BREAKING BARRIERS

Peter Carr **and** Andrew Chou **show how to price and hedge barrier claims using a static portfolio of vanilla options**

Static hedging was introduced in Bowie & Carr (1994) as a way to hedge barrier options when the underlying is a futures price with no drift. Derman, Ergener & Hani (1994) relaxed this drift restriction by introducing an algorithm for hedging barrier options in a binomial model, using options with a single strike but multiple expiries. By contrast, this article provides explicit formulas for static hedges in the standard Black-Scholes (1973) model, using options with the same expiry but multiple strikes.

Exotic options are very sophisticated instruments, and the techniques used to hedge and value them are fairly complex. The most popular are barrier options, which were introduced in the US over-the-counter markets years before vanilla options were listed (see Snyder, 1969). Barrier options are special cases of barrier securities, which may involve single or multiple barriers. Examples of the latter include double barrier options, rolldown calls and lookback options. For simplicity, we will focus on single barrier securities in this article. These allow for an arbitrary payout at maturity provided that the barrier has been touched (in-barrier securities) or not touched (out-barrier securities). They are usually further classified into "down securities" (barrier below spot) and "up securities" (barrier above spot).

The standard methodology for hedging and valuing barrier securities applies dynamic replication strategies in the underlying assets. We hope to add insight into these structures by looking at them in another way. In particular, we show how barrier securities can be broken up into more fundamental securities, which, in turn, can be created out of vanilla European options. This allows us to hedge path-dependent barrier securities with path-independent vanilla options, with trading in the vanilla options occurring only at the initiation and expiry[1] of the hedge. Due to the relative infrequency of trading, such hedges are commonly termed "static".

In common with dynamic hedging, static hedging provides valuation formulas for barrier securities. When both types of hedging strategies are cast in the same economic model, the formulas result in identical values. We will show how valuation formulas based on dynamic replication can be used to uncover the static hedge.

Since barrier option formulas have been around since Merton's seminal 1973 paper, they are now widely available (see Nelken, 1995, and Zhang, 1995, for surveys on exotic options). These formulas can be used to determine static hedges for a host of exotic options beyond those explicitly presented in this paper, such as double and partial barrier options.

We will illustrate our decomposition results with several commonly available types of barrier securities. We give explicit results for down barriers, including down calls, down puts and several types of binary options. By definition, a one-touch European-style binary put pays one dollar at maturity if the lower barrier has been hit, while a one-touch American-style binary pays a dollar at the first hitting time, if any. By contrast, a no-touch binary pays one dollar at maturity if the barrier is never reached. We indicate the static hedge for all three types of binaries.

Static benefits

While both dynamic and static replication strategies work perfectly well in theory in our model, there are several reasons why static hedging could be the method of choice in practice:

- First, a literal interpretation of dynamic replication requires continuous trading, which would generate ruinous transaction costs if implemented in practice. The standard compromise made for this problem is to trade periodically, which leads to acceptably low approximation error when the gamma of the security is low. However, barrier options often have regions of high gamma. These can be catastrophic to these periodically rebalanced strategies but are of no consequence to the static hedger, as long as the investor can trade when the barrier is first hit. Jumps across the barrier can induce sub- or super-replication for both types of strategies, with the superior strategy usually identifiable in advance.
- Second, dynamic replication requires estimation of the future carrying costs and volatility of the underlying asset. The error arising from using the wrong volatility in dynamic replication is directly proportional to the option's vega, which again is often high for barrier securities. By contrast, static replication needs only the implied volatility of the vanilla options at the entry of the static hedge and when the underlying is at the barrier. The volatility realised during the life of the hedge is of no consequence to the static hedger, except to the extent that it affects implied volatility.

The above benefits of static hedging may well be embedded in the implied volatility, which is typically greater than historical volatility. However, if this premium is paid at the initiation of the static hedge, it should be at least partially returned if the hedge is liquidated at the barrier. Thus, the main disadvantage of static hedging over dynamic hedging in practice appears to be the relative illiquidity of the standard options market compared with the market for the underlying asset. Perhaps this paper will help mitigate this disadvantage. In any case, empirical work is needed to compare the relative viability of these approaches.

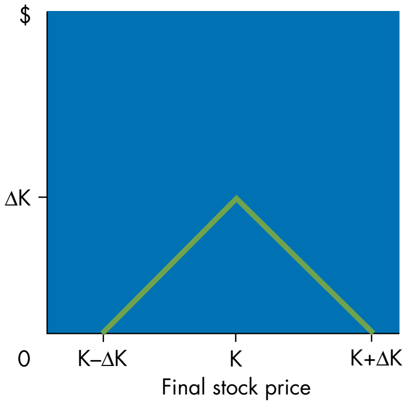
This paper will provide explicit formulas for static hedges in the standard Black-Scholes model using options with the same expiry date but multiple strikes. However, before we use the Black-Scholes model, let us develop a set of results in a more general setting. Thus, we initially assume only that markets are frictionless and arbitrage-free. To simplify notation, we also assume that investors can borrow or lend at a constant riskless rate r and that the underlying asset is a stock, with a constant dividend yield, d. These results can easily be extended to stochastic interest rates and dividend yields.

An implication of frictionless markets is that investors can trade continuously in all barrier securities. For our present purposes, hedges will only require a liquid market in knock-in options with the same trigger as the barrier security, but with any positive strike. Just as continuous trading is accepted as a reasonable approximation to reality even though markets close daily, we treat the availability of a continuum of strikes as an approximation of the over-the-counter market in barrier options. While we fully recognise that markets for barrier securities have limited liquidity in practice, we note that path-independent payouts arise as special cases[2] of the payouts from barrier securities. Given the advent of customisable options in the listed market and the emergence of a substantial over-the-counter mar-

[1] We define the expiry of the hedge as the earlier of maturity and the first hitting time of the barrier

[2] To achieve results for path-independent securities from the corresponding results for in-barrier securities, the barrier is moved to the spot. Conversely, to obtain results from out-barrier securities, the barrier is moved an infinite distance away from the spot

1. Payout of a down-and-in butterfly spread



ket in vanilla options, our liquidity assumption is tenable for this important special case.

We will show how a butterfly spread of down-and-in options can be used to create a fundamental security called a "down-and-in Arrow". We will then show that down-and-in options can be replicated with vanilla options. The net result is that a barrier security can be statically hedged with a portfolio of vanilla options.

Our decomposition of barrier securities into down-and-in options can be applied to the path-independent case by moving the barrier appropriately. In particular, we can decompose claims with an arbitrary path-independent payout into a static portfolio consisting of listed instruments such as bonds, forward contracts and vanilla options. A special case of this formula leads to a new decomposition of the claim value into intrinsic and time value. Besides being of intrinsic interest, our results on the path-independent case can be combined with our results for in-securities to obtain the corresponding static hedge and valuation formula for "out-securities".

Static replication with "Arrows"

A simple way to bet that the underlying will finish around K is to form a butterfly spread with vanilla calls, which costs:

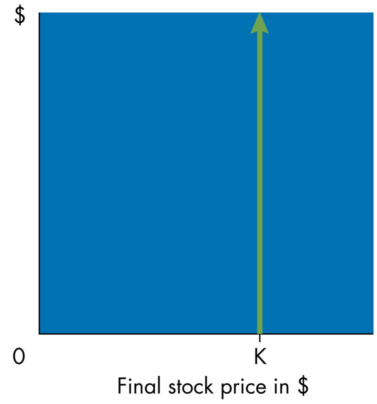
$$BS(K) = C(K - \Delta K) - 2C(K) + C(K + \Delta K)$$

where C(K) is the current price of a call struck at K. If we also wish to bet that a lower barrier was hit, the butterfly spread should be formed from down-and-in calls with the same barrier:

$$DIBS(K,H) = DIC(K - \Delta K,H) - 2DIC(K,H) + DIC(K + \Delta K,H)$$

where DIC(K,H) is the current price of a down-and-in call struck at K with barrier H. As long as the barrier has been hit, the final payout of this position is a triangle, as shown in figure 1.

2. Payout of a down-and-in Arrow



Note that the area under the triangle is:

$$\tfrac{1}{2} \times 2\Delta K \times \Delta K = (\Delta K)^2$$

Thus, this bet can be normalised so that the area under the triangle is one:

$$NDIBS(K,H) = \frac{DIC(K - \Delta K,H) - 2DIC(K,H) + DIC(K + \Delta K,H)}{(\Delta K)^2}$$

As ΔK approaches zero, the base of the triangular payout gets smaller and the height gets taller, so that the area is maintained at one. The limiting payout approaches a Dirac delta function. The securities providing this payout are called Arrow-Debreu securities, named after their founders, Nobel Prize winners Kenneth Arrow and Gerard Debreu. Given their heritage and their payout structure, we will refer to these securities as "down-and-in Arrows" (see figure 2). These securities are fundamental in the sense that an arbitrary payout from a down-and-in security may be easily decomposed into a portfolio of such securities. It follows that the arbitrary down-and-in security can be statically hedged and valued by a portfolio of down-and-in options.

Since the payout of a down-and-in Arrow is non-standard, it may be properly called a second-generation derivative security. The name is appropriate in another sense since the value of a down-and-in Arrow is given by the second derivative of the down-and-in call value with respect to its strike:

$$DIA(K,H) = \frac{\partial^2 DIC(K,H)}{\partial K^2} \qquad (1)$$

where DIA(K,H) denotes the current value of a down-and-in Arrow struck at K with barrier H.

We next show how Arrows can alternatively be observed from put values. Let DIB(H) be the value of a down-and-in bond which pays one dollar at T, so long as the stock price has hit the barrier H previously. Similarly, let DIS(H) denote the value of a down-and-in stock, which pays the stock price at T, as long as the barrier has been hit beforehand. There is a simple generalisation of put-call parity involving these contracts:

$$DIC(K,H) = DIS(H) - KDIB(H) + DIP(K,H) \qquad (2)$$

Differentiating twice with respect to the strike K implies that the down-and-in Arrow's value can alternatively be derived from down-and-in put values:

$$DIA(K,H) = \frac{\partial^2 DIP(K,H)}{\partial K^2} \qquad (3)$$

Now, let f(S) denote an arbitrary final payout received so long as the barrier has been hit. By buying and holding a portfolio of down-and-in Arrows of all strikes, with the number of Arrows at strike K given by f(K)dK, an investor can synthesise the payout f(S). Absence of arbitrage thereby requires that the value of the down-and-in claim DIV(H) paying f(S) at maturity is simply:

$$DIV(H) = \int_0^\infty f(K) DIA(K,H) dK \qquad (4)$$

When viewed as functions of their strike, down-and-in puts have zero value and slope at K=0, while down-and-in calls have zero value and slope at K = ∞. This observation motivates rewriting equation (4) as:

$$DIV(H) = \int_0^\kappa f(K) \frac{\partial^2 DIP(K,H)}{\partial K^2} dK + \int_\kappa^\infty f(K) \frac{\partial^2 DIC(K,H)}{\partial K^2} \qquad (5)$$

where κ is an arbitrary positive constant. Integrating by parts twice and using equation (2) yields the following decomposition of an arbitrary down-and-in claim into down-and-in versions of zeros, forward contracts[3] and options[4]:

$$\begin{aligned} DIV(H) = {} & f(\kappa) DIB(H) \\ & + f'(\kappa)\left[DIS(H) - \kappa DIB(H)\right] \\ & + \int_0^\kappa f''(K) DIP(K,H) dK \\ & + \int_\kappa^\infty f''(K) DIC(K,H) dK \end{aligned} \qquad (6)$$

Thus, to synthesise the payout f(S) received at T if the barrier has been hit, buy and hold a portfolio consisting of f(κ) down-and-in bonds; f′(κ) down-and-in forward contracts with delivery price κ; $f''(K)dK$ down-and-in puts of all strikes $K \leq \kappa$; and $f''(K)dK$ down-and-in calls of all strikes $K > \kappa$. Setting the barrier H to infinity in equation (6) yields the corresponding result for path-independent payouts as a special case:

$$V = f(\kappa)e^{-rT} + f'(\kappa)\left[Se^{-dT} - \kappa e^{-rT}\right] + \int_0^{\kappa} f''(K)P(K)\,dK + \int_{\kappa}^{\infty} f''(K)C(K)\,dK \qquad (7)$$

Further setting κ to the forward price $F \equiv Se^{(r-d)T}$ yields a new decomposition of a claim with an arbitrary path-independent payout into its intrinsic value, $f(F)e^{-rT}$, and its time value:

$$V = f(F)e^{-rT} + \int_0^{F} f''(K)P(K)\,dK + \int_F^{\infty} f''(K)C(K)\,dK \qquad (8)$$

The time value is expressed by the prices of out-of-the-money forward puts and calls. Since puts and calls with the same strike have the same time value, the time value of an arbitrary claim is simply a linear combination of the time values of an option, with coefficients given by the second derivative of the payout. If the payout is linear, then $f''(K) = 0$ for all K and there is no time value. Conversely, if the payout is globally convex ($f''(K) \geq 0$ for all K), then the time value is positive. Finally, note that if we restrict attention to Black's model then, as the underlying gets more volatile, the option values grow and so, therefore, does the time value.

If we set κ to infinity or zero in equation (6), then certain types of contracts can be eliminated from the static hedge, provided f behaves reasonably at these extremes. For example, setting κ to zero and H to infinity in equation (6) eliminates puts from the static hedge, provided that $f(0)$ and $f'(0)$ are bounded:

$$V = f(0)B + f'(0)Se^{-dT} + \int_0^{\infty} f''(K)C(K)\,dK \qquad (9)$$

If f is not smooth, generalised functions such as Heaviside step functions and their derivatives may be needed. For example, to replicate

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[3] Note that barrier forward contracts are easily synthesised by buying a barrier call and writing a barrier put

[4] Technically, (6) holds only for payouts f(K) which satisfy:

$$\lim_{K\downarrow 0} f(K)\frac{\partial DIP(K,H)}{\partial K} = 0 \quad \lim_{K\downarrow 0} f'(K)DIP(K,H) = 0$$

$$\lim_{K\uparrow \infty} f(K)\frac{\partial DIC(K,H)}{\partial K} = 0 \quad \lim_{K\uparrow 0} f'(K)DIC(K,H) = 0$$

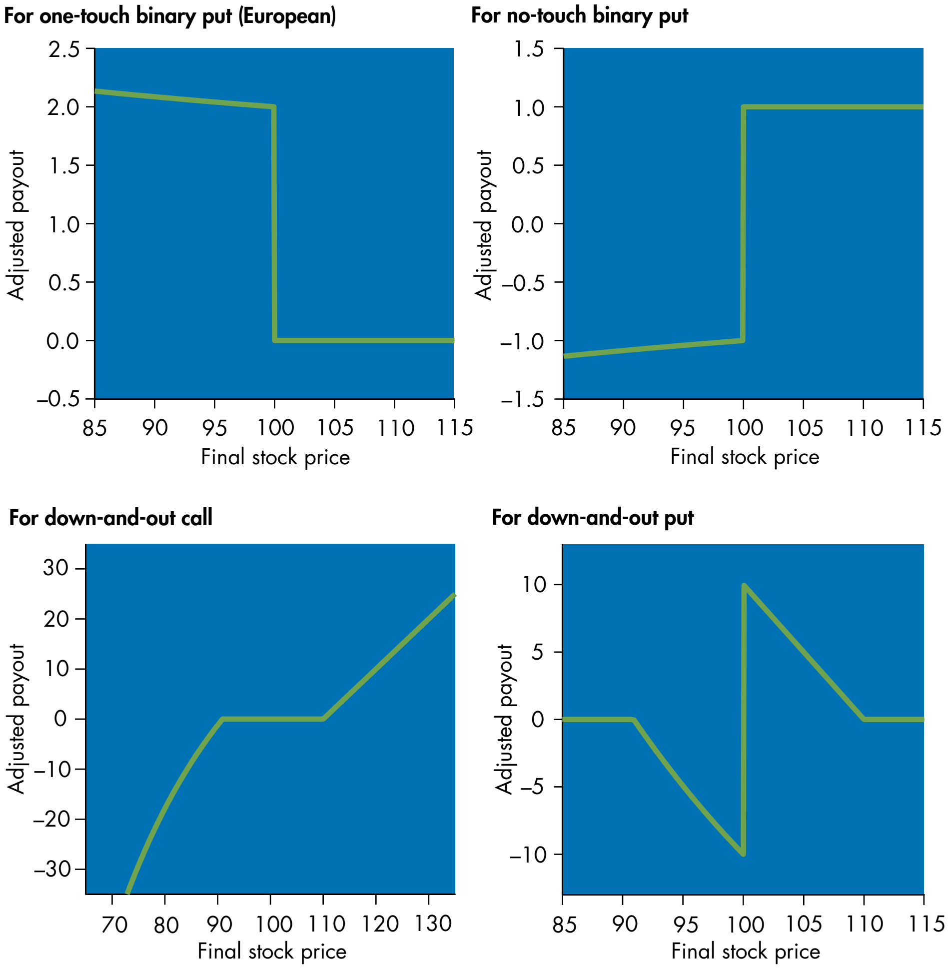
It is difficult to imagine payouts arising in practice that do not satisfy these restrictions

A. Adjusted payouts for down securities ($p = 1-(2(r-d)/\sigma^2)$)

Barrier security	Adjusted payout when $S_T > H$	Adjusted payout when $S_T < H$
No-touch binary put	1	$-(S_T/H)^p$
One-touch binary put (European)	0	$1+(S_T/H)^p$
Down-and-out call	$\max(S_T-K_C,0)$	$-(S_T/H)^p \max((H^2/S_T)-K_C,0)$
Down-and-out put	$\max(K_p-S_T,0)$	$-(S_T/H)^p \max(K_p-(H^2/S_T),0)$

3. Adjusted payouts for down securities

$r = 0.05$; $d = 0.03$; $\sigma = 0.15$; $K_c = K_p = 110$; $H = 100$



a vanilla put's payout of $f(S) = \max(0, K_p-S)$, equation (9) indicates that one should buy K zeros, sell e^{-dT} shares and buy a call struck at K_p. Thus, equation (7) generates put-call parity as a special case. Similarly, it can be used to generate the replication of digital options using vertical spreads.

To generate the corresponding results for down-and-out securities, subtract equation (6) from equation (7) and use in-out parity:

$$DOV(H) = f(\kappa)DOB(H) + f'(\kappa)\left[DOS(H) - \kappa DOB(H)\right] + \int_0^{\kappa} f''(K)DOP(K,H)dK + \int_{\kappa}^{\infty} f''(K)DOC(K,H)dK$$

To generate results for up-securities, replace D with U in all the above results.

Barrier securities can also be statically replicated with vanilla options, which is particularly useful because these are more liquid than barrier options in practice, and their prices are more transparent. However, the theoretical cost of this replication is the imposition of the rest of the Black-Scholes assumptions. We therefore assume henceforth that the stock price obeys a lognormal process with a constant volatility rate σ. Importantly, the price process is continuous, so that the underlying cannot jump across the barrier[5].

The hedging technique

Here we provide the main intuition behind our result. Although the actual technique is fairly direct, its simplicity may be lost in the details. Interested readers can find the full derivation in the appendix of the paper with the same title available on the World Wide Web at: *www.math.nyu.edu/research/carrp/papers*.

Consider a down-and-in call option. If the barrier is never reached, it will expire worthless at maturity. Upon reaching the barrier, it becomes identical to a vanilla call. To replicate this exotic, we want a portfolio of European options to imitate this behaviour. If the barrier is never reached, our portfolio should be worthless at maturity; at the barrier, it should always be equivalent to a call.

Depending on its strike, a down-and-in call can have payouts both above and below the barrier. For payouts below the barrier, the requirement that the in-barrier be touched is superfluous, and so we can replicate with European-style options as above. We will reflect the payouts above the barrier below the barrier. The reflected payouts will be constructed to match the values of the original whenever the stock price is at the barrier. Thus, we can also replicate the reflected payouts with European options to complete our static hedge.

More generally, suppose a European security has final payout $f(S_T)$. It can be shown (as on the Web) that the down-and-in version of this security with barrier H has the same value as a portfolio of European-style options with a payout of:

$$\hat{f}(S_T) \equiv \begin{cases} 0 & \text{if } S_T > H \\ f(S_T) + \left(\frac{S_T}{H}\right)^p f\left(\frac{H^2}{S_T}\right) & \text{if } S_T < H \end{cases}$$

[5] If jumps were possible, we could forecast whether our static portfolio would overvalue or undervalue the exotic

where the power:

$$p \equiv 1 - \frac{2(r-d)}{\sigma^2}$$

We call $\hat{f}(S_T)$ the *adjusted payout* for the down-and-in security. For a down-and-out security, in-out parity implies that the adjusted payout is:

$$\hat{f}(S_T) \equiv \begin{cases} f(S_T) & \text{if } S_T > H \\ -\left(\frac{S_T}{H}\right)^p f\left(\frac{H^2}{S_T}\right) & \text{if } S_T < H \end{cases}$$

In table A and figure 3, we show the adjusted payout for some common securities. The table shows that the adjusted payouts are usually not piecewise linear. Thus, exact replication using a finite number of European puts and calls is not usually possible. However, as figure 3 illustrates, the payouts are close to piecewise linear. Furthermore, a few special cases are worth mentioning. When $r=d$, then $p=1$ and all payouts are piecewise linear. The resulting payouts are identical to the results given in Bowie & Carr. Also, for:

$$r - d = \frac{\sigma^2}{2}$$

then $p = 0$ and the binary payouts are piecewise linear. In particular, a one-touch binary can be exactly replicated by two digitals.

Given the adjusted payout, the value of the replicating portfolio can be determined by risk-neutral valuation:

$$V(S,\tau) = \int_0^\infty \hat{f}(K)\, A(S,\tau;K)dK \qquad S > 0$$

where the value of an Arrow in the Black-Scholes model is:

$$A(S,\tau;K) = \exp(-r\tau)\frac{1}{K\sqrt{2\pi\sigma^2\tau}} \times \exp\left\{-\frac{\left(\ln(K/S) - \left(r - d - \frac{\sigma^2}{2}\right)\tau\right)^2}{2\sigma^2\tau}\right\}$$

The value of the down-and-in claim is obtained by restricting this value to stock prices above the barrier:

$$DIV(S,\tau;H) = V(S,\tau) \qquad S > H$$

The static hedge can also be derived in another way. We suppose that a pricing formula for a barrier security is known, either because it exists in the literature (see Rubinstein & Reiner 1991ii), or because it has been derived using dynamic replication arguments. This formula can then be used to generate a static hedge using vanilla options. The advantage of approaching static hedging in this manner is that it is very simple and the approach can be used to generate static hedges for a wide set of securities.

For simplicity, we again work with down securities only. We essentially work backwards from the results above. Thus, we assume we know the formula $D(S,\tau)$ for a down security as a function of the current stock price S and the time to maturity τ. The first step is to find the value of the replicating option portfolio for any stock price by simply removing the restriction that stock prices are above the barrier:

$$V(S,\tau) = D(S,\tau) \qquad S > 0 \qquad (11)$$

The second step is to obtain the adjusted payout that gave rise to this value. Since values converge to their payout at maturity, we simply take the limit of the value as the time to maturity approaches zero:

$$\hat{f}(S) = \lim_{\tau \downarrow 0} V(S,\tau) \qquad S > 0 \qquad (12)$$

The third step is to use equation (7) with $\kappa=H$ to uncover the requisite static position in bonds, forward contracts and vanilla options.

We illustrate this three step procedure with a down-and-in call struck at $K_C > H$. From

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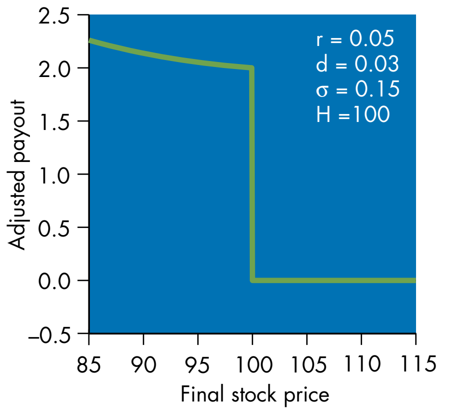
4. Adjusted payout for one-touch binary put (American)



Merton (1973), the valuation formula when S>H is:

$$DIC(S,\tau;H) = Se^{-d\tau}\left(\frac{S}{H}\right)^{p-2} N\left(\frac{\ln\left(\frac{H^2}{SK_C}\right)+\left(r-d+\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right) - K_C e^{-r\tau}\left(\frac{S}{H}\right)^{p} N\left(\frac{\ln\left(\frac{H^2}{SK_C}\right)+\left(r-d-\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right)$$

Removing the requirement that $S > H$, letting $\tau\downarrow 0$, and denoting the indicator function by $1(\cdot)$ gives:

$$\begin{aligned}\lim_{\tau\downarrow 0} DIC(S,\tau;H) &= S\left(\frac{S}{H}\right)^{p-2} 1\left(\frac{H^2}{S} > K_c\right) - K_c\left(\frac{S}{H}\right)^{p} 1\left(\frac{H^2}{S} > K_c\right)\\ &= \left(\frac{S}{H}\right)^{p}\left(\frac{H^2}{S} - K_c\right) 1\left(\frac{H^2}{S} > K_c\right)\\ &= \left(\frac{S}{H}\right)^{p} \max\left(0, \frac{H^2}{S} - K_c\right)\end{aligned}$$

Thus, using in-out parity, the adjusted payout for a down-and-out call agrees with table A (recall $K_C > H$). The third step is to statically replicate the down-and-in call's adjusted payout of:

$$\hat{f}(S) = \left(\frac{S}{H}\right)^{p} \max\left(0, \frac{H^2}{S} - K_c\right)$$

using listed instruments. Setting κ=H in equation (7) and replacing f by $\hat{f}$ gives:

$$V_0 = \hat{f}(H)e^{-rT} + \hat{f}'(H)\left[Se^{-dT} - He^{-rT}\right] + \int_0^H \hat{f}''(K)P(K)\,dK + \int_H^\infty \hat{f}''(K)C(K)\,dK \qquad (13)$$

The reader can verify that:

$$\hat{f}(H) = 0$$

$$\hat{f}'(H) = 0$$

$$\hat{f}''(K) = \left(\frac{H}{K_c}\right)^{p-2} \delta\left(K - \frac{H^2}{K_c}\right) + \left(\frac{K}{H}\right)^{p-2}(p-1)\left(\frac{p-2}{K} - \frac{pK_c}{H^2}\right) 1\left(K < \frac{H^2}{K_c}\right)$$

Applying equation (13), our replicating portfolio becomes:

$\left(\frac{H}{K_c}\right)^{p-2}$ puts at strike $\frac{H^2}{K_c}$ and

$\left(\frac{K}{H}\right)^{p-2}(p-1)\left(\frac{p-2}{K} - \frac{pK_c}{H^2}\right)dK$ puts at strike K for $K < \frac{H^2}{K_c}$.

To show how this approach can be used to generate adjusted payouts for other securities, consider the valuation of an American-style binary put paying a dollar at the first passage time to H. From Rubinstein & Reiner (1991ii), the valuation formula is:

$$ABP(S,\tau;H) = \left(\frac{S}{H}\right)^{\gamma+\varepsilon} N\left(\frac{\ln\left(\frac{H}{S}\right) - \varepsilon\sigma^2\tau}{\sigma\sqrt{\tau}}\right) + \left(\frac{S}{H}\right)^{\gamma-\varepsilon} N\left(\frac{\ln\left(\frac{H}{S}\right) + \varepsilon\sigma^2\tau}{\sigma\sqrt{\tau}}\right)$$

for S>H, where:

$$\gamma \equiv \frac{1}{2} - \frac{r-d}{\sigma^2},\quad \varepsilon \equiv \sqrt{\gamma^2 + \frac{2r}{\sigma^2}}$$

Removing the requirement that S>H and letting $\tau\downarrow 0$ gives the adjusted payout as in figure 4:

$$\lim_{\tau\downarrow 0} ABP(S,\tau;H) = \left[\left(\frac{S}{H}\right)^{\gamma+\varepsilon} + \left(\frac{S}{H}\right)^{\gamma+\varepsilon}\right] 1(S < H)$$

Conclusion

All down-and-in securities can be decomposed into a static portfolio of down-and-in Arrows. In the Black-Scholes model, the value of a down-and-in Arrow struck at some level K above the barrier H matches the value of a suitably weighted path-independent Arrow struck at the geometric reflection of K in H. It follows that the value of a down-and-in security can be represented by a static portfolio of Arrows struck below the barrier. Since any such portfolio can be created out of a static portfolio of European-style options, down-and-in securities can be statically hedged with vanilla options. In-out parity implies that the same result holds for out options. The valuation formulas derived via static replication match those obtained by dynamic replication. It follows that one can start from a formula obtained by dynamic replication and uncover the implicit static hedge.

In future analytical work in this area, we plan to explore the effects of imposing multiple barriers (see Carr & Chou, 1997). It will also be interesting to examine the static replication error arising from hedging with vanilla options when one can trade in only a finite number of strikes. One possibility is to attempt super-replication at the least cost. Another is to minimise mean squared error of the replicating portfolio's payout from the target. The latter approach is likely to permit lower offering prices and at least some of the risk can be diversified away. Finally, it should be interesting to conduct empirical tests comparing static and dynamic hedging under realistic market conditions. ■

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