STRONG CONVERGENCE RATES FOR MARKOVIAN REPRESENTATIONS OF FRACTIONAL BROWNIAN MOTION

PHILIPP HARMS

Abstract. Fractional Brownian motion can be represented as an integral over a family of Ornstein–Uhlenbeck processes. This representation naturally lends itself to numerical discretizations, which are shown in this paper to have strong convergence rates of arbitrarily high polynomial order. This explains the potential, but also some limitations of such representations as the basis of Monte Carlo schemes for fractional volatility models such as the rough Bergomi model.

Contents

1. Introduction

This paper establishes strong convergence rates for certain numerical approximations of fractional Brownian motion. These approximations are inspired by Markovian representations of fractional Brownian motion [12, 13, 28, 24] and of more general Volterra processes with singular kernels [29, 4, 2, 3, 14]. The motivation is to develop efficient Monte Carlo methods for fractional (or rough) volatility models [22, 6, 9, 7, 25], which have been introduced on the grounds of extensive empirical evidence [22, 6, 9] and theoretical results [5, 18, 17, 8]. The main result is the following.

Theorem 1. *For any time horizon $T \in (0,\infty)$, Hurst index $H \in (0,1/2)$, and desired convergence rate $r \in (0,\infty)$, the following statements hold:*

(a) *Volterra Brownian motion can be approximated at rate n^{-r} by a sum of n Ornstein–Uhlenbeck processes. More precisely, there are speeds of mean reversion $x_{n,i} \in (0,\infty)$ and weights $w_{n,i} \in (0,\infty)$, $1 \leq i \leq n$, such that for any Brownian*

Date: February 18, 2019.
2010 *Mathematics Subject Classification.* 60G22, 60G15, 65C05, 91G30 .
The author gratefully acknowledges support in the form of a Junior Fellowship of the Freiburg Institute of Advances Studies.

motion W, the continuous versions W^H and $W^{H,n}$ of the stochastic integrals

$$W_t^H := \int_0^t (t-s)^{H-1/2} dW_s, \quad W_t^{H,n} := \sum_{i=1}^n w_{n,i} \int_0^t e^{-(t-s)x_{n,i}} dW_s, \quad t \in [0,T].$$

satisfy

$$\forall p \in [1,\infty): \qquad \sup_{n\in\mathbb{N}} n^r \left\| W^H - W^{H,n} \right\|_{L^p(\Omega, C([0,T],\mathbb{R})} < \infty.$$

(b) *Under the above approximation, put prices in the rough Bergomi model converge at rate n^{-r}. More precisely, for any Brownian motion B, the stochastic exponentials*

$$S_t := 1 + \int_0^t S_s \exp(W_s^H) dB_s, \quad S_t^n := 1 + \int_0^t S_s^n \exp(W_s^{H,n}) dB_s, \quad t \in [0,T],$$

satisfy for all strikes $K \in [0,\infty)$ that

$$\sup_{n\in\mathbb{N}} n^r \left| \mathbb{E}\left[(K-S_T)_+\right] - \mathbb{E}\left[(K-S_T^n)_+\right] \right| < \infty.$$

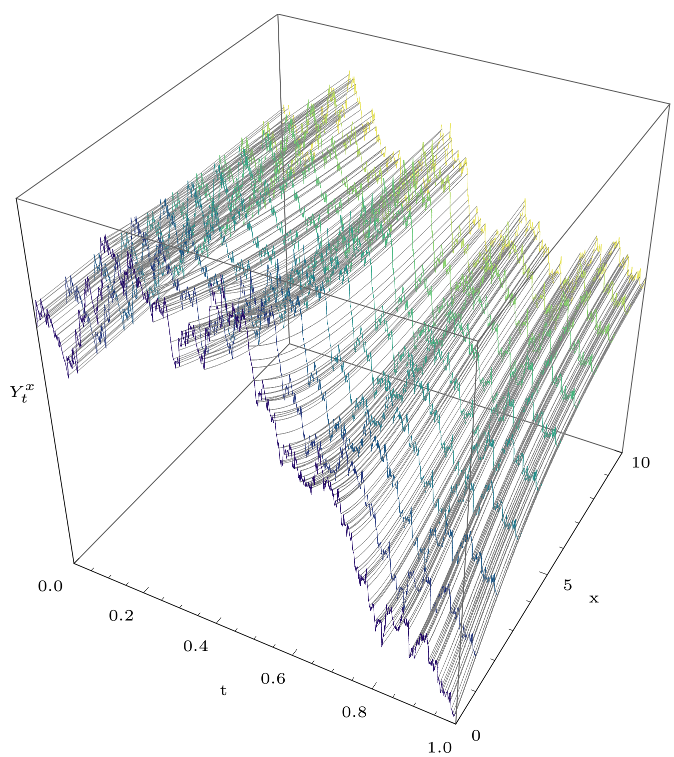

FIGURE 1. Volterra Brownian motion of Hurst index $H \in (0,1/2)$ can be represented as an integral $W_t^H = \int_0^\infty Y_t^x x^{-1/2-H} dx$ over a Gaussian random field Y_t^x. The smoothness of the random field in the spatial dimension x allows one to approximate this integral efficiently using high order quadrature rules.

Proof. **(a)** follows from the integral representation in Lemma 1 and its discretization in Lemma 2. More precisely, the m-point quadrature rule in Lemma 2 converges at any rate $r < \delta m/(1-\alpha-\beta+\delta+m) = 2Hm/3$, where the parameters $\alpha = H+1/2$, $\beta = m-1$, $\gamma = 1/2-H$, and $\delta = H$ are given by Lemma 1. Moreover, **(b)** follows from **(a)** and Lemma 3. □

The idea behind Theorem 1 is to represent Volterra Brownian motion as an integral over a Gaussian random field, as described in Lemma 1 and Figure 1. Thanks to the spatial smoothness of the random field, the integral can be approximated efficiently using high order quadrature rules, following and extending [13, 24, 1, 3]. A visual impression of the quality of this approximation can be obtained from Figure 2. The predicted convergence rate $r \approx 2Hm/3$ using m-point interpolatory quadrature closely matches the numerically observed one; see Figure 3.

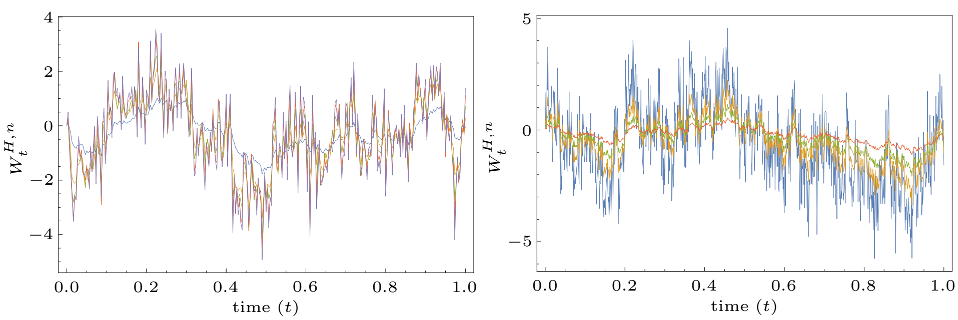

FIGURE 2. Dependence of the approximations on the number n of quadrature intervals and the Hurst index H. Left: varying the number $n \in \{2, 5, 10, 20, 40\} = \{$■, ■, ■, ■, ■$\}$ of quadrature intervals with fixed parameters $H = 0.1$, $m = 5$. Right: varying the Hurst index $H \in \{0.1, 0.2, 0.3, 0.4\} = \{$■, ■, ■, ■$\}$ with fixed parameters $n = 40$, $m = 5$.

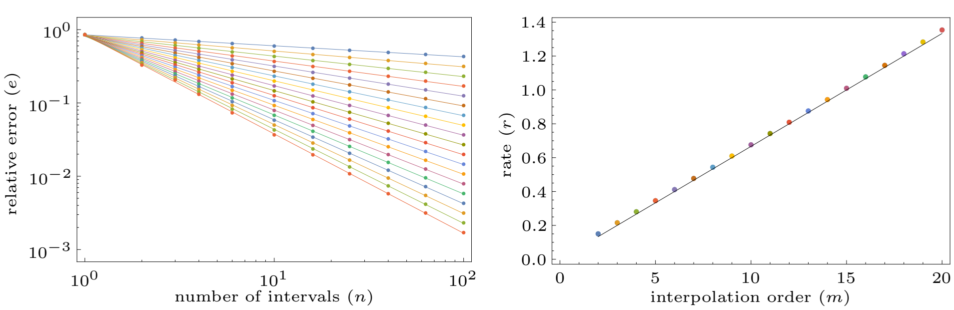

FIGURE 3. The predicted convergence rate $r \approx 2Hm/3$ with m-point interpolatory quadrature closely matches the numerically observed one. Left: relative error $e = \|W_1^H - W_1^{H,n}\|_{L^2(\Omega)}/\|W_1^H\|_{L^2(\Omega)}$ for $m \in \{2, 3, \ldots, 20\} = \{$■, ■, …, ■$\}$. Right: slopes of the lines in the left plot (dots) and predicted convergence rate (line).

A comparison to several alternative methods [26, 15, 10, 13] exhibits the potential, but also the limitations of integral representations as a basis for numerical simulation schemes. The ranking of these methods in terms of overall complexity depends on the desired accuracy and number of time points as shown in Table 1. Our scheme outperforms the others in situations where accuracy n^{-1} on a time grid of step size $\gg n^{-1/H}$ is desired. However, in fractional volatility modeling one typically wants accuracy n^{-1} on finer time grids of step size $\approx n^{-1/H}$ because this leads via piecewise constant interpolation to the same accuracy in the supremum norm. On these finer time grids our scheme achieves accuracy n^{-1} at complexity $n^{1/H+r}$ for arbitrarily small r. Using exponentially converging quadrature rules such as Chebychev [20, 19], one could at best hope to reduce this complexity down to $n^{1/H}\log n$. This is exactly the complexity of the hybrid scheme [10] and the circulant embedding method [16]. This complexity is optimal because it coincides with the complexity of convolution of $n^{1/H}$ numbers using the fast Fourier transform.

Method	Structure	Error	Complexity
Cholesky	Static	0	k^3
Hosking, Dieker [26, 15]	Recursive	0	k^2
Dietrich, Newsam [16]	Static	0	$k\log k$
Bennedsen, Lunde and Pakkanen [10]	Recursive	k^{-H}	$k\log k$
Carmona, Coutin, Montseny [13]	Recursive	n^{-1}	$kn^{9/(4H)}$
This paper	Recursive	n^{-1}	kn^r for all r

Table 1. Complexity of several numerical methods for sampling fractional Brownian motion $(W^H_{i/k})_{i\in\{1,\dots,k\}}$ with Hurst index $H\in(0,1/2)$ at k equidistant time points.

Our result has applications to fractional volatility modeling. One implication, which is spelled out in Theorem 1, is that put prices in the rough Bergomi model converge at the same rate as the underlying fractional volatility process. By put-call parity, this extends to call prices if the Brownian motions B and W are negatively correlated, as explained in Remark 2. A fully discrete Monte Carlo scheme for the rough Bergomi model can be obtained by discretizing the Ornstein–Uhlenbeck processes of Theorem 1 in time. This can be done efficiently because the covariance matrix of the Ornstein–Uhlenbeck increments has low numerical rank if the time steps are small.

Several directions for future generalization and improvement come to mind. Theorem 1 is proved by approximation in the Laplace domain, which implies convergence in the time domain by the continuity of the Laplace transform. As Volterra processes with Lipschitz drift and volatility coefficients depend continuously on the kernel in the L^2 norm, it would be interesting to check if similar convergence results hold also in this more general setting. The rate of convergence could potentially be improved using Chebychev quadrature, taking advantage of the real analyticity of the random field Y^x_t in the spatial variable x. Finally, following [10, 27], one could aim for more careful treatments of the singularity of the kernel near the diagonal and apply some variance reduction techniques.

2. Setting and notation

We will frequently make the following assumptions. Let $H \in (0, 1/2)$, let $\alpha = H + 1/2$, let μ be the sigma-finite measure $x^{-\alpha}dx$ on the interval $(0, \infty)$, let $p \in [1, \infty)$, let $T \in (0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ be a stochastic basis, and let $W, B \colon [0, T] \times \Omega \to \mathbb{R}$ be $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motions.

Moreover, we will use the following notation. The space $C^{0,\infty}([0, T] \times (0, \infty), \mathbb{R})$ carries the initial topology and differential structure with respect to the derivatives

$$\partial_x^k \colon C^{0,\infty}([0, T] \times (0, \infty), \mathbb{R}) \to C([0, T] \times K, \mathbb{R}), \qquad k \in \mathbb{N}, K \subset (0, \infty) \text{ compact},$$

and the spaces $C([0, T] \times K, \mathbb{R})$, $C([0, T], L^1(\mu))$, etc. carry the supremum norm. The space of real-valued Lipschitz functions $f \colon \mathbb{R} \to \mathbb{R}$ is denoted by $\mathrm{Lip}(\mathbb{R})$ and carries the norm $\|f\|_{\mathrm{Lip}(\mathbb{R})} = |f(0)| + \sup_{x\neq y} |f(y) - f(x)||y - x|^{-1}$.

3. Integral representation

This section establishes bounds on the tails and derivatives of the Markovian lift of Volterra Brownian motion [12, 13, 28, 24]. These bounds are used in the error analysis in Section 4. The meaning of the constants $\alpha, \beta, \gamma, \delta, m$ below is consistent throughout the paper.

Lemma 1. *Assume the setting of Section 2. Then there exists a measurable mapping*

$$Y \colon \Omega \to C^{0,\infty}([0, T] \times (0, \infty), \mathbb{R}) \cap C([0, T], L^1(\mu))$$

with the following properties:

(a) *Volterra Brownian motion is a linear functional of Y in the sense that*

$$\forall t \in [0, T] : \quad \mathbb{P}\left[\int_0^t (t - s)^{\alpha-1} dW_s = \int_0^\infty Y_t(x) \frac{dx}{x^\alpha}\right] = 1.$$

(b) *The following integrability conditions hold: for all $m \in \mathbb{N}_{>0}$, $\beta = m - 1$, $\gamma = 1 - \alpha$, and $\delta \in [0, \alpha - 1/2)$,*

$$\left\| \sup_{t\in[0,T]} \sup_{x\in(0,\infty)} \left|x^\beta \partial_x^m Y_t(x)\right| \right\|_{L^p(\Omega)} < \infty,$$

$$\sup_{x_0\in[0,1]} x_0^{-\gamma} \left\| \sup_{t\in[0,T]} \left|\int_0^{x_0} Y_t(x) \frac{dx}{x^\alpha}\right| \right\|_{L^p(\Omega)} < \infty,$$

$$\sup_{x_1\in[1,\infty)} x_1^{\delta} \left\| \sup_{t\in[0,T]} \left|\int_{x_1}^{\infty} Y_t(x) \frac{dx}{x^\alpha}\right| \right\|_{L^p(\Omega)} < \infty.$$

(c) *The following integrability condition holds: for each $\beta \in (0, 1/2)$,*

$$\left\| \sup_{t\in[0,T]} \sup_{x\in(0,\infty)} x^\beta |Y_t(x)| \right\|_{L^p(\Omega)} < \infty.$$

Proof. Let $Y \colon [0, T] \times (0, \infty) \times \Omega \to \mathbb{R}$ satisfy for each $t \in [0, T]$ and $x \in (0, \infty)$ that

$$Y_t(x) = \frac{1}{\Gamma(\frac{1}{2} - H)} \left(W_t - \int_0^t W_s x e^{-(t-s)x} ds\right).$$

Then $Y\colon \Omega \to C^{0,\infty}([0,T]\times(0,\infty),\mathbb{R})$ is well-defined and measurable because the right-hand side above is a smooth function of the sample paths of W, i.e., the following mapping is smooth:

$$C([0,T]) \ni w \mapsto \left((t,x)\mapsto w_t - \int_0^t w_s x e^{-(t-s)x} ds\right) \in C^{0,\infty}([0,T]\times(0,\infty),\mathbb{R}).$$

Moreover, $Y\colon \Omega \to C([0,T],L^1(\mu))$ is well-defined and measurable by [24, Theorem 2.11]. We briefly reproduce the argument here because it will be needed in the sequel. The expression

$$(*) \qquad \mathbb{E}\left[\int_0^\infty \sup_{t\in[0,T]} |Y_t(x)|\frac{dx}{x^\alpha}\right] = \int_0^\infty \mathbb{E}\left[\sup_{t\in[0,T]}|Y_t(x)|\right]\frac{dx}{x^\alpha}$$

is well-defined because the supremum is measurable by the continuity in t of $Y_t(x)$. Integration by parts and a continuity argument show that

$$\forall x\in(0,\infty): \quad \mathbb{P}\left[\forall t\in[0,T]: W_t - \int_0^t W_s x e^{-(t-s)x} ds = \int_0^t e^{-(t-s)x} dW_s\right] = 1,$$

which implies that

$$(*) = \frac{1}{\Gamma(\frac{1}{2}-H)}\int_0^\infty \mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t e^{-(t-s)x} dW_s\right|\right]\frac{dx}{x^\alpha}.$$

This can be bounded using the maximal inequality for Ornstein–Uhlenbeck processes of [23, Theorem 2.5 and Remark 2.6]: there is $C_1\in(0,4)$ such that

$$(*) \leq \frac{C_1}{\Gamma(\frac{1}{2}-H)}\int_0^\infty \sqrt{\frac{\log(1+Tx)}{x}}\frac{dx}{x^\alpha} < \infty.$$

Thus, Y has continuous sample paths in $L^1(\mu)$ by the dominated convergence theorem. To summarize, we have shown that the following mapping is well-defined and measurable,

$$Y\colon \Omega \to C^{0,\infty}([0,T]\times(0,\infty),\mathbb{R}) \cap C([0,T],L^1(\mu)),$$

where the intersection of the two spaces above carries the initial sigma algebra with respect to the inclusions. **(a)** follows from the above and the stochastic Fubini theorem: for each $t\in[0,T]$, one has almost surely that

$$\begin{aligned}\int_0^\infty Y_t(x)\frac{dx}{x^\alpha} &= \frac{1}{\Gamma(\frac{1}{2}-H)}\int_0^\infty\int_0^t e^{-(t-s)x} dW_s \frac{dx}{x^\alpha}\\ &= \frac{1}{\Gamma(\frac{1}{2}-H)}\int_0^t\int_0^\infty e^{-(t-s)x}\frac{dx}{x^\alpha} dW_s = \int_0^t (t-s)^\alpha dW_s.\end{aligned}$$

(b) can be seen as follows. Recall that $\beta = m-1$ and let

$$\begin{aligned}C_2 &= \sup_{\substack{t\in(-\infty,0]\\ x\in(0,\infty)}} |x^\beta \partial_x^m(xe^{tx})| = \sup_{\substack{t\in(-\infty,0]\\ x\in(0,\infty)}} |x^{m-1}\partial_x^m\partial_t e^{tx}|\\ &= \sup_{\substack{t\in(-\infty,0]\\ x\in(0,\infty)}} |x^{m-1}\partial_t(t^m e^{tx})| = \sup_{y\in(-\infty,0]}\left|my^{m-1}+y^m\right| e^y < \infty,\\ C_3 &= \sup_{x\in(0,\infty)} x^{-(\alpha-\frac{1}{2}-\delta)}\sqrt{\log(1+Tx)} < \infty,\end{aligned}$$

Using again the maximal inequality for Ornstein–Uhlenbeck processes of [23, Theorem 2.5 and Remark 2.6] and noting that $\log(1+Tx) \leq Tx$, one obtains the following three estimates:

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\sup_{x\in(0,\infty)}\left|x^{\beta}\partial_x^m Y_t(x)\right|\right] \\
&\qquad = \mathbb{E}\left[\sup_{t\in[0,T]}\sup_{x\in(0,\infty)}\left|\int_0^t W_s x^{m-1}\partial_x^m(xe^{-(t-s)x})ds\right|\right] \\
&\qquad \leq C_2 T\,\mathbb{E}\left[\sup_{t\in[0,T]}|W_t|\right] < \infty, \\
&\sup_{x_0\in[0,1]} x_0^{-\gamma}\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^{x_0} Y_t(x)x^{-\alpha}dx\right|\right] \\
&\qquad \leq C_1 \sup_{x_0\in[0,1]} x_0^{-\gamma}\int_0^{x_0}\sqrt{\frac{\log(1+Tx)}{x}}x^{-\alpha}dx \\
&\qquad \leq C_1 \sup_{x_0\in[0,1]} x_0^{-\gamma}\int_0^{x_0}\sqrt{T}x^{-\alpha}dx = C_1\sqrt{T}\gamma^{-1} < \infty, \\
&\sup_{x_1\in[1,\infty)} x_1^{\delta}\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_{x_1}^{\infty} Y_t(x)x^{-\alpha}dx\right|\right] \\
&\qquad \leq C_1 \sup_{x_1\in[1,\infty)} x_1^{\delta}\int_{x_1}^{\infty}\sqrt{\frac{\log(1+Tx)}{x}}x^{-\alpha}dx \\
&\qquad \leq C_1C_3 \sup_{x_1\in[1,\infty)} x_1^{\delta}\int_{x_1}^{\infty} x^{-1-\delta}dx = C_1C_3\delta^{-1} < \infty.
\end{aligned}$$

This shows **(b)** for $p = 1$. The generalization to $p \in [1,\infty)$ follow from the Kahane–Khintchine inequality applied to the Gaussian process Y.

(c) can be seen as follows. Let

$$C_4 = \mathbb{E}\left[\sup_{s,t\in[0,T]}\frac{|W_t - W_s|}{|t-s|^{\beta}}\right],$$

which is finite by the Hölder continuity of Brownian motion. Note that

$$\begin{aligned}
Y_t(x) &= W_t - \int_0^t W_t x e^{-(t-s)x}ds + \int_0^t (W_t - W_s)xe^{-(t-s)x}ds \\
&\leq (W_t - W_0)e^{-tx} + \int_0^t (W_t - W_s)xe^{-(t-s)x}ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\sup_{x\in(0,\infty)} x^{\beta}|Y_t(x)|\right] \\
&\qquad \leq C_4 \sup_{t\in[0,T]}\sup_{x\in(0,\infty)}\left((tx)^{\beta}e^{-tx} + \int_0^t (t-s)^{\beta}x^{\beta}xe^{-(t-s)x}ds\right)
\end{aligned}$$

$$= C_4 \sup_{y\in(0,\infty)} \left(y^\beta e^{-y} + \int_0^y z^\beta e^{-z} dz \right) \leq 2C_4.$$

This shows **(c)** for $p = 1$. The generalization to $p \in [1, \infty)$ follow from the Kahane–Khintchine inequality applied to the Gaussian process Y. □

4. Discretization

In this section, the measure μ in the integral representation of Volterra Brownian motion is approximated by a weighted sum of Dirac measures. More specifically, for each $n \in \mathbb{N}$, the positive half line is truncated to a finite interval $[\xi_{n,0}, \xi_{n,n}]$. This interval is then split into subintervals by a geometric sequence $(\xi_{n,i})_{i\in\{1,\dots,n\}}$, and on each subinterval $[\xi_{n,i}, \xi_{n,i+1}]$ the measure μ is approximated by an m-point interpolatory quadrature rule such as e.g. the Gauss rule. Classical error analysis for interpolatory quadrature rules (see e.g. [11]) then yields the desired convergence result.

Definition 1. Let $a, b \in \mathbb{R}$ satisfy $a < b$, let $w\colon [a, b] \to [0, \infty)$ be a continuous function such that $\int_a^b w(x)dx > 0$, and let $m \in \mathbb{N}_{>0}$. Then a measure μ on $[a, b]$ is called a non-negative m-point interpolatory quadrature rule on $[a, b]$ with respect to the weight function w if there are grid points $x_1, \dots, x_m \in [a, b]$ and weights $w_1, \dots, w_m \in [0, \infty)$ such that $\mu = \sum_{j=1}^m w_j \delta_{x_j}$ and

$$\forall k \in \{0, \dots, m-1\}: \qquad \int_a^b x^k w(x)\mu(dx) = \int_a^b x^k w(x)dx.$$

The assumptions of the following lemma are satisfied thanks to the bounds of Lemma 1, where the same constants $\alpha, \beta, \gamma, \delta, m$ are used.

Lemma 2. *Assume the setting of Section 2, let $m \in \mathbb{N}_{>0}$ and $\alpha, \beta, \gamma, \delta \in (0, \infty)$ satisfy $1 - \alpha - \beta + m > 0$, let*

$$Y : \Omega \to C^{0,m}([0, T] \times (0, \infty), \mathbb{R}) \cap C([0, T], L^1(\mu))$$

be a measurable function which satisfies the integrability conditions

$$\begin{aligned} \left\| \sup_{t\in[0,T]} \sup_{x\in[0,\infty)} \left| x^\beta \partial_x^m Y_t(x) \right| \right\|_{L^p(\Omega)} &< \infty, \\ \limsup_{x_0\downarrow 0} x_0^{-\gamma} \left\| \sup_{t\in[0,T]} \left| \int_0^{x_0} Y_t(x)x^{-\alpha}dx \right| \right\|_{L^p(\Omega)} &< \infty, \\ \limsup_{x_1\uparrow\infty} x_1^{\delta} \left\| \sup_{t\in[0,T]} \left| \int_{x_1}^{\infty} Y_t(x)x^{-\alpha}dx \right| \right\|_{L^p(\Omega)} &< \infty, \end{aligned}$$

let $r \in (0, \delta m/(1 - \alpha - \beta + \delta + m))$, for each $n \in \mathbb{N}$ and $i \in \{0, \dots, n-1\}$ let

$$\xi_{n,0} = n^{-r/\gamma}, \qquad \xi_{n,n} = n^{r/\delta}, \qquad \xi_{n,i} = \xi_{n,0}(\xi_{n,n}/\xi_{n,0})^{i/n},$$

let $\mu_{n,i}$ be a non-negative m-point interpolatory quadrature rule on $[\xi_{n,i}, \xi_{n,i+1}]$ with respect to the weight function $x \mapsto x^{-\alpha}$, and let $\mu_n = \sum_{i=0}^{n-1} \mu_{n,i}$. Then

$$\sup_{n\in\mathbb{N}} n^r \left\| \sup_{t\in[0,T]} \left| \int_0^\infty Y_t(x)x^{-\alpha}(\mu_n(dx) - dx) \right| \right\|_{L^p(\Omega)} < \infty.$$

Proof. We define the constants

$$\eta = \left(\frac{1}{r} - \frac{1-\alpha-\beta+m+\delta}{\delta m}\right) \Big/ \left(\frac{1}{\gamma} + \frac{1}{\delta}\right) \in (0,\infty),$$

$$C_1 = \frac{\pi^m}{m!2^m} \left\| \sup_{t\in[0,T]} \sup_{x\in[0,\infty)} |x^\beta Y_t^{(m)}(x)| \right\|_{L^p(\Omega)} \in (0,\infty),$$

$$C_2 = \sup_{\lambda\in(1,\infty)} \frac{\lambda-1}{\lambda^{1-\alpha-\beta+m}-1} \in (0, 1/(1-\alpha-\beta+m)],$$

$$C_3 = \sup_{\xi\in[1,\infty)} \sup_{n\in[\log\xi,\infty)} \left(\xi^{1/n}-1\right) n\xi^{-\eta} \in (0,\infty),$$

$$C_4 = \min\left\{ n\in\mathbb{N}; n \geq \log(\xi_{n,n}/\xi_{n,0}) = \left(\frac{r}{\gamma}+\frac{r}{\delta}\right)\log(n)\right\} < \infty,$$

where the upper bound on C_2 follows from Bernoulli's inequality

$$\forall\lambda\in[0,\infty): \quad \lambda^{1-\alpha-\beta+m} = (1+(\lambda-1))^{1-\alpha-\beta+m} \geq 1 + (1-\alpha-\beta+m)(\lambda-1),$$

and the upper bound on C_3 follows from the inequality

$$\forall\xi\in[1,\infty)\forall n\in[\log(\xi),\infty): \quad \xi^{1/n}-1 = \exp(\log(\xi)/n) - 1 \leq e\log(\xi)/n.$$

By [11, Theorem 4.2.3] one has for each $t\in[0,T]$, $n\in\mathbb{N}$, and $i\in\{0,\ldots,n-1\}$ that

$$\int_{\xi_{n,i}}^{\xi_{n,i+1}} Y_t(x)x^{-\alpha}(\mu_n(dx)-dx) = \int_{\xi_{n,i}}^{\xi_{n,i+1}} \partial_x^m Y_t(x) K_{n,i}(x)dx,$$

where the Peano kernel $K_{n,i}\colon[\xi_{n,i},\xi_{n,i+1}]\to\mathbb{R}$ is a measurable function which satisfies [11, Theorem 5.7.1]

$$\sup_{x\in[\xi_{n,i},\xi_{n,i+1}]} |K_{n,i}(x)| \leq \frac{\pi^m}{m!}\left(\frac{\xi_{n,i+1}-\xi_{n,i}}{2}\right)^m \sup_{x\in[\xi_{n,i},\xi_{n,i+1}]} x^{-\alpha}.$$

Thus, one has for each $n\in\mathbb{N}$ that

$$\begin{aligned}
(*) \quad & \left\| \sup_{t\in[0,T]} \left| \int_{\xi_{n,0}}^{\xi_{n,n}} Y_t(x)x^{-\alpha}(\mu_n(dx)-dx)\right| \right\|_{L^p(\Omega)} \\
&\leq \sum_{i=0}^{n-1} \left\| \sup_{t\in[0,T]} \left| \int_{\xi_{n,i}}^{\xi_{n,i+1}} Y_t(x)K_{n,i}(x)dx \right| \right\|_{L^p(\Omega)} \\
&\leq \sum_{i=0}^{n-1} \frac{\pi^m}{m!2^m} \left\| \sup_{\substack{t\in[0,T] \\ x\in[\xi_{n,i},\xi_{n,i+1}]}} |x^\beta Y_t^{(m)}(x)| \right\|_{L^p(\Omega)} \xi_{n,i}^{-\alpha-\beta}(\xi_{n,i+1}-\xi_{n,i})^{m+1} \\
&\leq C_1 \sum_{i=0}^{n-1} \xi_{n,i}^{-\alpha-\beta}(\xi_{n,i+1}-\xi_{n,i})^{m+1}.
\end{aligned}$$

This can be expressed as a geometric series: letting $\lambda_n = (\xi_{n,n}/\xi_{n,0})^{1/n}$, one has for each $n\in\mathbb{N}$ that

$$(*) = C_1 \xi_{n,0}^{1-\alpha-\beta+m}(\lambda_n-1)^{m+1} \sum_{i=0}^{n-1} \lambda_n^{i(1-\alpha-\beta+m)}$$

$$= C_1 \xi_{n,0}^{1-\alpha-\beta+m} (\lambda_n - 1)^{m+1} \frac{\lambda_n^{n(1-\alpha-\beta+m)} - 1}{\lambda_n^{1-\alpha-\beta+m} - 1}$$
$$= C_1 (\lambda_n - 1)^{m+1} \frac{\xi_{n,n}^{1-\alpha-\beta+m} - \xi_{n,0}^{1-\alpha-\beta+m}}{\lambda_n^{1-\alpha-\beta+m} - 1}.$$

Absorbing the denominator into one of the factors $(\lambda_n - 1)$ and discarding the term $\xi_{n,0}$ yields for each $n \in \mathbb{N}$ that

$$(*) \leq C_1 C_2 (\lambda_n - 1)^m \xi_{n,n}^{1-\alpha-\beta+m} = C_1 C_2 ((\xi_{n,n}/\xi_{n,0})^{1/n} - 1)^m \xi_{n,n}^{1-\alpha-\beta+m}.$$

For each $n \in \mathbb{N} \cap [C_4, \infty)$, this can be estimated by

$$\begin{aligned}(*) &\leq C_1 C_2 C_3^m n^{-m} (\xi_{n,n}/\xi_{n,0})^{\eta m} \xi_{n,n}^{1-\alpha-\beta+m} \\ &= C_1 C_2 C_3^m n^{-m+\eta m r(1/\gamma+1/\delta)+(1-\alpha-\beta+m)r/\delta} = C_1 C_2 C_3^m n^{-r}.\end{aligned}$$

Therefore, noting that $n^r = \xi_{n,0}^{-\gamma} = \xi_{n,n}^{\delta}$, one has

$$\begin{aligned}&\limsup_{n\to\infty} n^r \mathbb{E}\left[\sup_{t\in[0,T]} \left| \int_0^\infty Y_t(x) x^{-\alpha} (\mu_n(dx) - dx) \right| \right] \\ &\quad \leq \limsup_{n\to\infty} \xi_{n,0}^{-\gamma}\ \mathbb{E}\left[\sup_{t\in[0,T]} \left| \int_{(0,\xi_{n,0}]} Y_t(x) x^{-\alpha} dx \right| \right] \\ &\qquad + \limsup_{n\to\infty} \xi_{n,n}^{\delta}\ \mathbb{E}\left[\sup_{t\in[0,T]} \left| \int_{[\xi_{n,n},\infty)} Y_t(x) x^{-\alpha} dx \right| \right] \\ &\qquad + \sup_{n\in\mathbb{N}} n^r \mathbb{E}\left[\sup_{t\in[0,T]} \left| \int_{\xi_{n,0}}^{\xi_{n,n}} Y_t(x) x^{-\alpha} (\mu_n(dx) - dx) \right| \right] < \infty. \qquad \square\end{aligned}$$

Remark 1. *The choice of the quadrature rule in Lemma 2 is admittedly somewhat arbitrary but produces good results. The use of the geometric grid $\xi_{n,i}$ goes back to [13] and simplifies the error analysis compared to more complex subdivisions which distribute the error more equally. It would be interesting to explore if the holomorphicity of $x \mapsto Y_t(x)$ permits the use of quadrature rules with exponential convergence rates such as Chebychev quadrature; see the discussion in Section 3.*

5. Rough Bergomi model

The prices of put options in the rough Bergomi model converge at the same rate as the approximated Volterra processes. This holds not only for the Ornstein–Uhlenbeck approximations of Lemma 2, but in full generality for any approximations in the $L^2([0,T] \times \Omega)$ norm with exponential moment bounds.

Lemma 3. *Assume the setting of Section 2, let $V, \tilde{V}, S, \tilde{S} \colon [0,T] \times \Omega \to \mathbb{R}$ be continuous stochastic processes with $V_0 = \tilde{V}_0 = 0$ and*

$$\forall t \in [0,T]: \quad S_t = 1 + \int_0^t S_s \exp(V_s) dW_s, \quad \tilde{S}_t = 1 + \int_0^t \tilde{S}_s \exp(\tilde{V}_s) dW_s,$$

and let $f \colon (0,\infty) \to \mathbb{R}$ be a measurable function such that $f \circ \exp \in \operatorname{Lip}(\mathbb{R})$. Then

$$\begin{aligned}\left|\mathbb{E}[f(S_T)] - \mathbb{E}[f(\tilde{S}_T)]\right| &\leq \|f \circ \exp\|_{\operatorname{Lip}(\mathbb{R})} \big(\sqrt{T} + 6\big) \\ &\quad \times \|\exp(2|V|) + \exp(2|\tilde{V}|)\|_{L^2(\Omega, C([0,T]))} \|V - \tilde{V}\|_{L^2([0,T]\times\Omega)}.\end{aligned}$$

Proof. It is sufficient to control the log prices in L^1 because

$$\big|\mathbb{E}[f(S_T)] - \mathbb{E}[f(\tilde{S}_T)]\big| \leq \|f \circ \exp\|_{\mathrm{Lip}(\mathbb{R})} \|\log(S_T) - \log(\tilde{S}_T)\|_{L^1(\Omega)}.$$

The basic inequality

$$\forall x, y \in \mathbb{R}: \quad \big|\exp(x) - \exp(y)\big| \leq \big(\exp(x) + \exp(y)\big)|x - y|$$

and the Burkholder–Davis–Gundy inequality imply that

$$\begin{aligned}
&\|\log(S_T) - \log(\tilde{S}_T)\|_{L^1(\Omega)} \\
&\quad = \left\| -\frac{1}{2}\int_0^T \big(\exp(2V_t) - \exp(2\tilde{V}_t)\big)dt + \int_0^T \big(\exp(V_t) - \exp(\tilde{V}_t)\big)dW_t \right\|_{L^1(\Omega)} \\
&\quad \leq \left\| \frac{1}{2}\int_0^T \big(\exp(2V_t) + \exp(2\tilde{V}_t)\big)(2V_t - 2\tilde{V}_t)dt \right\|_{L^1(\Omega)} \\
&\qquad + 6\left\| \sqrt{\int_0^T \big(\exp(V_t) + \exp(\tilde{V}_t)\big)^2 (V_t - \tilde{V}_t)^2 dt} \right\|_{L^1(\Omega)} \\
&\quad \leq \|\exp(2|V|) + \exp(2|\tilde{V}|)\|_{L^2(\Omega, C([0,T]))} \\
&\qquad \times \left(\left\| \int_0^T (V_t - \tilde{V}_t)dt \right\|_{L^2(\Omega)} + 6\left\| \sqrt{\int_0^T (V_t - \tilde{V}_t)^2 dt} \right\|_{L^2(\Omega)} \right) \\
&\quad \leq \|\exp(2|V|) + \exp(2|\tilde{V}|)\|_{L^2(\Omega, C([0,T]))} \big(\sqrt{T} + 6\big) \|V - \tilde{V}\|_{L^2([0,T]\times\Omega)}. \qquad \square
\end{aligned}$$

Remark 2. *For each $K \in (0, \infty)$ the put-option payoff*

$$f\colon (0,\infty) \to \mathbb{R}, \qquad x \mapsto (K - x)_+,$$

satisfies the assumption of Lemma 3 that $f \circ \exp \in \mathrm{Lip}(\mathbb{R})$ because

$$\sup_{\substack{x,y\in\mathbb{R}\\ x\neq y}} \frac{|f(e^y) - f(e^x)|}{|y - x|} \leq e^K < \infty.$$

The call-option payoff does not have this property, but the prices of call options can be obtained by put-call parity if W and B are negatively correlated because this implies that S is a martingale [21].

References

[1] E. Abi Jaber. *Lifting the Heston model.* HAL: `hal-01890751`.

[2] E. Abi Jaber and O. El Euch. *Markovian structure of the Volterra Heston model.* 2018. arXiv: `1803.00477`.

[3] E. Abi Jaber and O. El Euch. *Multi-factor approximation of rough volatility models.* 2018. arXiv: `1801.10359`.

[4] E. Abi Jaber, M. Larsson, and S. Pulido. *Affine Volterra processes.* 2017. arXiv: `1708.08796`.

[5] E. Alòs, J. A. León, and J. Vives. "On the short-time behavior of the implied volatility for jump-diffusion models with stochastic volatility". In: *Finance and Stochastics* 11.4 (2007), pp. 571–589.

[6] C. Bayer, P. Friz, and J. Gatheral. "Pricing under rough volatility". In: *Quantitative Finance* 16.6 (2016), pp. 887–904.

[7] C. Bayer et al. *A regularity structure for rough volatility*. 2017. arXiv: 1710.07481.

[8] C. Bayer et al. *Short-time near-the-money skew in rough volatility models*. arXiv: 1703.05132.

[9] M. Bennedsen, A. Lunde, and M. S. Pakkanen. *Decoupling the short-and long-term behavior of stochastic volatility*. 2016. arXiv: 1610.00332.

[10] M. Bennedsen, A. Lunde, and M. S. Pakkanen. "Hybrid scheme for Brownian semistationary processes". In: *Finance and Stochastics* 21.4 (2017), pp. 931–965.

[11] H. Brass and K. Petras. *Quadrature theory*. Vol. 178. Mathematical Surveys and Monographs. Providence, RI: American Mathematical Society, 2011.

[12] P. Carmona and L. Coutin. "Fractional Brownian motion and the Markov property". In: *Electronic Communications in Probability* 3 (1993), pp. 95–107.

[13] P. Carmona, L. Coutin, and G. Montseny. "Approximation of some gaussian processes". In: *Statistical Inference for Stochastic Processes* 3.1 (2000), pp. 161–171.

[14] C. Cuchiero and J. Teichmann. *Generalized Feller processes and Markovian lifts of stochastic Volterra processes: the affine case*. 2018. arXiv: 1804.10450.

[15] T. Dieker. "Simulation of fractional Brownian motion". MA thesis. University of Twente, 2004.

[16] C. R. Dietrich and G. N. Newsam. "Fast and exact simulation of stationary Gaussian processes through circulant embedding of the covariance matrix". In: *SIAM Journal on Scientific Computing* 18.4 (1997), pp. 1088–1107.

[17] M. Forde and H. Zhang. "Asymptotics for rough stochastic volatility models". In: *SIAM Journal on Financial Mathematics* 8.1 (2017), pp. 114–145.

[18] M. Fukasawa. "Asymptotic analysis for stochastic volatility: martingale expansion". In: *Finance and Stochastics* 15.4 (2011), pp. 635–654.

[19] M. Gaß, K. Glau, M. Mahlstedt, and M. Mair. "Chebyshev Interpolation for Parametric Option Pricing". In: *Working Paper* (2016). arXiv: 1511.00884.

[20] M. Gaß, K. Glau, and M. Mair. "Magic Points in Finance: Empirical Interpolation for Parametric Option Pricing (first version 2015)". In: *Working Paper* (2016). arXiv: 1511.00884.

[21] P. Gassiat. *On the martingale property in the rough Bergomi model*. 2018. arXiv: 1811.10935.

[22] J. Gatheral, T. Jaisson, and M. Rosenbaum. *Volatility is rough*. 2014. arXiv: 1410.3394.

[23] S. E. Graversen and G. Peskir. "Maximal inequalities for the Ornstein-Uhlenbeck process". In: *Proceedings of the American Mathematical Society* (2000), pp. 3035–3041.

[24] P. Harms and D. Stefanovits. *Affine representations of fractional processes with applications in mathematical finance*. 2016. arXiv: 1510.04061.

[25] B. Horvath, A. Jacquier, and A. Muguruza. *Functional central limit theorems for rough volatility*. 2017. arXiv: 1711.03078.

[26] J. R. Hosking. "Modeling persistence in hydrological time series using fractional differencing". In: *Water resources research* 20.12 (1984), pp. 1898–1908.

[27] R. McCrickerd and M. S. Pakkanen. “Turbocharging Monte Carlo pricing for the rough Bergomi model”. In: *Quantitative Finance* (2018), pp. 1–10.

[28] A. A. Muravlev. “Representation of a fractional Brownian motion in terms of an infinite-dimensional Ornstein-Uhlenbeck process”. In: *Russian Mathematical Surveys* 66.2 (2011), pp. 439–441.

[29] L. Mytnik and T. S. Salisbury. *Uniqueness for Volterra-type stochastic integral equations*. 2015. arXiv: `1502.05513`.

Department of Stochastics, University of Freiburg
E-mail address: `philipp.harms@stochastik.uni-freiburg.de`