Model Free Results on Volatility Derivatives

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February 27, 2006

Outline

- I VIX Pricing
- II Models
- III Lower Bound
- IV Conclusion

Historical Volatility Products

- Historical variance: $\frac{1}{n}\sum_{i=1}^{n}(\ln(\frac{S_i}{S_{i-1}}))^2$
- OTC products:
  - Volatility swap
  - Variance swap
  - Corridor variance swap
  - Options on volatility/variance
  - Volatility swap again
- Listed Products
  - Futures on realized variance

Implied Volatility Products

- Definition
  - Implied volatility: input in Black-Scholes formula to recover market price: $BS(S,\sigma^{impl},r,K,T) = C_{K,T}^{Market}$
  - Old VIX: proxy for ATM implied vol
  - New VIX: proxy for variance swap rate
- OTC products
  - Swaps and options
- Listed products
  - VIX Futures contract
  - Volax

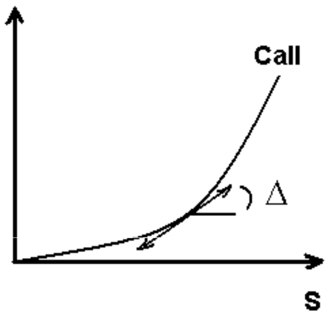
I VIX Futures Pricing

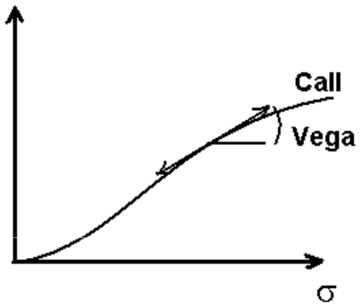
Vanilla Options

Simple product, but complex mix of underlying and volatility:

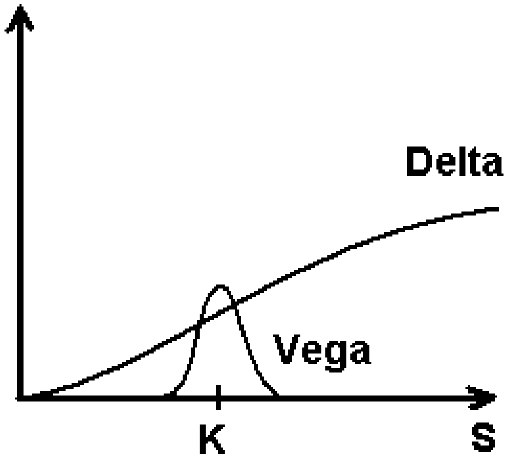
Call option has :

- Sensitivity to S : Δ
- Sensitivity to σ : Vega





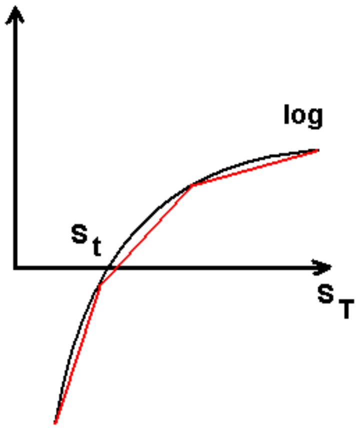
These sensitivities vary through time, spot and vol :



Volatility Games

To play pure volatility games (eg bet that S&P vol goes up, no view on the S&P itself):

- Need of constant sensitivity to vol;
- Achieved by combining several strikes;
- Ideally achieved by a log profile : (variance swaps)



Log Profile

Under BS: dS = σS dW, $E\left[\ln\frac{S_T}{S_0}\right] = -\frac{\sigma^2}{2}T$

For all S, $\ln(\frac{S}{S_0}) = \frac{S-S_0}{S_0} - \int_0^{S_0}\frac{(K-S)^+}{K^2}dK - \int_{S_0}^{\infty}\frac{(S-K)^+}{K^2}dK$

The log profile is decomposed as:

$$\tfrac{1}{S_0}\ Futures\ - \int_0^{S_0}\frac{P_{K,T}}{K^2}dK - \int_{S_0}^{\infty}\frac{C_{K,T}}{K^2}dK$$

In practice, finite number of strikes ⇨ CBOE definition:

$$VIX_t^2 \equiv \frac{2}{T}\sum\frac{K_{i+1}-K_{i-1}}{2K_i^2}e^{rT}X(K_i,T) - \frac{1}{T}(\frac{F}{K_0}-1)^2$$

Put if K_i<F,

Call otherwise

FWD adjustment

Option prices for one maturity

SPX ↓1095.45 +8.33 1094.97/1095.68 Index OMON

At 16:59 Op 1087.12 Hi 1095.69 Lo 1087.12

Template List | Edit | Contract Months | Security List | SPX Index | Go

Option Monitor: S&P 500 INDEX

Center 1095.45 Number of Strikes 18 -or- % from Center Exchange C (Composite)

CALLS

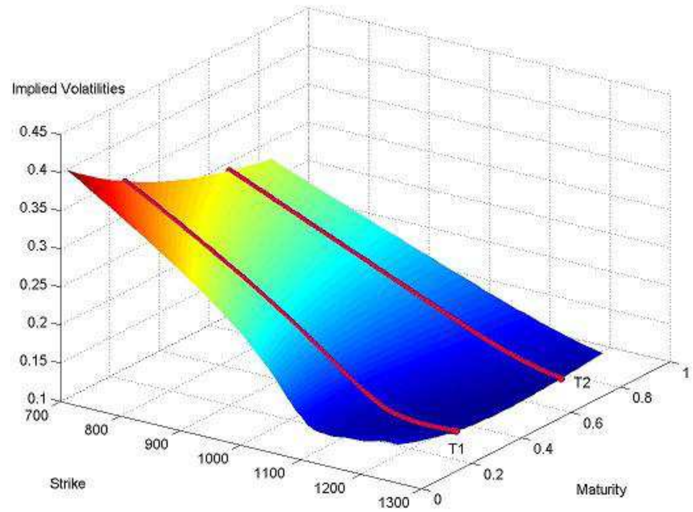
Ticker	Strike	Bid	Ask	Last	Volume
SPX 22 MAY 04 (Contract Size: 100)					
1) SPQ+EH	1040	53.00	55.00		
2) SPQ+EJ	1050	43.80	45.80	47.00	4385
3) SPQ+EL	1060	34.90	36.90	36.00	1
4) SPQ+EM	1065	30.60	32.60	30.10	1
5) SPQ+EN	1070	26.60	28.60	28.00	10
6) SPQ+EO	1075	22.70	24.70	23.80	111
7) SPQ+EP	1080	18.50	20.00	19.20	117
8) SPQ+ER	1090	12.90	14.40	13.00	783
9) SPT+ET	1100	8.10	8.10	8.70	11438
10) SPT+EB	1110	4.60	5.00	4.90	683
11) SPT+EC	1115	3.30	3.60	3.20	738
12) SPT+ED	1120	2.25	2.95	3.00	1239
13) SPT+EE	1125	1.65	2.10	1.90	3978
14) SPT+EF	1130	1.15	1.40	1.35	461
15) SPT+EG	1135	.65	1.05	.90	1521
16) SPT+EH	1140	.50	.60	.65	1548
17) SPT+EI	1145	.30	.50	.50	1
18) SPT+EJ	1150	.30	.40	.30	6754

PUTS

Ticker	Strike	Bid	Ask	Last	Volume
SPX 22 MAY 04 (Contract Size: 100)					
19) SPQ+QH	1040	1.25	1.75	1.35	3078
20) SPQ+QJ	1050	2.00	2.50	2.40	3283
21) SPQ+QL	1060	3.00	3.70	3.20	2382
22) SPQ+QM	1065	3.80	4.50	4.00	795
23) SPQ+QN	1070	4.70	5.40	4.80	2023
24) SPQ+QO	1075	6.00	6.30	6.30	6859
25) SPQ+QP	1080	7.50	8.00	8.20	1468
26) SPQ+QR	1090	11.00	11.90	11.70	1923
27) SPT+QT	1100	15.30	16.00	16.90	15701
28) SPT+QB	1110	21.40	23.40	22.10	1266
29) SPT+QC	1115	25.10	27.10	26.00	24
30) SPT+QD	1120	29.10	31.10	30.00	131
31) SPT+QE	1125	33.30	35.20	31.50	1532
32) SPT+QF	1130	37.70	39.70	40.00	34
33) SPT+QG	1135	42.30	44.30	43.50	12
34) SPT+QH	1140	47.00	49.00	48.30	85
35) SPT+QI	1145	51.80	53.80	52.50	25
36) SPT+QJ	1150	56.70	58.70	54.20	27

Perfect Replication of $VIX_{T_1}^2$

$$
\begin{aligned}
VIX_{T_1}^2 &= -\frac{2}{\delta T}\, price_t\left[\ln\frac{S_{T_1+\delta T}}{S_{T_1}}\right] \\
&= price_t\left[\frac{2}{\delta T}\ln\frac{S_{T_1}}{S_0} - \frac{2}{\delta T}\ln\frac{S_{T_1+\delta T}}{S_0}\right] \\
&= price_t\left[PF\right]
\end{aligned}
$$

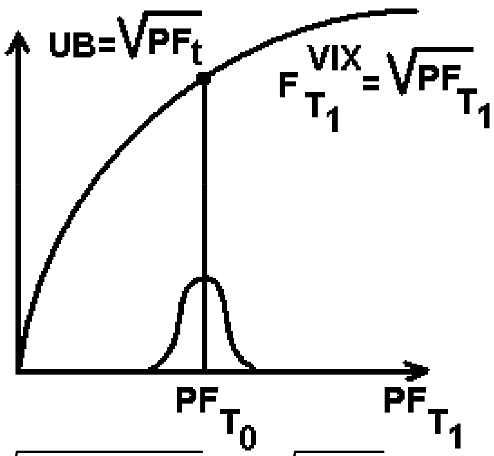


We can buy today a PF which gives VIX^2_{T1} at T_1:

buy T_2 options and sell T_1 options.

Theoretical Pricing of VIX Futures F^{VIX} before launch

- F^{VIX}_t: price at t of receiving $\sqrt{PF_{T_1}} = VIX_{T_1} = F^{VIX}_{T_1}$ at T_1 .



$$F_t^{VIX} = E_t[\sqrt{PF_T}] \leq \sqrt{E_t[PF_T]} = \sqrt{PF_t} = Upper\ \ Bound(UB)$$

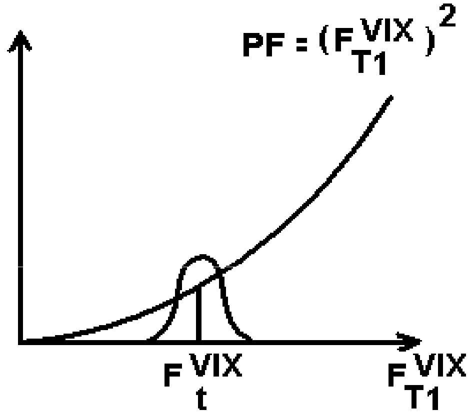
•The difference between both sides depends on the variance of PF (vol vol), which is difficult to estimate.

Pricing of F^{VIX} after launch

Much less transaction costs on F than on PF (by a factor of at least 20)

$\Rightarrow$Replicate PF by F

instead of F by PF!



$$PF_{T_1} = (F_{T_1}^{VIX})^2 = (F_t^{VIX})^2 + 2\int_t^{T_1}(F_s^{VIX} - F_t^{VIX})dF_s^{VIX} + QV_{t,T_1}^{F^{VIX}}$$

$$PF_t = E_t[(F_{T_1}^{VIX})^2] = E_t[F_{T_1}^{VIX}]^2 + Var_t[F_{T_1}^{VIX}]$$

$$\Rightarrow F_t^{VIX} = E_t[F_{T_1}^{VIX}] = \sqrt{PF_t - Var_t[F_{T_1}^{VIX}]}\left(\leq \sqrt{PF_t} = UB\right)$$

Bias estimation

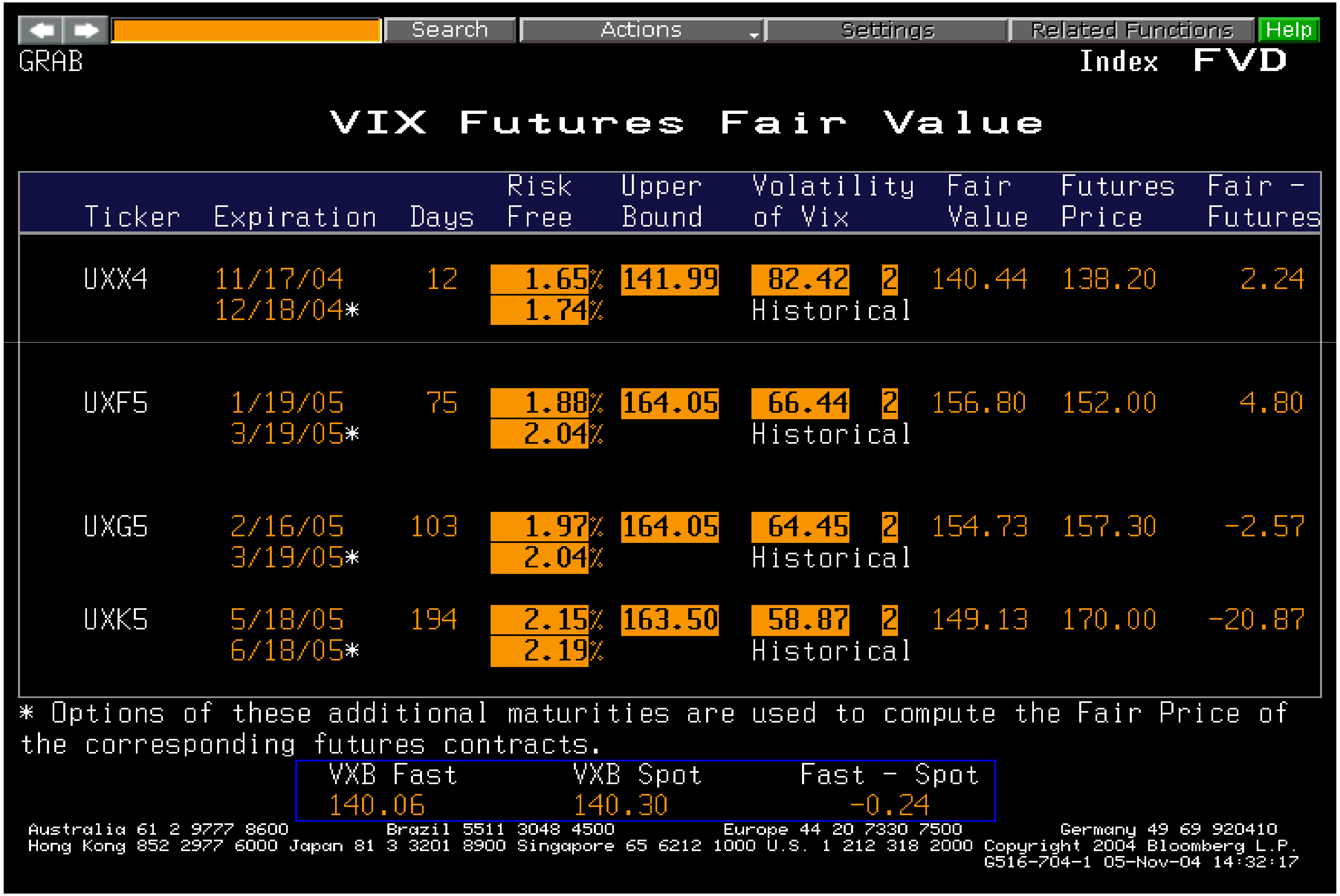
$$F_t^{VIX} = \sqrt{UB^2 - Var_t[F_{T_1}^{VIX}]}$$

$Var[F_{T_1}]$ can be estimated by combining the historical volatilities of F and Spot VIX.

Seemingly circular analysis :

F is estimated through its own volatility!

VIX Fair Value Page



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GRAB — Index **FVD**

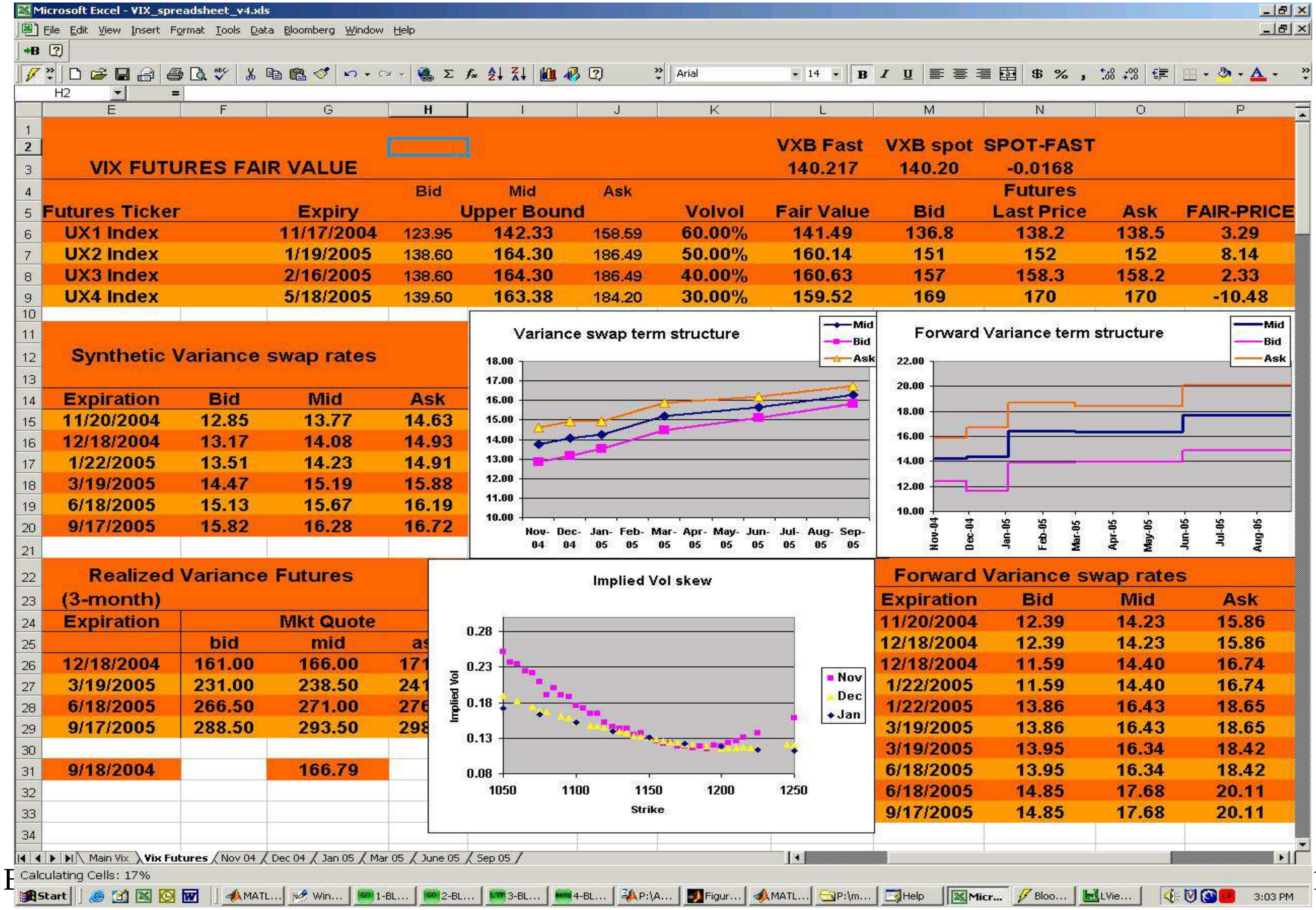
VIX Futures Fair Value

Ticker	Expiration	Days	Risk Free	Upper Bound	Volatility of Vix		Fair Value	Futures Price	Fair - Futures
UXX4	11/17/04	12	1.65%	141.99	82.42	2	140.44	138.20	2.24
	12/18/04*		1.74%		Historical				
UXF5	1/19/05	75	1.88%	164.05	66.44	2	156.80	152.00	4.80
	3/19/05*		2.04%		Historical				
UXG5	2/16/05	103	1.97%	164.05	64.45	2	154.73	157.30	-2.57
	3/19/05*		2.04%		Historical				
UXK5	5/18/05	194	2.15%	163.50	58.87	2	149.13	170.00	-20.87
	6/18/05*		2.19%		Historical				

* Options of these additional maturities are used to compute the Fair Price of the corresponding futures contracts.

VXB Fast	VXB Spot	Fast - Spot
140.06	140.30	-0.24

Behind The Scene



Microsoft Excel - VIX_spreadsheet_v4.xls

VIX FUTURES FAIR VALUE							VXB Fast	VXB spot	SPOT-FAST		
							140.217	140.20	-0.0168		
		Bid	Mid	Ask				Futures			
Futures Ticker	Expiry		Upper Bound		Volvol	Fair Value	Bid	Last Price	Ask	FAIR-PRICE	
UX1 Index	11/17/2004	123.95	142.33	158.59	60.00%	141.49	136.8	138.2	138.5	3.29	
UX2 Index	1/19/2005	138.60	164.30	186.49	50.00%	160.14	151	152	152	8.14	
UX3 Index	2/16/2005	138.60	164.30	186.49	40.00%	160.63	157	158.3	158.2	2.33	
UX4 Index	5/18/2005	139.50	163.38	184.20	30.00%	159.52	169	170	170	-10.48	

Synthetic Variance swap rates

Expiration	Bid	Mid	Ask
11/20/2004	12.85	13.77	14.63
12/18/2004	13.17	14.08	14.93
1/22/2005	13.51	14.23	14.91
3/19/2005	14.47	15.19	15.88
6/18/2005	15.13	15.67	16.19
9/17/2005	15.82	16.28	16.72

Realized Variance Futures (3-month)

Expiration	Mkt Quote		
	bid	mid	as
12/18/2004	161.00	166.00	171
3/19/2005	231.00	238.50	241
6/18/2005	266.50	271.00	276
9/17/2005	288.50	293.50	298
9/18/2004		166.79	

Forward Variance swap rates

Expiration	Bid	Mid	Ask
11/20/2004	12.39	14.23	15.86
12/18/2004	12.39	14.23	15.86
12/18/2004	11.59	14.40	16.74
1/22/2005	11.59	14.40	16.74
1/22/2005	13.86	16.43	18.65
3/19/2005	13.86	16.43	18.65
3/19/2005	13.95	16.34	18.42
6/18/2005	13.95	16.34	18.42
6/18/2005	14.85	17.68	20.11
9/17/2005	14.85	17.68	20.11

VIX Summary

- VIX Futures is a FWD volatility between future dates T_1 and T_2.
- Depends on volatilities over T_1 and T_2.
- Can be locked in by trading options maturities T_1 and T_2.
- 2 problems :
  - Need to use all strikes (log profile)
  - Locks in σ^2, not σ ⇨ need for convexity adjustment and dynamic hedging.

II Volatility Modeling

Volatility Modeling

- Neuberger (90): Quadratic variation can be replicated by delta hedging Log profiles
- Dupire (92): Forward variance synthesized from European options. Risk neutral dynamics of volatility to fit the implied vol term structure. Arbitrage pricing of claims on Spot and on vol
- Heston (93): Parametric stochastic volatility model with quasi closed form solution
- Dupire (96), Derman-Kani (97): non parametric stochastic volatility model with perfect fit to the market (HJM approach)

Volatility Modeling 2

- Matytsin (99): Parametric stochastic volatility model with jumps to price vol derivatives
- Carr-Lee (03), Friz-Gatheral (04): price and hedge of vol derivatives under assumption of uncorrelated spot and vol increments
- Duanmu (04): price and hedge of vol derivatives under assumption of volatility of variance swap
- Dupire (04): Universal arbitrage bounds for vol derivatives under the sole assumption of continuity

Variance swap based approach (Dupire (92), Duanmu (04))

- V = QV(0,T) is replicable with a delta hedged log profile (parabola profile for absolute quadratic variation)
  - Delta hedge removes first order risk
  - Second order risk is unhedged. It gives the quadratic variation
- V is tradable and is the underlying of the vol derivative, which can be hedged with a position in V
- Hedge in V is dynamic and requires assumptions on

$$V_t \equiv E_t[V] = QV_{0.t} + E_t[QV_{t,T}]$$

Stochastic Volatility Models

- Typically model the volatility of volatility (volvol). Popular example: Heston (93)

$$\frac{dS_t}{S_t} = \sqrt{v_t}\, dW_t$$

$$dv_t = \kappa(v_\infty - v_t)dt + \alpha\sqrt{v_t}\, dZ_t$$

- Theoretically: gives unique price of vol derivatives (1st equation can be discarded), but does not provide a natural unique hedge
- Problem: even for a market calibrated model, disconnection between volvol and real cost of hedge.

Link Skew/Volvol

- A pronounced skew imposes a high spot/vol correlation and hence a high volvol if the vol is high
- As will be seen later, non flat smiles impose a lower bound on the variability of the quadratic variation
- High spot/vol correlation means that options on S are related to options on vol: do not discard 1^{st} equation anymore

From now on, we assume 0 interest rates, no dividends and V is the quadratic variation of the price process (not of its log anymore)

Skew⇔volvol

To make it simple:

$$\left.\begin{array}{l} Skew < 0 \Rightarrow \rho(S,\sigma) < 0 \\ \sigma >> 0 \Rightarrow S \ moves \end{array}\right] \Rightarrow \sigma \ moves \Rightarrow volvol >> 0 \Rightarrow LB$$

Carr-Lee approach

- Assumes
  - Continuous price
  - Uncorrelated increments of spot and of vol
- Conditionally to a path of vol, X(T) is normally distributed, $= X_0 + \sqrt{V}g$ (g: normal sample)
- Then it is possible to recover from the risk neutral density of X(T) the risk neutral density of V
- Example: $E[(X_T - X_0)^{2n}] = E[V^n g^{2n}] = \mu_{2n} E[V^n]$
- Vol claims priced by expectation and perfect hedge
- Problem: strong assumption, imposes symmetric smiles not consistent with market smiles
- Extensions under construction

III Lower Bound

Spot Conditioning

- Claims can be on the forward quadratic variation QV_{T_1,T_2}
- Extreme case: $f(v_T)$ where v_T is the instantaneous variance at T
- If f is convex,

$$E[f(v_T)] = E[E[f(v_T \mid X_T = K)]] \geq E[f(E[v_T \mid X_T = K])] = E\left[f(v_{loc}(K,T))\right]$$

Which is a quantity observable from current option prices

X(T) not normal => V not constant

- Main point: departure from normality for X(T) enforces departure from constancy for V, or:

  smile non flat => variability of V
- Carr-Lee: conditionally to a path of vol, X(T) is gaussian
- Actually, in general, if X is a continuous local martingale
  – QV(T) = constant => X(T) is gaussian
  – Not: conditional to QV(T) = constant, X(T) is gaussian
  – Not: X(T) is gaussian => QV(T) = constant

The Main Argument

- If you sell a convex claim on X and delta hedge it, the risk is mostly on excessive realized quadratic variation
- Hedge: buy a Call on V!
- Classical delta hedge (at a constant implied vol) gives a final PL that depends on the Gammas encountered
- Perform instead a “business time” delta hedge: the payoff is replicated as long as the quadratic variation is not exhausted

Trader's Puzzle

- You <u>know</u> in advance that the total realized historical volatility over the quarter will be 10%
- You sell a 3 month Put at 15% implied

- Are you sure you can make a profit?

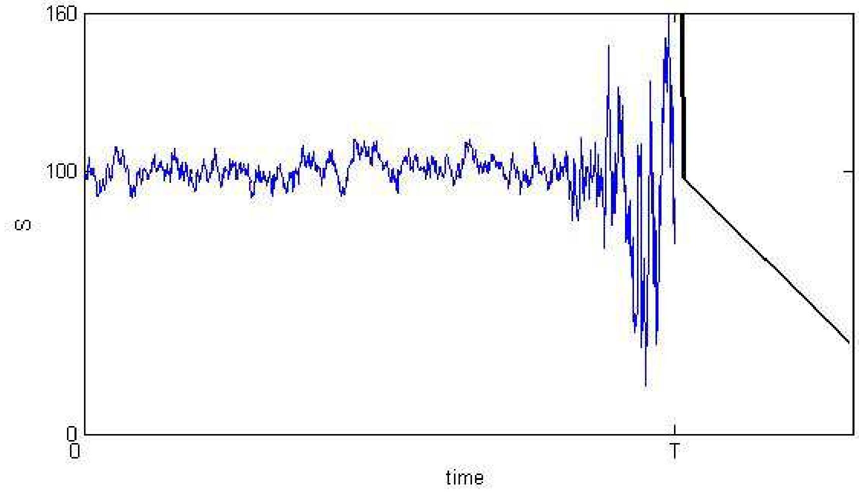
Answers

- <u>Naïve answer:</u> YES

  Δ hedging with 10% replicates the Put at a lower cost →
  Profit = Put(15%) - Put(10%)

- <u>Classical answer:</u> NO

  Big moves close to the strike at maturity incur losses because **Γ<< 0**.



- <u>Correct answer:</u> YES

  Adjust the Δ hedge according to realized volatility so far

  → Profit = Put(15%) – Put(10%)

Delta Hedging

- Extend f(x) to f(x,v) as the Bachelier (normal BS) price of f for start price x and variance v:

$$f(x,v) \equiv E^{x,v}[f(X)] \equiv \frac{1}{\sqrt{2\pi v}} \int f(y) e^{-\frac{(y-x)^2}{2v}} dy$$

with f(x,0) = f(x)

- Then, $f_v(x,v) = \frac{1}{2} f_{xx}(x,v)$
- We explore various delta hedging strategies

Calendar Time Delta Hedging

- Delta hedging with constant vol: P&L depends on the path of the volatility and on the path of the spot price.

$$df(X_t, \sigma^2.(T-t)) = f_x dX_t - \sigma^2 f_v dt + \frac{1}{2} f_{xx} dQV_{0,t}$$

$$= f_x dX_t + \frac{1}{2} f_{xx}(dQV_{0,t} - \sigma^2 dt)$$

- Calendar time delta hedge: replication cost of $f(X_t, \sigma^2.(T-t))$

$$f(X_0, \sigma^2.T) + \frac{1}{2}\int_0^t f_{xx}(dQV_{0,u} - \sigma^2 du)$$

- In particular, for sigma = 0, replication cost of $f(X_t)$

$$f(X_0) + \frac{1}{2}\int_0^t f_{xx} dQV_{0,u}$$

Business Time Delta Hedging

- Delta hedging according to the quadratic variation: P&L that depends only on quadratic variation and spot price

$$df(X_t, L - QV_{0,t}) = f_x dX_t - f_v dQV_{0,t} + \frac{1}{2} f_{xx} dQV_{0,t} = f_x dX_t$$

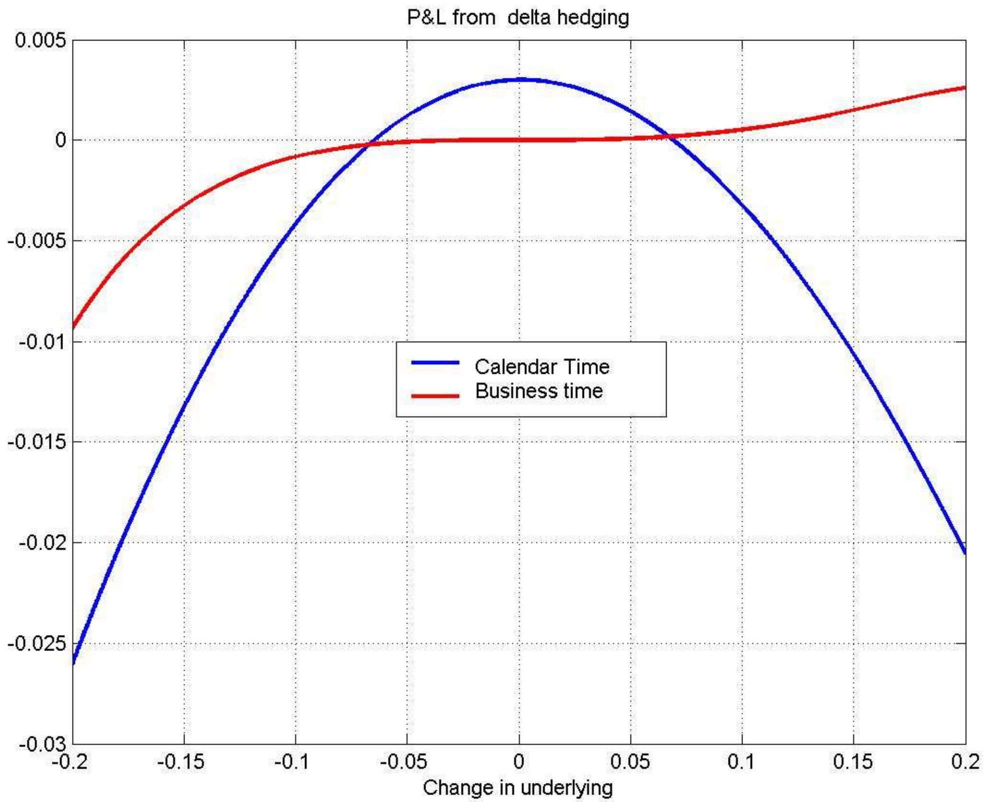
- Hence, for $QV_{0,T} \leq L$,

$$f(X_t, L - QV_{0,t}) = f(X_0, L) + \int_0^t f_x(X_u, L - QV_{0,u})\, dX_t$$

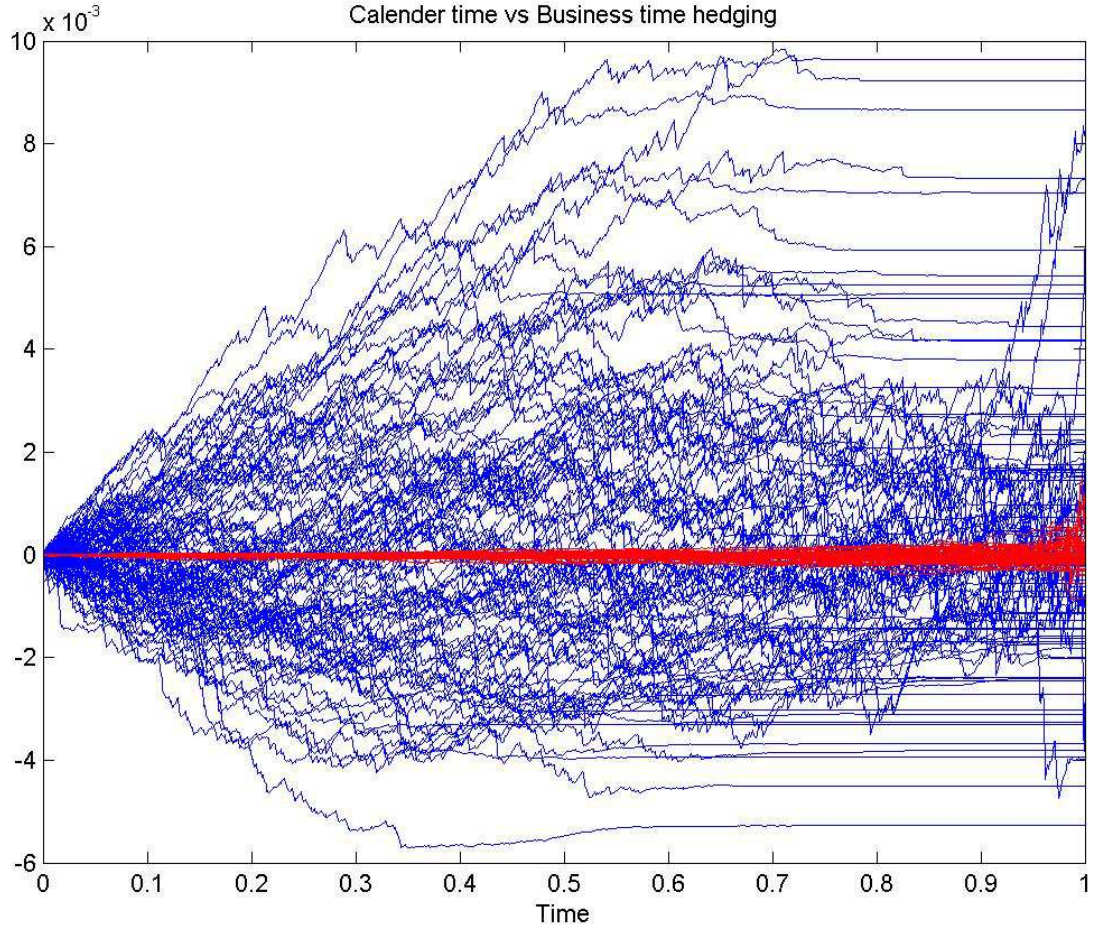
And the replicating cost of $f(X_t, L - QV_{0,t})$ is $f(X_0, L)$

$f(X_0, L)$ finances exactly the replication of f until $\tau : QV_{0,\tau} = L$

Daily P&L Variation



Tracking Error Comparison



Hedge with Variance Call

- Start from $f(X_0, L)$ and delta hedge f in "business time"
- If V < L, you have been able to conduct the replication until T and your wealth is $f(X_T, L-V) \geq f(X_T)$
- If V > L, you "run out of quadratic variation" at τ < T. If you then replicate f with 0 vol until T, extra cost:

$$\frac{1}{2}\int_{\tau}^{T} f''(X_t)dQV_t \leq \frac{M_f}{2}\int_{\tau}^{T} dQV_t = \frac{M_f}{2}(V-L)$$

where $M_f \equiv \sup\{f''(x)\}$

- After appropriate delta hedge, $f(X_0, L) + \frac{M_f}{2} Call_L^V$

dominates $f(X_T)$ which has a market price $f(X_0, L^f)$

Super-replication

$$f(X_0, L^f) \le f(X_0, L) + \frac{M_f}{2} C_L^V$$

$1) V_T \le L$

$f(X_0, L)$ $\Delta^{BT} hedge$ $f(X_T, L - V_T) \ge f(X_T)$

0 T

$2) V_T > L$

$f(X_0, L)$ $\Delta^{BT} hedge$ $f(X_\tau)$ $\frac{M_f}{2}(V - L)$ $\ge f(X_T)$

0 τ T

$V_\tau = L$

Lower Bound for Variance Call

- C_L^V : price of a variance call of strike L. For all f,

$$C_L^V \geq \frac{2}{M_f}(f(X_0, L^f) - f(X_0, L))$$

- We maximize the RHS for, say, $M_f \leq 2$
- We decompose f as

$$f(x) = f(X_0) + (x - X_0) f'(X_0) + \int f''(K)\, Vanilla_K(x)\, dK$$

Where $Vanilla_K(x) \equiv K - x$ if $K \leq X_0$ and $x - K$ otherwise

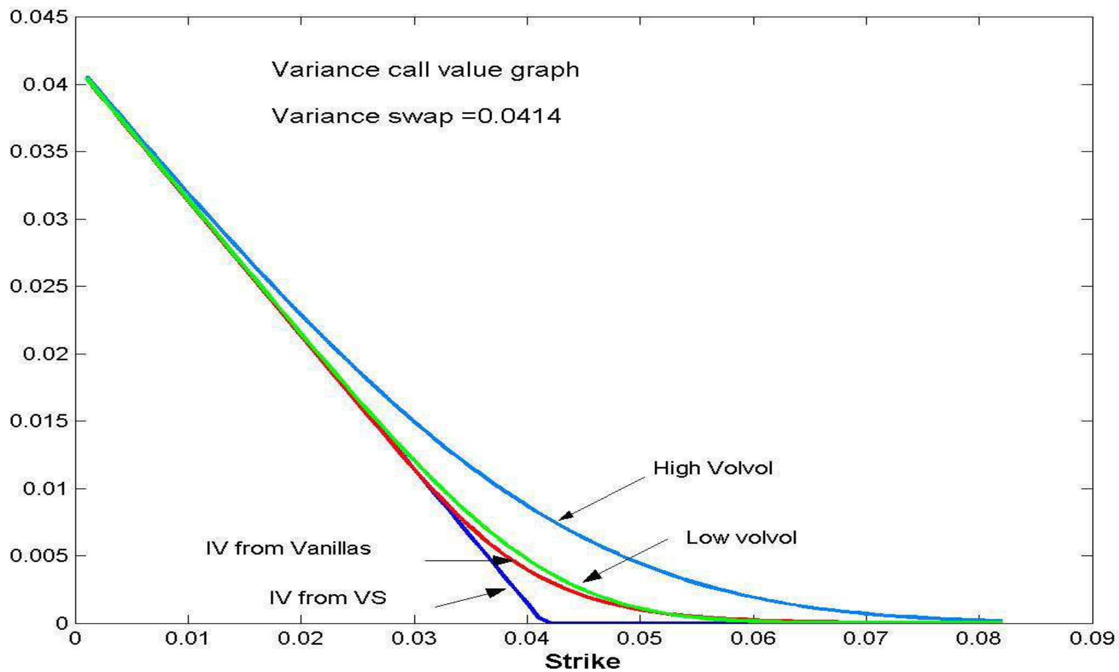
Then, $C_L^V \geq \int f''(K)(Van_K(L^K) - Van_K(L))\, dK$

Where $Van_K(L^K)$ is the price of $Vanilla_K(x)$ for variance v and L^K is the market implied variance for strike K

Lower Bound Strategy

- Maximum when f" = 2 on $A \equiv \{K : L^K \geq L\}$, 0 elsewhere
- Then, $f(x) = 2\int_A Vanilla_K(x)\, dK$ (truncated parabola)

and $C_L^V \geq 2\int_A (Van_K(L^K) - Van_K(L))\, dK$



Arbitrage Summary

- If a Variance Call of strike L and maturity T is below its lower bound:
- 1) at t = 0,
  - Buy the variance call
  - Sell all options with implied vol $\geq \sqrt{\frac{L}{T}}$
- 2) between 0 and T,
  - Delta hedge the options in business time
  - If $\tau < T$, then carry on the hedge with 0 vol
- 3) at T, sure gain

IV Conclusion

- Skew denotes a correlation between price and vol, which links options on prices and on vol
- Business time delta hedge links P&L to quadratic variation
- We obtain a lower bound which can be seen as the real intrinsic value of the option
- Uncertainty on V comes from a spot correlated component (IV) and an uncorrelated one (TV)
- It is important to use a model calibrated to the whole smile, to get IV right and to hedge it properly to lock it in