latter will be derived in a follow-up paper using different mathematical methods.

[2] For example: Infinity Q Alpha Fund SEC Form N-CSR 31 Aug. 2020, pp. 5, 8–, Assenagon Alpha Annual Report 31 Jan. 2020, p. 7.

FIGURE 1. Sample terms of an industry dispersion option. Source: large investment bank.

Interactive Bankers, N.A. "Bankers you can talk to"

Sample Term sheet

3-year Dispersion Warrant on five shares in USD quanto

The following product is a warrant where the investor receives a Bonus linked to the performance of five stocks compared to the basket minus a Strike Level. The product has no capital protection at any time and there can be a partial or total loss of any capital invested. Investment is therefore highly speculative and should only be considered by investors who can afford to lose their entire investment amount.

Issuer & Guarantor	Interactive Bankers, N.A. (credit rating Aa3, unsecured)
Issue Type	Warrant
Issue Amount	USD 3,000,000
Number of Warrants	3,000
Notional Amount per Warrant (N)	1 Warrant = USD 1,000
Settlement Currency	USD quanto
Issue Price per Warrant	USD 60
Listing	None
Trade Date (T)	[today]
Strike Date	T
Issue Date	T + 5 days
Redemption Date	T + 3 years

Underlying Shares

i	Name	Ticker	$\text{Share}^i_{\text{initial}}$	Weight w_i
1	Apple	AAPL	[114]	20%
2	Microsoft	MSFT	[210]	20%
3	Airbus	AIR	[64]	20%
4	Yamaha	7951	[5000]	20%
5	Beyond Meat	BYND	[170]	20%

Settlement Amount On the Redemption Date, the Issuer will pay to the holder the following amount in U.S. dollars:

$$N \times \text{Bonus}$$

Where

$$\text{Bonus} = \max\left(0\%, \text{Dispersion} - \text{Strike}\right)$$

$$\text{Dispersion} = \sum_{i=1}^{5} w_i \times \text{abs}\left(\frac{\text{Share}^i_{\text{final}}}{\text{Share}^i_{\text{initial}}} - \text{Basket}_{\text{final}}\right)$$

With

Strike = 20%

$\text{Basket}_{\text{initial}} = 1$

$\text{Basket}_{\text{final}} = \sum_{i=1}^{5} w_i \times \frac{\text{Share}^i_{\text{final}}}{\text{Share}^i_{\text{initial}}}$

$\text{Share}^i_{\text{initial}}$ with i from 1 to 5 is the official closing price of Underlying Share i on the Strike Date

$\text{Share}^i_{\text{initial}}$ with i from 1 to 5 is the official closing price of Underlying Share i on the Redemption Date

Business Day Convention	Following Business Day
Governing law	U.S. law

calls[3] across all possible basket weights $y_1, y_2, \ldots, y_n$ that replicate the target payoff up to a fixed amount of cash c:

$$F\left(\sqrt{\textstyle\sum_{i=1}^n x_i^2}\right) = c + \int \cdots \int \left(\textstyle\sum_{i=1}^n x_i y_i - k\right)^+ \varphi(y_1, \cdots, y_n)\,\mathrm{d}y_1 \cdots \mathrm{d}y_n,$$

where $k > 0$ is a fixed moneyness parameter, $t^+ := \max(0, t)$ denotes the positive part of a real number t and $\int \cdots \int$ denotes a multiple integral over a suitable domain. For maximum generality we let all our variables x_i, y_i be positive or negative real numbers and we leave the definition of asset performance unspecified with the important caveat that the replicant option payoffs $(\sum_i x_i y_i - k)^+$ are defined accordingly. A typical definition would be the gross returns to maturity or the price ratios of n underlying assets[4].

Switching to vector notations, in the language of functional analysis we want to solve the multidimensional integral equation of the first kind

$$F(|\mathbf{x}|) - c = \int_{\mathbb{R}^n} (\mathbf{x} \cdot \mathbf{y} - k)^+ \varphi(\mathbf{y})\,\mathrm{d}\mathbf{y}, \qquad \mathbf{x} \in \mathbb{R}^n, \tag{1}$$

for the unknown function $\varphi(\mathbf{y})$ and constant c. Here, $\mathbf{x} \cdot \mathbf{y} := \sum_i x_i y_i$ denotes the canonical dot product of Euclidean space $\mathbb{R}^n$ with associated norm $|\mathbf{x}| := \sqrt{\mathbf{x} \cdot \mathbf{x}}$, and $(\mathbf{x} \cdot \mathbf{y} - k)^+$ is the known integral kernel. This inverse problem is mathematically nontrivial and may be viewed as a multidimensional generalization of the Breeden and Litzenberger (1978) and Carr and Madan (1998) inverse problems for a particular class of sophisticated, multi-asset options.

1.1. **Background and review.** Evidence of research interest in static option replication strategies from practitioners and academics alike can be found in the work of Dupire (1993), Derman, Ergener, and Kani (1994), Pelsser (2003), Baldeaux and Rutkowski (2010), to name just a few. This interest is justified by the resulting model-free price for the target instrument, even if actual arbitrage enforcement could be difficult to implement due to liquidity and transaction cost issues. The most successful illustration of this approach is the decomposition of the log-contract into a continuous portfolio of out-of-the-money calls and puts on the S&P 500 index, whose discretization underpins the calculation of the VIX. While the majority of such calls and puts are illiquid and do not trade very often, the VIX is widely regarded as an excellent, model-free gauge of aggregate implied volatility and estimate of the fair price of a variance swap.

Further practical motivations for decomposing a dispersion option as a sum of basket calls may include technical limitations of risk systems which are often designed to work with simpler instruments, in which case having a payoff equivalence can save a lot of time and reprogramming costs; as well as specific hedging needs of large derivatives issuers to offload excess covariance or correlation risk accumulated by selling simpler multi-asset options[5].

In other related literature, Baxter (1998, p. 13) mentions a generalization of the Breeden and Litzenberger formula to a vector of assets in $\mathbb{R}^n$ based on Fourier transforms, while Lipton (2001,

[3]This includes basket call options on all n underlying assets, as well as any subset: single-asset calls (case where all weights y_i but one are zero), two-asset calls (case where all weights y_i but two are zero), and so forth.

[4]Another possible definition of asset performances could be the time series of n daily returns $x_1, \ldots, x_n$ with respect to a single asset, in which case s would be the asset's realized volatility. However, the corresponding replicant options would then be based on various weighted sums of daily returns resembling *cliquet options* which are less compelling than basket calls in terms of practical applications.

[5]An interesting price property of dispersion calls and puts is that they are short correlation instruments which help issuers reduce their correlation risk exposure.

pp. 291–292) proposes a generalization of the Carr and Madan formula for two assets using Radon transforms. Expanding on the latter approach, Carr and Laurence (2011) derive a multi-asset version of the Dupire (1993) local volatility formula, while Austing (2011) uses standard calculus tools to replicate basket options using best-of and worst-of options. Recently, Pötz (2020) investigates efficient basket option pricing with Chebyshev quadrature techniques, and Cui and Xu (2021) derive a multi-asset extension of the Carr and Madan formula as multiple integrals of products of call options.

1.2. **Results and organization of this paper.** Our main contribution is to establish that any standard dispersion option with sufficiently regular payoff is replicated by a continuous portfolio of vanilla basket calls, and consequently admits a model-free arbitrage price so long as the prices of basket call options of arbitrary basket weights are known. We also provide closed-form solutions to replicate the dispersion call, zero-strike dispersion call, and dispersion put. To achieve this result, we relied on a fair amount of technical machinery presented in Appendix A, leveraging on existing fractional calculus techniques in relation to Radon transforms which we adapted to our needs. In addition, we overcame a substantial mathematical limitation that the payoff function satisfy $F'(0) = 0$ by isolating the first-order term which we proved to be replicable with zero-strike basket calls.

The remainder of our paper is organized as follows: In section 2 we discuss the concept of constrained and unconstrained continuous portfolios of vanilla basket calls. In sections 3 and 4 we derive solutions for the replication of standard dispersion calls and puts. In section 5 we extend our results to arbitrary target payoff functions. In section 6 we consider a numerically tractable application for the "Mexican hat" dispersion straddle. In section 7 we show how the dispersion call decomposition may be expanded as continuous portfolios of various basket securities in finite quantities, before discussing the consequences of our results for the pricing of dispersion options in our concluding section 8.

2. Continuous portfolios of vanilla basket calls

In this opening section, we discuss the financial interpretation of the multiple integral $\int_{\mathbb{R}^n} (\mathbf{x} \cdot \mathbf{y} - k)^+ \varphi(\mathbf{y})\, \mathrm{d}\mathbf{y}$ to the right-hand side of integral equation (1) as a continuous portfolio of vanilla basket calls indexed by basket weights $\mathbf{y}$. For maximum generality the basket weights $\mathbf{y}$ in equation (1) are unconstrained, contrary to industry practice where they typically sum to 1. As a result, the moneyness control parameter k is *not* interpreted as a direct strike price. When the sum of weights is positive, correspondence is easily obtained by simple standardization:

$$(\mathbf{x} \cdot \mathbf{y} - k)^+ = \left(\sum_{i=1}^{n} y_i \right) \left(\mathbf{x} \cdot \frac{\mathbf{y}}{\sum_i y_i} - \frac{k}{\sum_i y_i} \right)^+, \qquad \textstyle\sum_i y_i > 0,$$

which is a quantity $\sum_i y_i$ of basket calls with standardized basket weights $\mathbf{y}/\sum_i y_i$ summing to 1 and strike price $k/\sum_i y_i$. Another consequence of letting basket weights unconstrained within $\mathbb{R}^n$ is that some weights may be negative, resulting in a long-short basket[6] which is uncommon in

[6]Short-only basket calls are also possible and better interpreted as long-only basket puts with negative moneyness parameter $-k$.

the derivatives industry. However, a long-short basket may be viewed as a "spread" between two long-only baskets:

$$(\mathbf{x}\cdot\mathbf{y}-k)^+ = \left(\sum_{y_i>0} x_i y_i - \sum_{y_i<0} x_i |y_i| - k\right)^+,$$

in which case k is interpreted as a "residual" strike price. Such call and put options on the performance spread between two assets, also known *outperformance options*, are well understood by practitioners. Again, weights may be standardized to sum to 1 within each basket for better correspondence with industry practice.

It is possible to introduce constraints on basket weights as alternative formulations of our replication problem (1), at the greater risk of finding no solution. For example, a long-only constraint can be expressed as an integral over $\mathbb{R}^n_+$ rather than $\mathbb{R}^n$. More complex types of constraints $\mathbf{y}\in\mathcal{S}\subseteq\mathbb{R}^n$, such as weights summing to 1, are best expressed as a *surface integral*

$$\int_{\mathcal{S}} (\mathbf{x}\cdot\mathbf{y}-k)^+\varphi(\mathbf{y})\,\mathrm{d}\mathbf{y},$$

where $\mathrm{d}\mathbf{y}$ now denotes the infinitesimal change in surface area. Two particularly important types of constraints encountered in this paper are

- *Unit sum of weights* corresponding to the hyperplane $\mathcal{S} := \{\mathbf{y}\in\mathbb{R}^n : \sum_i y_i = 1\}$ with surface integral

$$\int_{\sum_i y_i=1} (\mathbf{x}\cdot\mathbf{y}-k)^+\varphi(\mathbf{y})\,\mathrm{d}\mathbf{y} = \int_{\mathbf{y}\cdot\mathbf{e}=1} (\mathbf{x}\cdot\mathbf{y}-k)^+\varphi(\mathbf{y})\,\mathrm{d}\mathbf{y} = \int_{\mathbb{R}^n} \delta(\mathbf{y}\cdot\mathbf{e}-1)(\mathbf{x}\cdot\mathbf{y}-k)^+\varphi(\mathbf{y})\,\mathrm{d}\mathbf{y},$$

  where $\mathbf{e} := (1,\cdots,1)$ is the first diagonal vector of $\mathbb{R}^n$ and δ is Dirac's delta function. This type of surface integral is known as a *Radon transform* and may be financially interpreted as a *standardized continuous portfolio* of basket calls.

- *Unit sum of squares of weights*: $\sum_i y_i^2 = 1$, or $|\mathbf{y}| = 1$ in vector notation. While this type of constraint is uncommon in the industry, it is well known to mathematicians as a surface integral over the unit hypersphere[7]

$$\int_{|\mathbf{u}|=1} (\mathbf{x}\cdot\mathbf{u}-k)^+\varphi(\mathbf{u})\,\mathrm{d}\mathbf{u} = \int_{\mathbb{S}^{n-1}} (\mathbf{x}\cdot\mathbf{u}-k)^+\varphi(\mathbf{u})\,\mathrm{d}\mathbf{u},$$

  where we use the letter $\mathbf{u}$ instead of $\mathbf{y}$ to emphasize it is a unit vector. From a quantitative finance perspective we may name the above a *normalized continuous portfolio* of basket calls.

3. Replication of standard dispersion calls

The standard dispersion call pays off $F(|\mathbf{x}|) := (|\mathbf{x}|-K)^+, K>0$. The payoff function $F(s)$ is *sufficiently regular* and satisfies $F'(0)=0$, and the replication problem (1) has a *regular solution*[8]:

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[7]The n-dimensional unit hypersphere, or simply unit sphere, is an object of algebraic dimension n as subset of vector space $\mathbb{R}^n$. In the academic literature, it is often denoted $\mathbb{S}^{n-1}$ in reference to its *geometric* dimension $n-1$ as easily visualized for $n=2$ or 3. Unlike the n exponent in $\mathbb{R}^n$ which denotes the Cartesian product $\mathbb{R}\times\cdots\times\mathbb{R}$, the $n-1$ superscript in $\mathbb{S}^{n-1}$ does not appear to have a particular meaning and some authors indeed prefer the subscript notation $\mathbb{S}_{n-1}$.

[8]See Appendix A, definitions (D1), (D2) and proposition A.4 for technical details.

Proposition 1. The standard dispersion call is replicable with vanilla basket calls as per equation (1) with $c = 0$ and

$$\varphi_{\mathrm{C}}\left(\mathbf{y}; \frac{k}{K}\right) = \begin{cases} \dfrac{2}{\pi^{(n-1)/2}} \delta^{\left(\frac{n-1}{2}\right)}\left(\dfrac{k^2}{K^2} - |\mathbf{y}|^2\right), & n \text{ odd}, \\ (-1)^{n/2} \dfrac{2\,\Gamma\left(\frac{n-1}{2}\right)}{\pi^{(n+1)/2}} \dfrac{H\left(\frac{k^2}{K^2} - |\mathbf{y}|^2\right)}{\left(\frac{k^2}{K^2} - |\mathbf{y}|^2\right)^{(n+1)/2}} \,\P, & n \text{ even}, \end{cases} \tag{2}$$

where δ is Dirac's delta function, H is Heaviside's step function, Γ is Euler's gamma function, and the pilcrow symbol ¶ indicates a *pseudofunction* subject to Hadamard regularization (Kanwal, 2004, pp. 71–74).

Remark. The solution vanishes as $K \to 0$ and thus cannot be used to replicate the zero-strike dispersion call with payoff $|\mathbf{x}|$, as predicted by proposition A.4.

Proof. Substituting $F''(s) = \delta(s - K)$ into equation (A.7), then sifting and simplifying,

$$\begin{aligned} \phi_{\mathrm{C}}(r) &= \begin{cases} \dfrac{1}{\pi^{(n-1)/2}} \left(\dfrac{\mathrm{d}}{\mathrm{d}r^2}\right)^{\frac{n-1}{2}} r^n \delta(r - K), & n \text{ odd}, \\ \dfrac{2}{\pi^{n/2}} \left(\dfrac{\mathrm{d}}{\mathrm{d}r^2}\right)^{n/2} \displaystyle\int_0^r \dfrac{s^{n+1}\delta(s-K)}{\sqrt{r^2 - s^2}}\,\mathrm{d}s, & n \text{ even}; \end{cases} \\ &= \begin{cases} \dfrac{K^n}{\pi^{(n-1)/2}} \left(\dfrac{\mathrm{d}}{\mathrm{d}r^2}\right)^{\frac{n-1}{2}} \delta(r - K), & n \text{ odd}, \\ \dfrac{2K^{n+1}}{\pi^{n/2}} \left(\dfrac{\mathrm{d}}{\mathrm{d}r^2}\right)^{n/2} \dfrac{H(r-K)}{\sqrt{r^2 - K^2}}, & n \text{ even}. \end{cases} \end{aligned}$$

Substituting $H(r - K) = H(r^2 - K^2)$ together with its chain rule version $\delta(r - K) = 2K\delta(r^2 - K^2)$ into the above expression,

$$\begin{aligned} \phi_{\mathrm{C}}(r) &= \begin{cases} \dfrac{2K^{n+1}}{\pi^{(n-1)/2}} \left(\dfrac{\mathrm{d}}{\mathrm{d}r^2}\right)^{\frac{n-1}{2}} \delta(r^2 - K^2), & n \text{ odd}, \\ \dfrac{2K^{n+1}}{\pi^{n/2}} \left(\dfrac{\mathrm{d}}{\mathrm{d}r^2}\right)^{n/2} \dfrac{H(r^2-K^2)}{\sqrt{r^2 - K^2}}, & n \text{ even}; \end{cases} \\ &= \begin{cases} \dfrac{2K^{n+1}}{\pi^{(n-1)/2}} \delta^{\left(\frac{n-1}{2}\right)}(r^2 - K^2), & n \text{ odd}, \\ \dfrac{2K^{n+1}}{\pi^{n/2}} \left(-\dfrac{1}{2}\right)\left(-\dfrac{3}{2}\right)\cdots\left(-\dfrac{n-1}{2}\right) \dfrac{H(r^2-K^2)}{(r^2 - K^2)^{(n+1)/2}} \,\P, & n \text{ even}. \end{cases} \end{aligned}$$

Substituting $k \mapsto k/r$, dividing both sides by r^{n+1}, homogenizing the delta function and simplifying yields expression (2) as stated for $\varphi(\mathbf{y}; k) = \phi(k/|\mathbf{y}|)/|\mathbf{y}|^{n+1}$. □

Corollary. By the chain rule for the derivative of the delta function (Kanwal, 2004, p. 50) we have in dimensions 2 and 3:

$$\varphi_{\mathrm{C}}\left(\mathbf{y}; \frac{k}{K}\right) = \begin{cases} \dfrac{2}{\pi}\delta'\left(\dfrac{k^2}{K^2} - |\mathbf{y}|^2\right) = -\dfrac{K^3/k^3}{2\pi}\delta\left(|\mathbf{y}| - \dfrac{k}{K}\right) - \dfrac{K^2/k^2}{2\pi}\delta'\left(|\mathbf{y}| - \dfrac{k}{K}\right), & n = 3, \\ -\dfrac{2}{\pi} \dfrac{H\left(\frac{k^2}{K^2} - |\mathbf{y}|^2\right)}{\left(\frac{k^2}{K^2} - |\mathbf{y}|^2\right)^{3/2}} \,\P, & n = 2. \end{cases} \tag{3}$$

Remark. We may validate the solution for $n = 3$ by inserting it into equation (A.10) together with $c = F(0) = 0$; substituting $r \mapsto \sqrt{r}$ and simplifying; then integrating by parts and sifting to obtain

$$\begin{aligned} F(|\mathbf{x}|) &= 2\int_0^\infty r^3 \delta'\big(k^2/K^2 - r^2\big)\, \frac{(|\mathbf{x}| - k/r)^{+2}}{|\mathbf{x}|}\, \mathrm{d}r \\ &= \int_0^\infty \delta'\big(k^2/K^2 - r\big)\, \frac{(|\mathbf{x}|\sqrt{r} - k)^{+2}}{|\mathbf{x}|}\, \mathrm{d}r \\ &= \int_0^\infty \delta\big(k^2/K^2 - r\big)\, \frac{2\,(r|\mathbf{x}| - k)^{+}}{2\sqrt{r}}\, \mathrm{d}r = (|\mathbf{x}| - K)^+, \end{aligned}$$

as required.

4. Replication of standard dispersion puts

As noted in the remark to proposition 1, the zero-strike standard dispersion call with payoff $|\mathbf{x}|$ does not admit a regular solution. By put-call parity, this issue applies to standard dispersion puts as well. Fortunately, this limitation may be circumvented by including *zero-strike* basket calls with payoff $(\mathbf{x} \cdot \mathbf{y})^+$ in the replicant kernel, in which case we have the decompositions given below.

Proposition 2. The zero-strike standard dispersion call is replicated with an equally weighted normalized portfolio of zero-strike basket calls, as follows:

$$|\mathbf{x}| = \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n-1)/2}} \int_{|\mathbf{u}|=1} (\mathbf{x} \cdot \mathbf{u})^+ \,\mathrm{d}\mathbf{u}. \tag{4}$$

Proof. By slice integration (Rubin, 2015, p. 29),

$$\int_{|\mathbf{u}|=1} (\mathbf{x} \cdot \mathbf{u})^+ \,\mathrm{d}\mathbf{u} = \left|\mathbb{S}^{n-2}\right| |\mathbf{x}| \int_{-1}^{1} t^+ (1 - t^2)^{(n-3)/2}\, \mathrm{d}t,$$

where $\left|\mathbb{S}^{n-2}\right| = 2\pi^{(n-1)/2}/\Gamma[(n-1)/2]$ is the surface area of the $(n-1)$-dimensional unit sphere. Solving the integral, simplifying and rearranging yields the identity as stated. □

Corollary. By put-call parity, the standard dispersion put with payoff $F(|\mathbf{x}|) := (K - |\mathbf{x}|)^+$ is replicated by a combination of cash, a short normalized continuous portfolio of zero-strike basket calls replicating the zero-strike dispersion call, and a long continuous portfolio of basket calls replicating the standard dispersion call:

$$(K - |\mathbf{x}|)^+ = K - \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n-1)/2}} \int_{|\mathbf{u}|=1} (\mathbf{x} \cdot \mathbf{u})^+ \,\mathrm{d}\mathbf{u} + \int_{\mathbb{R}^n} \varphi_{\mathrm{C}}\left(\mathbf{y}; \frac{k}{K}\right) (\mathbf{x} \cdot \mathbf{y} - k)^+ \,\mathrm{d}\mathbf{y}, \tag{5}$$

where $\varphi_{\mathrm{C}}(\mathbf{y}; k/K)$ is given by formula (2).

5. General replication of standard dispersion options

Having established that standard dispersion calls and puts are replicable with vanilla basket calls, it follows from the Carr and Madan formula that any standard dispersion option with well-behaved payoff $F(|\mathbf{x}|)$ is replicable as well:

$$F(|\mathbf{x}|) = F(s_0) + F'(s_0)(|\mathbf{x}| - s_0) + \int_0^{s_0} F''(K)\,(K - |\mathbf{x}|)^+ \,\mathrm{d}K + \int_{s_0}^{\infty} F''(K)\,(|\mathbf{x}| - K)^+ \,\mathrm{d}K,$$

where $s_0 \geq 0$ is an arbitrary split level. The following pair of theorems gives the general solution to replication problem (1) for any sufficiently regular payoff function.

Theorem 1 (GENERAL DECOMPOSITION)**.** Any standard dispersion option paying off $F(|\mathbf{x}|)$, where F is sufficiently regular payoff function, is replicated with a combination of cash, a normalized continuous portfolio of zero-strike basket calls, and a continuous portfolio of positive-strike basket calls, as follows:

$$F(|\mathbf{x}|) = F(0) + F'(0)\frac{\Gamma(\frac{n+1}{2})}{\pi^{(n-1)/2}}\int_{|\mathbf{u}|=1}(\mathbf{x}\cdot\mathbf{u})^+\,\mathrm{d}\mathbf{u} + \int_{\mathbb{R}^n}(\mathbf{x}\cdot\mathbf{y}-k)^+\frac{\phi(k/|\mathbf{y}|)}{|\mathbf{y}|^{n+1}}\,\mathrm{d}\mathbf{y}, \tag{6}$$

where

$$\phi(r) = \begin{cases} \dfrac{1}{\pi^{(n-1)/2}}\left(\dfrac{\mathrm{d}}{\mathrm{d}r^2}\right)^{\frac{n-1}{2}} r^n F''(r), & n \text{ odd}, \\ \dfrac{2}{\pi^{n/2}}\dfrac{\mathrm{d}}{\mathrm{d}r^2}\displaystyle\int_0^r \frac{s}{\sqrt{r^2-s^2}}\left(\frac{\mathrm{d}}{\mathrm{d}s^2}\right)^{\frac{n-2}{2}}[s^n F''(s)]\,\mathrm{d}s, & n \text{ even}. \end{cases} \tag{7}$$

Proof. Let $F_1(s) \coloneqq F(s) - F'(0)s$. Then F_1 is sufficiently regular with $F_1(0) = F(0), F_1'(0) = 0$, and F_1'' coincides with F''. By proposition A.4, a regular solution φ exists for F_1. By proposition A.2 this solution is given as $\varphi(\mathbf{y};k) = \dfrac{\phi(k/|\mathbf{y}|)}{|\mathbf{y}|^{n+1}}$, and we have

$$F_1(|\mathbf{x}|) = F(0) + \int_{\mathbb{R}^n}(\mathbf{x}\cdot\mathbf{y}-k)^+\varphi(\mathbf{y};k)\,\mathrm{d}\mathbf{y}.$$

Substituting $F_1(|\mathbf{x}|) \coloneqq F(|\mathbf{x}|) - F'(0)|\mathbf{x}|$, then equation (4) and rearranging yields the decomposition (6) as stated. □

Mathematically, including zero-strike basket calls in the replicant kernel is equivalent to extending the solution space to *singular solutions*[9]:

Theorem 2 (GENERAL SOLUTION)**.** Any standard dispersion option with payoff $F(|\mathbf{x}|)$, where F is sufficiently regular, is replicated with vanilla basket calls as per equation (1) with

$$\begin{cases} c = F(0), \\ \varphi_{\mathrm{F}}(\mathbf{y};k) = \dfrac{\phi(k/|\mathbf{y}|)}{|\mathbf{y}|^{n+1}} + F'(0)\dfrac{\Gamma(\frac{n+1}{2})}{\pi^{(n-1)/2}}\dfrac{\delta(1/|\mathbf{y}|)}{|\mathbf{y}|^{n+2}}, \end{cases} \tag{8}$$

where $\phi(r)$ is given by formula (7).

Remark. The singular term in $\dfrac{\delta(1/|\mathbf{y}|)}{|\mathbf{y}|^{n+2}}$ may be viewed as a corrective term to inversion formula (A.7) when allowing for singular solutions. When $F'(0) = 0$, both formulas coincide and the solution is regular, as for the standard dispersion call.

Proof. Plugging the proposed solution (8) into the right-hand side of equation (1) and splitting the integral,

$$\int_{\mathbb{R}^n}(\mathbf{x}\cdot\mathbf{y}-k)^+\varphi(\mathbf{y})\,\mathrm{d}\mathbf{y} = F(0) + F'(0)\frac{\Gamma(\frac{n+1}{2})}{\pi^{(n-1)/2}}\int_{\mathbb{R}^n}\frac{\delta(1/|\mathbf{y}|)}{|\mathbf{y}|^{n+2}}\,\mathrm{d}\mathbf{y} + \int_{\mathbb{R}^n}\frac{\phi(k/|\mathbf{y}|)}{|\mathbf{y}|^{n+1}}\,\mathrm{d}\mathbf{y}. \tag{9}$$

[9] Appendix A, definition (D2).

Switching to cylindrical coordinates in the first integral, simplifying, and homogenizing the delta function ; substituting $r \mapsto k/r$; then sifting,

$$\begin{aligned}\int_{\mathbb{R}^n} (\mathbf{x}\cdot\mathbf{y} - k)^+ \frac{\delta(1/|\mathbf{y}|)}{|\mathbf{y}|^{n+2}}\,\mathrm{d}\mathbf{y} &= \int_0^\infty \delta\left(\frac{k}{r}\right)\frac{k}{r^2}\,\mathrm{d}r \int_{|\mathbf{u}|=1}\left(\mathbf{x}\cdot\mathbf{u} - \frac{k}{r}\right)^+ \mathrm{d}\mathbf{u} \\ &= \int_0^\infty \delta(r)\,\mathrm{d}r \int_{|\mathbf{u}|=1} (\mathbf{x}\cdot\mathbf{u} - r)^+\,\mathrm{d}\mathbf{u} \\ &= \int_{|\mathbf{u}|=1} (\mathbf{x}\cdot\mathbf{u})^+\,\mathrm{d}\mathbf{u} = \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n-1)/2}}|\mathbf{x}|,\end{aligned}$$

where we used identity (4) in the last step. Substituting the above into equation (9) and then decomposition (6) yields $F(|\mathbf{x}|)$ as required. □

Corollary. The replication problem (1) admits singular solutions for the following dispersion options:

(a) For the zero-strike standard dispersion call with payoff $F(|\mathbf{x}|) \coloneqq |\mathbf{x}|$,

$$\varphi_{\mathrm{C}}(\mathbf{y};\infty) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n-1)/2}}\frac{\delta(1/|\mathbf{y}|)}{|\mathbf{y}|^{n+2}}.$$

(b) For the standard dispersion put with payoff $F(|\mathbf{x}|) \coloneqq (K - |\mathbf{x}|)^+$,

$$\begin{cases} c = K, \\ \varphi_{\mathrm{P}}(\mathbf{y};k/K) = \varphi_{\mathrm{C}}(\mathbf{y};k/K) - \dfrac{\Gamma(\frac{n+1}{2})}{\pi^{(n-1)/2}}\dfrac{\delta(1/|\mathbf{y}|)}{|\mathbf{y}|^{n+2}}, \end{cases}$$

where $\varphi_{\mathrm{C}}(\mathbf{y};k/K)$ is given by formula (2).

6. Numerical application: replication of the "Mexican hat" dispersion straddle

The "Mexican hat" dispersion straddle option with payoff $F(|\mathbf{x}|) \coloneqq 1 - e^{-|\mathbf{x}|^2}$ is a good example of a continuous and bounded payoff function for which the replication problem has a "nice", numerically tractable continuous solution. The payoff function $F(s) \coloneqq 1 - e^{-s^2}$ is clearly sufficiently regular and satisfies $F(0) = F'(0) = 0$. Substituting $F''(s) = 2(1-2s^2)e^{-s^2}$ into formula (7),

$$\phi(r) = \begin{cases} \dfrac{2}{\pi^{(n-1)/2}}\left(\dfrac{\mathrm{d}}{\mathrm{d}r^2}\right)^{\frac{n-1}{2}} e^{-r^2}(r^n - 2r^{n+2}), & n \text{ odd}, \\ \dfrac{4}{\pi^{n/2}}\dfrac{\mathrm{d}}{\mathrm{d}r^2}\displaystyle\int_0^r \frac{s}{\sqrt{r^2-s^2}}\left(\frac{\mathrm{d}}{\mathrm{d}s^2}\right)^{\frac{n-2}{2}}\left[e^{-s^2}(s^n - 2s^{n+2})\right]\mathrm{d}s, & n \text{ even}. \end{cases}$$

For ease of exposure we merely proceed with the cases $n = 2, 3$ whereby

$$\phi(r) = \begin{cases} \dfrac{2}{\pi}\dfrac{\mathrm{d}}{\mathrm{d}r^2}e^{-r^2}(r^3 - 2r^5), & n = 3, \\ \dfrac{4}{\pi}\dfrac{\mathrm{d}}{\mathrm{d}r^2}\displaystyle\int_0^r \frac{s}{\sqrt{r^2-s^2}}e^{-s^2}(s^2 - 2s^4)\,\mathrm{d}s, & n = 2. \end{cases}$$

As shown in Appendix C, the integral above solves to $r + r^3 - (1 + r^2 + 2r^4)\mathfrak{D}(r)$ wherein $\mathfrak{D}(r) \coloneqq e^{-r^2}\int_0^r e^{t^2}\,\mathrm{d}t$ is Dawson's function. Substituting this expression together with $\frac{\mathrm{d}}{\mathrm{d}r^2} = \frac{1}{2r}\frac{\mathrm{d}}{\mathrm{d}r}$ into the above equation,

$$\phi(r) = \begin{cases} \frac{1}{r\pi}\frac{\mathrm{d}}{\mathrm{d}r} e^{-r^2}(r^3 - 2r^5), & n = 3, \\ \frac{2}{r\pi}\frac{\mathrm{d}}{\mathrm{d}r}\left[r + r^3 - (1 + r^2 + 2r^4)\mathfrak{D}(r)\right] & n = 2; \end{cases}$$
$$= \begin{cases} \frac{r}{\pi}(3 - 12r^2 + 4r^4)e^{-r^2}, & n = 3, \\ \frac{4}{\pi}\left[r - r^3 + (2r^4 - 3r^2)\mathfrak{D}(r)\right], & n = 2. \end{cases}$$

The solution φ to replication problem (1) is thus

$$\varphi(\mathbf{y}; k) = \frac{\phi(k/|\mathbf{y}|)}{|\mathbf{y}|^{n+1}} = \begin{cases} \frac{k}{\pi|\mathbf{y}|^5}\left(3 - \frac{12k^2}{|\mathbf{y}|^2} + \frac{4k^4}{|\mathbf{y}|^4}\right)e^{-k^2/|\mathbf{y}|^2}, & n = 3, \\ \frac{4k}{\pi|\mathbf{y}|^4}\left(1 - \frac{k^2}{|\mathbf{y}|^2}\right) - \frac{8k^2}{\pi|\mathbf{y}|^5}\left(\frac{3}{2} - \frac{k^2}{|\mathbf{y}|^2}\right)\mathfrak{D}\left(\frac{k}{|\mathbf{y}|}\right), & n = 2. \end{cases}$$

Figure 2 shows the payoff F and its replicating solution ϕ for $n = 2$ assets. For $n = 3$, the solution may be verified by inserting it into equation (A.10) to obtain

$$F(|\mathbf{x}|) = c + \int_0^\infty \frac{k}{r^2}\left(3 - \frac{12k^2}{r^2} + \frac{4k^4}{r^4}\right)e^{-k^2/r^2}\frac{(|\mathbf{x}| - k/r)^{+2}}{|\mathbf{x}|}\,\mathrm{d}r,$$

which solves to the target payoff $1 - e^{-|\mathbf{x}|^2}$ as required after substituting $r \mapsto k/r$ and $c = F(0) = 0$.

7. Theoretical application: tractable expansion of the dispersion call decomposition

Solution formula (2) for replicating a dispersion call is mathematically correct but it involves generalized functions that present a singularity at $|\mathbf{y}| = k/K$ implying infinite quantities of basket calls to buy or sell. This would typically not be an issue in theoretical pricing applications thanks to the dampening effect of the expectation operator, but it is an issue for discretization, numerical integration, and of course trading. There is a well-known parallel in the single-asset case whereby a binary option with payoff $F(x) = H(x - K)$ may be represented as the limit-case of a levered call spread with strikes $K - \varepsilon$ and K:

$$H(x - K) = \frac{\mathrm{d}}{\mathrm{d}x}(x - K)^+ = \lim_{\varepsilon \to 0}\frac{1}{\varepsilon}\left[(x + \varepsilon - K)^+ - (x - K)^+\right].$$

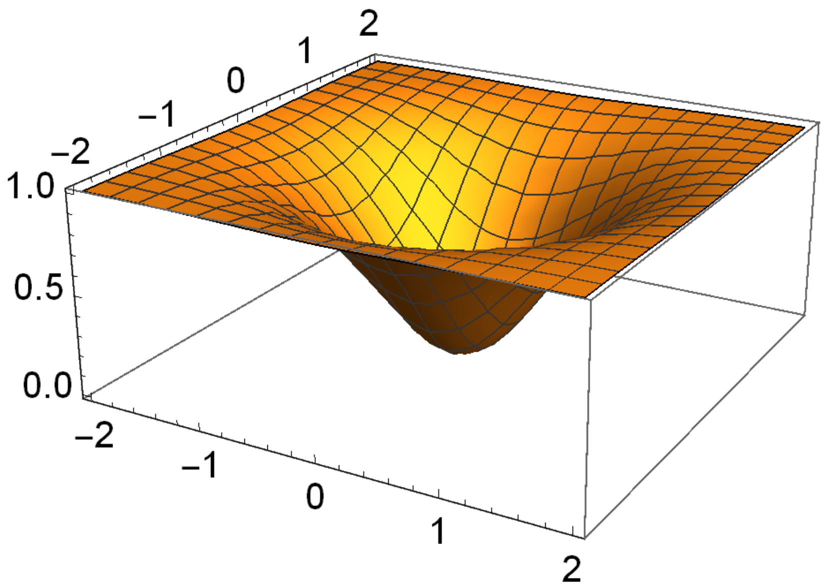
An equivalent mathematical representation of the above is

$$H(x - K) = \int_0^\infty \delta(\kappa - K)\,\mathrm{d}(x - \kappa)^+ = \int_0^\infty \delta'(\kappa - K)(x - \kappa)^+\,\mathrm{d}\kappa,$$

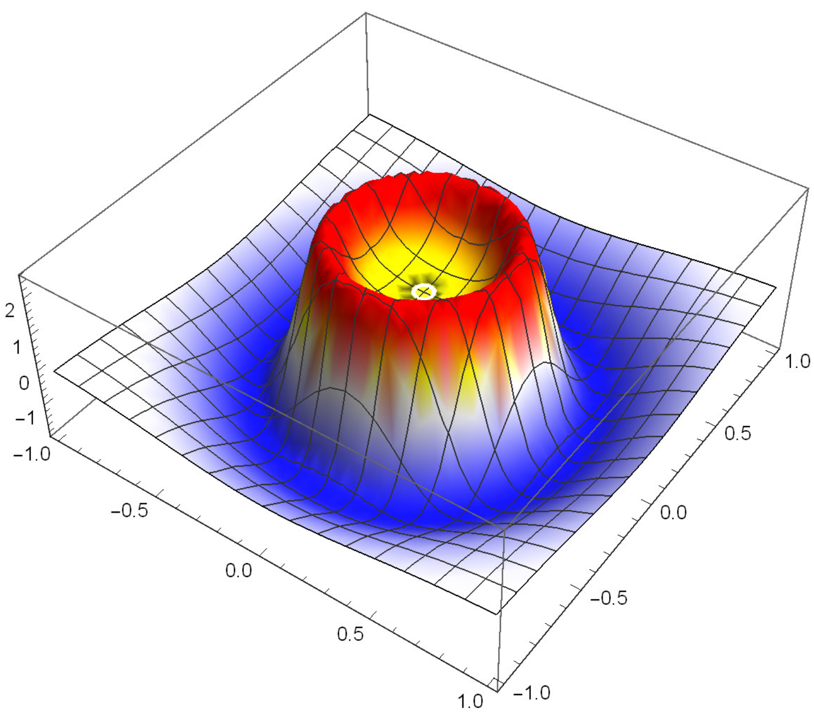
where the second expression stems from integration by parts and is consistent with the Carr and Madan formula at origin. Similarly, standard dispersion calls may be replicated with *ad hoc* continuous portfolios of vanilla basket calls, binary basket calls and so forth in finite quantities, as discussed below.

FIGURE 2. Two-asset "Mexican hat" straddle dispersion payoff $F(x_1, x_2) = 1 - e^{-x_1^2 - x_2^2}$ and its replicating solution $\phi(y_1, y_2)$ as quantity of basket calls with moneyness parameter $k = 1$.

2A. Payoff function



2B. Replicating solution



7.1. **Odd dimension.** We begin with the case $n = 3$ before discussing the general case. Substituting solution (3) into equation (A.9) and switching the order of integration, then sifting, we get

$$\begin{aligned}
(|\mathbf{x}| - K)^+ &= -\frac{K^2}{2\pi k^2}\int_{|\mathbf{u}|=1}\int_0^\infty r^2\left[\frac{K}{k}\delta\left(r - \frac{k}{K}\right) + \delta'\left(r - \frac{k}{K}\right)\right](r\mathbf{x}\cdot\mathbf{u} - k)^+\,\mathrm{d}r\,\mathrm{d}\mathbf{u} \\
&= -\frac{K^2}{2\pi k^2}\int_{|\mathbf{u}|=1}\left[\frac{k}{K}\left(\frac{k}{K}\mathbf{x}\cdot\mathbf{u} - k\right)^+ + \int_0^\infty r^2\delta'\left(r - \frac{k}{K}\right)(r\mathbf{x}\cdot\mathbf{u} - k)^+\,\mathrm{d}r\right]\mathrm{d}\mathbf{u} \\
&= \int_{|\mathbf{u}|=1}\left[\frac{1}{\pi}(\mathbf{x}\cdot\mathbf{u} - K)^+ - \frac{K}{2\pi}H(\mathbf{x}\cdot\mathbf{u} - K)\right]\mathrm{d}\mathbf{u}, \qquad n = 3,
\end{aligned}$$

where we integrated by parts, sifted and simplified terms in the last step. Thus, in dimension $n = 3$, the standard dispersion call option with dispersion strike K is replicated by a normalized continuous portfolio of long basket calls in quantity $1/\pi$ and short binary basket calls in quantity $K/2\pi$, with fixed moneyness parameter K and basket weights $\mathbf{u} = (u_1, u_2, u_3)$ subject to the constraint $u_1^2 + u_2^2 + u_3^3 = 1$.

It is worth observing that the presence of binary options in the above decomposition provides some insight into the dynamic hedging challenges for dispersion calls: for every binary option near the money, its delta price sensitivity becomes very large and the delta-hedging strategy prescribed by standard option theory is not feasible. In practice this issue can be mitigated by replacing binary options with tight call spreads so as to obtain an "overhedge" for the issuer — see e.g. Demeterfi et al. (1999, pp. 37–39), Taleb (1997, pp. 286–290), Bossu (2014, pp. 1, 2, 37–39) for further details. The following proposition gives the general form of the expansion:

Proposition 3. In odd dimension $n \geq 3$, the standard dispersion call is replicated by a normalized continuous portfolio of vanilla basket calls and their payoff derivatives up to order $(n-1)/2$, such as binary basket calls (step function), basket Arrow-Debreu securities[10] (delta function) and higher-order derivatives, as follows:

$$(|\mathbf{x}| - K)^+ = \frac{1}{\pi^{(n-1)/2}}\int_{|\mathbf{u}|=1}\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^{\frac{n-1}{2}}\left[r^{\frac{n-2}{2}}\left(\sqrt{r}\mathbf{x}\cdot\mathbf{u} - K\right)^+\right]\Bigg|_{r=1}\mathrm{d}\mathbf{u}, \qquad n \text{ odd},$$

where $\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^{\frac{n-1}{2}}\left[r^{\frac{n-2}{2}}\left(\sqrt{r}\mathbf{x}\cdot\mathbf{u} - K\right)^+\right]$ may be further expanded using Leibniz's product rule.

Proof. Substituting the solution formula (2) into equation (A.9) and switching the order of integration; then substituting $r \mapsto \frac{k}{K}\sqrt{r}$ and simplifying,

$$\begin{aligned}
(|\mathbf{x}| - K)^+ &= \frac{2}{\pi^{(n-1)/2}}\int_{|\mathbf{u}|=1}\int_0^\infty \delta^{\left(\frac{n-1}{2}\right)}\left(\frac{k^2}{K^2} - r^2\right)r^{n-1}(r\mathbf{x}\cdot\mathbf{u} - k)^+\,\mathrm{d}r\,\mathrm{d}\mathbf{u} \\
&= \frac{1}{\pi^{(n-1)/2}}\int_{|\mathbf{u}|=1}\int_0^\infty \delta^{\left(\frac{n-1}{2}\right)}(1 - r)\,r^{\frac{n-2}{2}}\left(\sqrt{r}\mathbf{x}\cdot\mathbf{u} - K\right)^+\mathrm{d}r\,\mathrm{d}\mathbf{u}, \qquad n \text{ odd}.
\end{aligned}$$

Integrating by parts $(n-1)/2$ times and sifting yields the decomposition as stated. □

[10] Alternatively, $\delta(\mathbf{x}\cdot\mathbf{u} - K)$ may be interpreted as the limit-case of a levered butterfly spread $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\left[(\mathbf{x}\cdot\mathbf{u} - K + \varepsilon)^+ - 2(\mathbf{x}\cdot\mathbf{u} - K)^+ + (\mathbf{x}\cdot\mathbf{u} - K - \varepsilon)^+\right]$.

Corollary. For $n = 5$ the decomposition expands as

$$(|\mathbf{x}| - K)^+ = \frac{1}{\pi^2} \int_{|\mathbf{u}|=1} \frac{\mathrm{d}^2}{\mathrm{d}r^2} \Big[r^{3/2} \left(\sqrt{r}\mathbf{x} \cdot \mathbf{u} - K \right)^+ \Big] \Big|_{r=1} \mathrm{d}\mathbf{u}$$
$$= \int_{|\mathbf{u}|=1} \left[\frac{2}{\pi^2} (\mathbf{x} \cdot \mathbf{u} - K)^+ + \frac{5K}{4\pi^2} H(\mathbf{x} \cdot \mathbf{u} - K) + \frac{1}{4\pi^2} \delta(\mathbf{x} \cdot \mathbf{u} - K) \right] \mathrm{d}\mathbf{u}, \qquad n = 5.$$

7.2. **Even dimension.** In even dimension the standard dispersion call also decomposes into continuous portfolios of vanilla basket calls, binary basket calls and higher-order payoff derivatives, after *ad hoc* Hadamard regularization of the pseudofunction written in solution formula (2). We illustrate below how this is done in dimension $n = 2$.

Proposition 4. In dimension $n = 2$, the standard dispersion call is replicated by a normalized continuous portfolio of vanilla and binary basket calls together with a constrained portfolio of basket Arrow-Debreu securities, as follows:

$$(|\mathbf{x}| - K)^+ = \frac{1}{2} \int_{|\mathbf{u}|=1} \Big[(\mathbf{x} \cdot \mathbf{u} - K)^+ + K H(\mathbf{x} \cdot \mathbf{u} - K) \Big] \mathrm{d}\mathbf{u}$$
$$- \frac{K^2}{\pi} \int_{|\mathbf{y}| \le 1} \frac{\arcsin|\mathbf{y}|}{|\mathbf{y}|^3} \delta(\mathbf{x} \cdot \mathbf{y} - K) \, \mathrm{d}\mathbf{y}, \qquad n = 2.$$

Proof. Substituting solution formula (3) into equation (A.9),

$$(|\mathbf{x}| - K)^+ = -\frac{2}{\pi} \int_0^\infty r \frac{H(k^2/K^2 - r^2)}{(k^2/K^2 - r^2)^{3/2}} \P \, \mathrm{d}r \int_{|\mathbf{u}|=1} (r\mathbf{x} \cdot \mathbf{u} - k)^+ \, \mathrm{d}\mathbf{u}, \qquad n = 2,$$

subject to Hadamard regularization of the singularity at $r = k/K$. Substituting $r \mapsto \frac{k}{K}\sqrt{1-r}$ and simplifying; regularizing; and then integrating by parts,

$$(|\mathbf{x}| - K)^+ = -\frac{1}{\pi} \int_0^1 \frac{H(r)}{r^{3/2}} \P \, \mathrm{d}r \int_{|\mathbf{u}|=1} \left(\sqrt{1-r}\mathbf{x} \cdot \mathbf{u} - K \right)^+ \mathrm{d}\mathbf{u}$$
$$= -\frac{1}{\pi} \lim_{\varepsilon \to 0} \left[\int_\varepsilon^1 \frac{\mathrm{d}r}{r^{3/2}} \int_{|\mathbf{u}|=1} \left(\sqrt{1-r}\mathbf{x} \cdot \mathbf{u} - K \right)^+ \mathrm{d}\mathbf{u} - \frac{2}{\sqrt{\varepsilon}} \int_{|\mathbf{u}|=1} (\mathbf{x} \cdot \mathbf{u} - K)^+ \, \mathrm{d}\mathbf{u} \right]$$
$$= \frac{1}{2\pi} \int_0^1 \frac{\mathrm{d}r}{\sqrt{r(1-r)}} \int_{|\mathbf{u}|=1} (\mathbf{x} \cdot \mathbf{u}) H\left(\sqrt{1-r}\mathbf{x} \cdot \mathbf{u} - K \right) \mathrm{d}\mathbf{u}, \qquad n = 2,$$

which is a convergent improper integral. Substituting $r \mapsto 1 - r^2$ and then integrating by parts and sifting,

$$(|\mathbf{x}| - K)^+ = \frac{1}{\pi} \int_0^1 \frac{\mathrm{d}r}{\sqrt{1-r^2}} \int_{|\mathbf{u}|=1} (\mathbf{x} \cdot \mathbf{u}) H(r\mathbf{x} \cdot \mathbf{u} - K) \, \mathrm{d}\mathbf{u}$$
$$= \frac{1}{2} \int_{|\mathbf{u}|=1} (\mathbf{x} \cdot \mathbf{u}) H(\mathbf{x} \cdot \mathbf{u} - K) \, \mathrm{d}\mathbf{u}$$
$$- \frac{K^2}{\pi} \int_0^1 \frac{\arcsin r}{r^2} \, \mathrm{d}r \int_{|\mathbf{u}|=1} \delta(r\mathbf{x} \cdot \mathbf{u} - K) \, \mathrm{d}\mathbf{u}, \qquad n = 2.$$

The second term above may be rewritten as the surface integral over the unit disk $\{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 \leq 1\}$,

$$-\frac{K^2}{\pi} \int_{|\mathbf{y}|\leq 1} \frac{\arcsin|\mathbf{y}|}{|\mathbf{y}|^3} \delta(\mathbf{x} \cdot \mathbf{y} - K)\, \mathrm{d}\mathbf{y}.$$

Substituting the above together with $(\mathbf{x} \cdot \mathbf{u})H(\mathbf{x} \cdot \mathbf{u} - K) = (\mathbf{x} \cdot \mathbf{u} - K)^+ + KH(\mathbf{x} \cdot \mathbf{u} - K)^+$ into the prior expression, we obtain the decomposition as stated. □

8. Consequences for arbitrage pricing and conclusions

It is standard industry practice to price a given European multi-asset option with an *ad hoc* model capturing the option's idiosyncratic risks in terms of dynamic hedging, together with empirical "street adjustments" compensating for certain unavoidable risks such as payoff discontinuities. In the early days, a wide range of multi-asset options would typically be priced using a multi-asset Black-Scholes or local volatility model with constant correlation (e.g. Bossu, 2014, pp. 82–84) — for instance: basket calls or puts, best-of and worst-of calls or puts, quanto options. Recently, the derivatives industry appears to have shifted toward local correlation and stochastic correlation models that better reflect complex joint dynamics between asset prices, particularly for best-of and worst-of options. Evidence of this shift can be found in the works of Langnau (2010), Reghai (2010), Austing (2011), among others.

Dispersion options are typically viewed as risky instruments to hedge that require a sophisticated pricing model, perhaps featuring stochastic volatility and correlation, and jumps. The replication results in this paper indicate that this view may not be entirely justified. Instead, the existence of a static replicating portfolio suggests standard dispersion options should be priced with the same model used for vanilla basket calls, under penalty of arbitrage. However, the presence of potentially discontinuous payoffs such as binary baskets calls in the replicating portfolio, as found for the dispersion call in section 7, together with the dynamic hedging challenges associated with negative basket weights, might still justify some street adjustments not accounted for by our theory.

Overall, the results presented in this paper constitute a first step toward extending the seminal work of Carr and Madan (1998) and Breeden and Litzenberger (1978) to the static replication and pricing of multi-asset options, leveraging on advanced mathematical tools and theory such as Radon transforms that have vast potential for further applications in quantitative finance and indeed other scientific fields.

Appendix A. Classical solution formulas

General references for this section are especially Rubin (2015, pp. 26–68, 127–143), and Deans (1983), Natterer (2001).

A.1. Conversion to one-dimensional fractional integral equation of the first kind.

Proposition A.1. If the target dispersion payoff function $F(s)$ is twice differentiable (possibly in a generalized sense), the multidimensional inverse problem (1) of replicating a dispersion option with basket calls converts to the one-dimensional fractional integral equation of the first kind:

$$f(s) = \frac{2}{\Gamma[(n-1)/2]} \int_0^s r\phi(r) \left(s^2 - r^2\right)^{\frac{n-3}{2}} \mathrm{d}r, \qquad \text{(A.1)}$$

where Γ is Euler's gamma function, $f(s) \coloneqq \dfrac{s^n F''(s)}{\pi^{(n-1)/2}}$ and $\varphi(\mathbf{y};k) \equiv \dfrac{\phi(k/|\mathbf{y}|)}{|\mathbf{y}|^{n+1}}$.

Proof. Writing integral equation (1) in cylindrical coordinates $\mathbf{x} \mapsto s\mathbf{v}$ where $s \coloneqq |\mathbf{x}| \geq 0$ is a nonnegative real number and $\mathbf{v} \coloneqq \mathbf{x}/|\mathbf{x}|$ is a unit vector of $\mathbb{R}^n$ yields

$$F(s) - c = \int_{\mathbb{R}^n} (s\mathbf{v}\cdot\mathbf{y} - k)^+ \varphi(\mathbf{y})\,\mathrm{d}\mathbf{y}, \qquad s \geq 0, |\mathbf{v}| = 1. \tag{A.2}$$

Differentiating both sides twice against s and then sifting we obtain

$$\begin{aligned} F''(s) &= \int_{\mathbb{R}^n} (\mathbf{v}\cdot\mathbf{y})^2 \delta(s\mathbf{v}\cdot\mathbf{y} - k)\varphi(\mathbf{y})\,\mathrm{d}\mathbf{y} \\ &= \frac{k^2}{s^2}\int_{\mathbb{R}^n} \delta(s\mathbf{v}\cdot\mathbf{y} - k)\varphi(\mathbf{y})\,\mathrm{d}\mathbf{y}. \end{aligned}$$

Multiplying both sides by s^2/k^2 and switching back to Cartesian coordinates yields

$$\frac{|\mathbf{x}|^2}{k^2} F''(|\mathbf{x}|) = \int_{\mathbb{R}^n} \delta(\mathbf{x}\cdot\mathbf{y} - k)\varphi(\mathbf{y})\,\mathrm{d}\mathbf{y}, \qquad \mathbf{x} \in \mathbb{R}^n,$$

which is a Radon transform inverse problem of the target *radial function* $s \mapsto F''(s)\,s^2/k^2$ with Cartesian parameters $(\mathbf{x}, k) \in \mathbb{R}^{n+1}$. Conversion of the transform with cylindrical parameters $(\mathbf{x}, k) \in \mathbb{S}^{n-1} \times \mathbb{R}$ into a *modified Erdélyi-Kober fractional integral* is covered in Rubin (2015, pp. 140–142). In particular the solution φ, if it exists, is also radial, i.e. $\varphi(\mathbf{y}) = \psi(|\mathbf{y}|)$ where $\psi(r)$ is a function of a single variable. The case at hand with Cartesian parameters is straightforwardly adapted as follows. Rewriting the integral to the right-hand side of the above equation in cylindrical coordinates $\mathbf{y} \mapsto r\mathbf{u}, |\mathbf{u}| = 1$,

$$\frac{|\mathbf{x}|^2}{k^2} F''(|\mathbf{x}|) = \int_0^\infty r^{n-1}\psi(r)\,\mathrm{d}r \int_{|\mathbf{u}|=1} \delta(r\mathbf{x}\cdot\mathbf{u} - k)\,\mathrm{d}\mathbf{u}, \tag{A.3}$$

where the inner integral is a surface integral over the n-dimensional unit sphere $\mathbb{S}^{n-1} \coloneqq \{\mathbf{u} \in \mathbb{R}^n : |\mathbf{u}| = 1\}$ introduced in section 2. By *slice integration* (Rubin, 2015, p. 29), this spherical integral collapses to

$$\begin{aligned} \int_{|\mathbf{u}|=1} \delta(r\mathbf{x}\cdot\mathbf{u} - k)\,\mathrm{d}\mathbf{u} &= \left|\mathbb{S}^{n-2}\right| \int_{-1}^{1} \delta(r|\mathbf{x}|t - k)\left(1 - t^2\right)^{\frac{n-3}{2}}\mathrm{d}t \\ &= \frac{\left|\mathbb{S}^{n-2}\right|}{r|\mathbf{x}|^{n-2}} \left(|\mathbf{x}|^2 - \frac{k^2}{r^2}\right)^{\frac{n-3}{2}} H\!\left(r - \frac{k}{|\mathbf{x}|}\right), \end{aligned}$$

where $\left|\mathbb{S}^{n-2}\right| = 2\pi^{(n-1)/2}/\Gamma[(n-1)/2]$ is the surface area of the $(n-1)$-dimensional unit sphere, and H is Heaviside's step function. Substituting into equation (A.3) and simplifying,

$$\frac{|\mathbf{x}|^2}{k^2} F''(|\mathbf{x}|) = \frac{\left|\mathbb{S}^{n-2}\right|}{|\mathbf{x}|^{n-2}} \int_{k/|\mathbf{x}|}^{\infty} r^n\,\psi(r)\left(|\mathbf{x}|^2 - \frac{k^2}{r^2}\right)^{\frac{n-3}{2}} \frac{\mathrm{d}r}{r^2},$$

which, for fixed $k > 0$, is a radial equation as both sides are functions of $|\mathbf{x}|$ only. Substituting $r \mapsto k/r$ and $\left|\mathbb{S}^{n-2}\right| = 2\pi^{(n-1)/2}/\Gamma[(n-1)/2]$, simplifying and rearranging,

$$\frac{|\mathbf{x}|^n F''(|\mathbf{x}|)}{\pi^{(n-1)/2}} = \frac{2}{\Gamma[(n-1)/2]} \int_0^{|\mathbf{x}|} r\left(\frac{k}{r}\right)^{n+1} \psi\!\left(\frac{k}{r}\right)\left(|\mathbf{x}|^2 - r^2\right)^{\frac{n-3}{2}} \mathrm{d}r. \tag{A.4}$$

The above integral is a *left-sided* modified Erdélyi-Kober fractional integral of the function $\phi(r) \coloneqq (k/r)^{n+1}\psi(k/r)$. Finally, substituting $s \coloneqq |\mathbf{x}|, f(s) \coloneqq s^n F''(s)/\pi^{(n-1)/2}$ and $\phi(r)$ into the above and then simplifying yields the fractional integral equation (A.1) as stated. □

Remark. Throughout this paper we handle the parameter k, which appears to the right-hand side of integral equation (A.2) but not to the left-hand side, as a *constant parameter* susceptible to appear in the solution $\varphi(\mathbf{y})$, which we denote $\varphi(\mathbf{y}; k)$ every so often to emphasize its parametric dependence. In contrast, other authors tend to assume that φ is independent from k, which is more restrictive.

A.2. Inversion formulas.

Proposition A.2. Provided that the following standard and fractional derivatives exist (possibly in a generalized sense), the solution to the fractional integral equation (A.1) is given as

$$\phi(r) = \begin{cases} \left(\dfrac{\mathrm{d}}{\mathrm{d}r^2}\right)^{\frac{n-1}{2}} f(r), & n \text{ odd}, \\ \dfrac{2}{\sqrt{\pi}} \dfrac{\mathrm{d}}{\mathrm{d}r^2} \displaystyle\int_0^r \dfrac{s}{\sqrt{r^2 - s^2}} \left(\dfrac{\mathrm{d}}{\mathrm{d}s^2}\right)^{\frac{n-2}{2}} f(s)\,\mathrm{d}s, & n \text{ even}. \end{cases} \tag{A.5}$$

Proof. See Rubin (2015, pp. 65–68) for inversion of modified Erdélyi-Kober fractional integral operators with particular focus on *right-sided* operators in theorem 2.44, and pp. 142–143 for an application to the Radon transform of radial functions. The case of left-sided operators is similar and illustrated in appendix B. □

Remark. A variant of the above formula for n even has the differential operator taken out of the integral:

$$\phi(r) = \begin{cases} \left(\dfrac{\mathrm{d}}{\mathrm{d}r^2}\right)^{\frac{n-1}{2}} f(r), & n \text{ odd}, \\ \dfrac{2}{\sqrt{\pi}} \left(\dfrac{\mathrm{d}}{\mathrm{d}r^2}\right)^{n/2} \displaystyle\int_0^r \dfrac{s f(s)}{\sqrt{r^2 - s^2}}\,\mathrm{d}s, & n \text{ even}. \end{cases} \tag{A.6}$$

Corollary. The solution $\varphi(\mathbf{y})$ to the replication problem (1), if it exists, is given as $\varphi(\mathbf{y}; k) = \dfrac{\phi(k/|\mathbf{y}|)}{|\mathbf{y}|^{n+1}}$ where

$$\phi(r) = \begin{cases} \dfrac{1}{\pi^{(n-1)/2}} \left(\dfrac{\mathrm{d}}{\mathrm{d}r^2}\right)^{\frac{n-1}{2}} r^n F''(r), & n \text{ odd}, \\ \dfrac{2}{\pi^{n/2}} \dfrac{\mathrm{d}}{\mathrm{d}r^2} \displaystyle\int_0^r \dfrac{s}{\sqrt{r^2 - s^2}} \left(\dfrac{\mathrm{d}}{\mathrm{d}s^2}\right)^{\frac{n-2}{2}} [s^n F''(s)]\,\mathrm{d}s, & n \text{ even}. \end{cases} \tag{A.7}$$

Proof. Immediate from substituting $f(s) \coloneqq s^n F''(s)/\pi^{(n-1)/2}$ into equation (A.5). □

A.3. **Existence and uniqueness of solutions.** In most function spaces, fractional integrals are injective linear operators and thus if a solution ϕ exists it is unique almost everywhere. However, a solution may not always exist, particularly if we impose smoothness or regularity requirements such as continuity, as shown below.

Proposition A.3. For $n \geq 3$, the left-sided modified Erdélyi-Kober fractional integral operator $I : \phi \mapsto I\phi$ where

$$I\phi(s) \coloneqq \frac{2}{\Gamma[(n-1)/2]} \int_0^s r\phi(r)(s^2 - r^2)^{(n-3)/2} \, \mathrm{d}r, s > 0,$$

is an endomorphism of the space of continuous functions over $(0, \infty)$.

Proof. Substituting $r \mapsto s\sqrt{r}$ and simplifying,

$$I\phi(s) = s^{n-3} \frac{1}{\Gamma[(n-1)/2]} \int_0^1 \phi(s\sqrt{r}) \, (1-r)^{\frac{n-3}{2}} \, \mathrm{d}r, \tag{A.8}$$

which is continuous in s for $n \geq 3$ if $\phi(r)$ is continuous in r (see also Luchko and Trujillo, 2007, th. 2.2). □

Corollary. (a) If the payoff function F is not twice continuously differentiable over $(0, \infty)$, then the replication problem (1) has no continuous solution φ.
(b) If $\phi(r) = O(r^3)$ then F'' is continuous at the origin.

Proof. (a) Contraposition of proposition A.3 when $I\phi(s) = f(s) \coloneqq s^n F''(s)/\pi^{(n-1)/2}$.
(b) Replace $I\phi(s)$ with $f(s) \coloneqq s^n F''(s)/\pi^{(n-1)/2}$ into equation (A.8), divide both sides by s^n and let $s \to 0$.

□

Here, it is worth emphasizing that many payoff functions that are relevant to finance, beginning with calls and puts, are not twice continuously differentiable over the entire domain $(0, \infty)$; we must therefore look for solutions outside of classical theory such as generalized functions. Fortunately, the inversion formulas of Section A.2 are compatible with generalized functions (see e.g. Kanwal, 2004, p. 22 for a definition) and yield a solution for standard dispersion calls with payoff $F(s) \coloneqq (s-K)^+$ as shown in Section 3. However, the following proposition shows that the *zero-strike* call $F(s) \coloneqq s$ (and thus puts $F(s) \coloneqq (K-s)^+$ by reason of put-call parity) is not replicable in this manner. We resolve this impasse in Sections 4 and 5 by including singular generalized functions in the solution space.

Definition. (D1) A payoff function $F(s)$ is *sufficiently regular* if it is twice continuously differentiable at the origin and the associated function $\phi(r)$ defined in formula (A.7) exists as a regular generalized function with $\phi(r) = O(r^3)$ as $r \to 0$.
(D2) A solution $\varphi(\mathbf{y})$ to the replication problem (1) is *regular* if there is a regular generalized function ϕ such that $\varphi(\mathbf{y}; k) = \dfrac{\phi(k/|\mathbf{y}|)}{|\mathbf{y}|^{n+1}}$ and $\phi(r) = O(r^3)$ as $r \to 0$. Otherwise it is *singular*.

Remark. In odd dimension n, if F is piecewise-differentiable of order $(n+3)/2$, then $\phi(r)$ exists as a regular generalized function.

Proposition A.4. Let F be a sufficiently regular payoff function. The replication problem (1) has a regular solution if and only if $F'(0) = 0$.

Proof. If $\varphi(\mathbf{y})$ is a regular solution to the replication problem (1), we may differentiate both sides of equation (A.2) against s and let $s \to 0$ to get $F'(0) = 0$. Conversely, assume that $F'(0) = 0$. The solution ϕ given in (A.7) is known to solve equation (A.1); integrate the latter twice to retrace our steps and retrieve equation (A.2) up to a linear term λs where $\lambda = F'(0) = 0$. Hence, $\varphi(\mathbf{y}; k) = \phi(k/|\mathbf{y}|)/|\mathbf{y}|^{n+1}$ is a regular solution to the replication problem. □

Corollary. There is no regular solution to the replication problem (1) for the class of affine standard dispersion options with payoff $F(|\mathbf{x}|) \coloneqq c + \lambda|\mathbf{x}|, \lambda \neq 0$.

Proof. Immediate from $F'(0) = \lambda \neq 0$. Alternatively, substitute $F''(s) = 0$ into formula (A.7) to obtain a degenerate $\phi \equiv 0$. □

A.4. **Another one-dimensional conversion.** In the spirit of our previous paper (2021), we present another conversion of the multidimensional integral equation (1) with basket call kernel $(\mathbf{x} \cdot \mathbf{y} - k)^+$ to a one-dimensional equation with integral kernel $G(|\mathbf{x}|, r)$ indexed by $r \in (0, \infty)$. This alternative expression can be handy to validate a solution φ obtained by the fractional calculus methods used earlier.

Proposition A.5. The replication problem (1) converts to the Fredholm integral equation of the first kind

$$F(s) - c = \int_0^\infty G(s, r)\psi(r)\,\mathrm{d}r, \qquad s \geq 0,$$

with integral kernel

$$G(s,r) = \left|\mathbb{S}^{n-2}\right| \left[\frac{rs}{n-1}\left(r^2 - \frac{k^2}{s^2}\right)^{(n-1)/2} - \frac{k}{2} r^{n-1}\mathrm{B}\left(1 - \frac{k^2}{r^2 s^2}; \tfrac{n-1}{2}, \tfrac{1}{2}\right)\right] H(rs - k),$$

where $s \coloneqq |\mathbf{x}|, \varphi(\mathbf{y}) \equiv \psi(|\mathbf{y}|)$, and $\mathrm{B}(x; a, b) \coloneqq \int_0^x t^{a-1}(1-t)^{b-1}\,\mathrm{d}t$ is the incomplete beta function.

Proof. Switching to cylindrical coordinates $\mathbf{y} \mapsto r\mathbf{u}, |\mathbf{u}| = 1$ and rearranging, equation (1) becomes

$$F(|\mathbf{x}|) - c = \int_0^\infty \psi(r)\,\mathrm{d}r \int_{|\mathbf{u}|=1} r^{n-1}\left(r\mathbf{x} \cdot \mathbf{u} - k\right)^+ \mathrm{d}\mathbf{u}. \tag{A.9}$$

By slice integration the sphere integral collapses to

$$\int_{|\mathbf{u}|=1} r^{n-1}(r\mathbf{x} \cdot \mathbf{u} - k)^+ \,\mathrm{d}\mathbf{u} = r^n |\mathbf{x}| \left|\mathbb{S}^{n-2}\right| \int_{-1}^1 \left(t - \frac{k}{r|\mathbf{x}|}\right)^+ \left(1 - t^2\right)^{(n-3)/2} \mathrm{d}t.$$

Denoting $\alpha \coloneqq k/(rs)$ for $k < rs$, splitting the integral at $t = \alpha$, applying the reverse chain rule to one split integral and substituting $t \mapsto \sqrt{1-t}$ inside the other,

$$\begin{aligned}
\int_{-1}^1 (t - \alpha)^+ \left(1 - t^2\right)^{(n-3)/2} \mathrm{d}t &= \int_\alpha^1 t\left(1 - t^2\right)^{(n-3)/2} \mathrm{d}t - \alpha \int_\alpha^1 \left(1 - t^2\right)^{(n-3)/2} \mathrm{d}t \\
&= \frac{1}{n-1}\left(1 - \alpha^2\right)^{(n-1)/2} - \frac{\alpha}{2}\int_0^{1-\alpha^2} t^{(n-3)/2}(1-t)^{-1/2}\,\mathrm{d}t \\
&= \frac{1}{n-1}\left(1 - \alpha^2\right)^{(n-1)/2} - \frac{\alpha}{2}\mathrm{B}(1 - \alpha^2; \tfrac{n-1}{2}, \tfrac{1}{2}),
\end{aligned}$$

where we recognized the incomplete beta function $\mathrm{B}(x; a, b)$ in the last step. Substituting $\alpha \coloneqq k/(rs)$, multiplying both sides by $r^n s\left|\mathbb{S}^{n-2}\right|$ and simplifying yields the formula for $G(s, r)$ as stated. □

Corollary. For $n = 3$ we have the simpler expression

$$G(s,r) = \frac{\pi r^3}{s}\left(s - \frac{k}{r}\right)^{+2}, \qquad n = 3.$$

Proof. By slice integration, the sphere integral in equation (A.9) simplifies to

$$\int_{|\mathbf{u}|=1} r^{n-1}(r\mathbf{x}\cdot\mathbf{u} - k)^{+}\,\mathrm{d}\mathbf{u} = \int_{-1}^{1}\left(t - \frac{k}{r|\mathbf{x}|}\right)^{+}\mathrm{d}t = \frac{1}{2}\left(1 - \frac{k}{r|\mathbf{x}|}\right)^{+2},$$

which solves to $\frac{\pi r^3}{s}\left(s - \frac{k}{r}\right)^{+2}$ as stated. □

In other words, a dispersion option payoff $F(|\mathbf{x}|)$ on three underlying assets may be replicated with cash and a continuous portfolio of "smooth dispersion calls" indexed by $r \in (0,\infty)$ as

$$F(|\mathbf{x}|) = c + \int_0^\infty \pi r^3 \psi(r)\frac{(|\mathbf{x}| - k/r)^{+2}}{|\mathbf{x}|}\,\mathrm{d}r, \qquad n = 3, \tag{A.10}$$

provided that a solution $\varphi(\mathbf{y}) \equiv \psi(|\mathbf{y}|)$ to integral equation (1) exists in the first place.

Appendix B. Inversion of modified Erdélyi-Kober fractional integral equation

We show how the left-sided modified Erdélyi-Kober fractional integral equation

$$f(x) = 2\int_0^x yg(y)(x^2 - y^2)^{\frac{n-3}{2}}\,\mathrm{d}y, \qquad x \geq 0, \tag{B.1}$$

is solved for $g(y)$ by repeated differentiation against x^2, together with further analysis when n is even. When $n \geq 5$ is odd, the exponent $m := \frac{n-3}{2}$ is a positive integer and we may differentiate both sides against x^2 to obtain

$$\frac{\mathrm{d}}{\mathrm{d}x^2}[f(x)] = 2xg(x)(x^2 - x^2)^m\frac{\mathrm{d}x}{\mathrm{d}x^2} + 2m\int_0^x yg(y)(x^2 - y^2)^{m-1}\,\mathrm{d}y,$$

where we used Leibniz's integral rule. Since $m > 0$ when $n \geq 5$ the first term vanishes, and we may iterate this process to write

$$\left(\frac{\mathrm{d}}{\mathrm{d}x^2}\right)^m[f(x)] = 2\,m!\int_0^x yg(y)\,\mathrm{d}y,$$

which is also satisfied when $n = 3, m = 0$ with the conventions $(\mathrm{d}/\,\mathrm{d}x^2)^0 = \mathrm{id}$ and $0! = 1$. Differentiating against x, dividing both sides by $2\,m!x$ and substituting $\frac{1}{2x}\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x^2}$ we recover

$$\frac{1}{m!}\left(\frac{\mathrm{d}}{\mathrm{d}x^2}\right)^{m+1}[f(x)] = g(x), \qquad n \text{ odd},$$

which solves the integral equation for $n \geq 3$ odd. For $n \geq 2$ even, the exponent $\frac{n-3}{2}$ is now an integer and a half, and equation (B.1) is a proper fractional integral equation. Half-integrating both sides yields

$$\int_0^x f(s)\frac{s}{\sqrt{x^2 - s^2}}\,\mathrm{d}s = 2\int_0^x \frac{s}{\sqrt{x^2 - s^2}}\int_0^s yg(y)(s^2 - y^2)^{\frac{n-3}{2}}\,\mathrm{d}y\,\mathrm{d}s.$$

Switching the order of integration[11], then substituting $s \mapsto \sqrt{y^2 + (x^2 - y^2)s}$ and simplifying,

$$\begin{aligned}\int_0^x f(s)\frac{s}{\sqrt{x^2-s^2}}\,\mathrm{d}s &= 2\int_0^x yg(y)\int_y^x s\frac{(s^2-y^2)^{\frac{n-3}{2}}}{\sqrt{x^2-s^2}}\,\mathrm{d}s\,\mathrm{d}y \\ &= \int_0^x yg(y)(x^2-y^2)^{\frac{n-2}{2}}\int_0^1 s^{\frac{n-3}{2}}(1-s)^{-1/2}\,\mathrm{d}s\,\mathrm{d}y \\ &= \mathrm{B}\big(\tfrac{n-1}{2},\tfrac{1}{2}\big)\int_0^x yg(y)(x^2-y^2)^{\frac{n-2}{2}}\,\mathrm{d}y,\end{aligned}$$

where we recognized the inner integral as a beta function $\mathrm{B}(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ in the last step. The exponent $\frac{n-2}{2}$ in the above expression being an integer, we may repeatedly differentiate both sides against x^2 as we did in odd dimension to obtain

$$\left(\frac{\mathrm{d}}{\mathrm{d}x^2}\right)^{n/2}\int_0^x f(s)\frac{s}{\sqrt{x^2-s^2}}\,\mathrm{d}s = \tfrac{n-2}{2}!\,\mathrm{B}\big(\tfrac{n-1}{2},\tfrac{1}{2}\big)\,g(x).$$

Simplifying and rearranging yields the solution

$$g(x) = \frac{1}{\Gamma[(n-1)/2]\sqrt{\pi}}\left(\frac{\mathrm{d}}{\mathrm{d}x^2}\right)^{n/2}\int_0^x f(s)\frac{s}{\sqrt{x^2-s^2}}\,\mathrm{d}s, \qquad n \text{ even}.$$

Appendix C. Calculation of the solution for the "Mexican hat" payoff in dimension $n = 2$

Substituting $s \mapsto r\sin\theta$ and simplifying, then substituting $\sin^2\theta = 1 - \cos^2\theta$ and simplifying,

$$\begin{aligned}&\int_0^r \frac{s}{\sqrt{r^2-s^2}}e^{-s^2}(s^2-2s^4)\,\mathrm{d}s \\ &= \int_0^{\pi/2} r\sin\theta\, e^{-r^2\sin^2\theta}(r^2\sin^2\theta - 2r^4\sin^4\theta)\,\mathrm{d}\theta \\ &= \int_0^{\pi/2} r\sin\theta\, e^{r^2\cos^2\theta - r^2}\left[r^2 - r^2\cos^2\theta - 2\left(r^2 - r^2\cos^2\theta\right)^2\right]\mathrm{d}\theta.\end{aligned}$$

Substituting $t = r\cos\theta$ and simplifying; expanding the square; separating terms; integrating by parts twice and simplifying,

$$\begin{aligned}\int_0^r \frac{s}{\sqrt{r^2-s^2}}e^{-s^2}(s^2-2s^4)\,\mathrm{d}s &= e^{-r^2}\int_0^r e^{t^2}\left[r^2 - t^2 - 2(r^2-t^2)^2\right]\mathrm{d}t \\ &= (r^2 - 2r^4)\mathfrak{D}(r) + e^{-r^2}\int_0^r 2te^{t^2}\left(2r^2t - t/2 - t^3\right)\mathrm{d}t \\ &= r + r^3 - (1 + r^2 + 2r^4)\mathfrak{D}(r),\end{aligned}$$

where $\mathfrak{D}(r) \coloneqq e^{-r^2}\int_0^r e^{t^2}\,\mathrm{d}t$.

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[11]By Fubini's theorem this is licit if g is absolutely integrable, but not only.

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