Bond Yield Curve Convexity Trading

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Abstract

Bond Yield curve is an important indicator of the borrowing costs and lending returns, is also one of the most observed indicator by traders in fixed income trading desk among investment banks. The shape of the yield curve can be normal, flat or inverted. In most cases, bond yield curve is concavely shaped. There are theories to explain the economical meaning for a normal, flat or inverted yield curve. However, there is a lack of explanation for the concave shape of the yield curve. We in this article try to provide an explanation by constructing arbitrage portfolio under the assumption that the yield curve moves in parallel. We can show that under this assumption, zero coupon bond yield curve should be concavely shaped. We will also reach the same conclusion for swap curves. In this process, we also discover some interesting properties which were never discussed in the literature.

In fixed income sector, the yield curve is probably the most observed indicators by traders and economists. A yield curve plots interest rates across different contract maturities from the short end to the long end. For

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each currency, the corresponding curve shows the relationship between the level of the interest rates (or cost of borrowing) and the time to maturity. For example, the U.S. dollar interest rates paid on U.S. Treasury securities for various maturities are plotted as the US treasury curve.

The shape of the yield curve gives an idea of future interest rate changes and economic activity. There are three main types of yield curve shapes: normal, flat and inverted. A normal yield curve is one in which longer maturity bonds have a higher yield compared to shorter-term bonds. An inverted yield curve is one in which the shorter-term yields are higher than the longer-term yields. In a flat or humped yield curve, the shorter-term and longer-term yields are very close to each other.

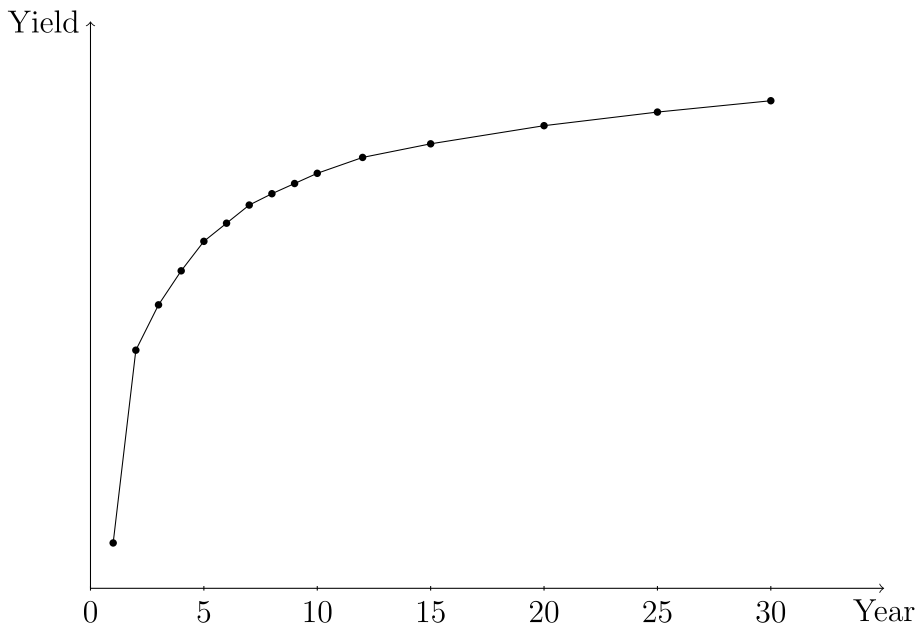

Figure 1: US Yield Curve as of 2018-08-15

There are various literatures on modeling yield curve. For example, Nelson and Siegel [7] was widely cited to give a parametric model to describe the shape of the yield curve. In the model, they use the following parametrization to model the forward rates

$$f(t) = \beta_1 + \beta_2 e^{-\frac{t}{\tau}} + \beta_3 \frac{t}{\tau} e^{-\frac{t}{\tau}}$$

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where τ is a nonlinear parameter and $\beta_1, \beta_2, \beta_3$ are linear parameters of the model.

Rezende and Ferreira [18] added a third hump to obtain five-factor model as follows:

$$f(t) = \beta_1 + \beta_2 e^{-\frac{t}{\tau_1}} + \beta_3 \frac{t}{\tau_1} e^{-\frac{t}{\tau_1}} + \beta_4 \frac{t}{\tau_2} e^{-\frac{t}{\tau_2}} + \beta_5 \frac{t}{\tau_3} e^{-\frac{t}{\tau_3}}$$

Modeling the term structure of the yield curve has draw a lot of attention as well. Vasicek [16] used the following Ornstein-Uhlenbeck process to model the short rate

$$dr_t = \lambda(\theta - r_t)dt + \sigma dW_t$$

The merit of this model is that short rate has mean - reversion property. The Cox, Ingersoll and Ross [17] used a slightly different version

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t$$

One of the problem of Vasicek and CIR model is that they are not flexible enough to fit into the initial yield curve. Ho and Lee [11] have been the first to propose and exogenous term structure model.

$$dr_t = \theta(t)dt + \sigma dW_t$$

Hull and White [12] extended the Vasicek model by assuming the short rate evolves under the risk neutral measure according to

$$dr_t = (\theta(t) - \kappa r_t)dt + \sigma(t)dW_t$$

While the structure can successfully calibrate to the initial term structure, it has to assume that the entire movement of the term structure are governed by one source of risk. Eventually the Heath, Jarrow and Morton [14] gave the arbitrage free dynamics of the forward rate.

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_T(t)$$

where

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds$$

All these models are trying to describe the future movements of the interest rate, but none of these models tries to answer the question: why the yield is usually concavely shaped.

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In another direction, Letterman and Scheinkman [19] found that the term structure is driven by a small number of common factors by using principal components analysis. It is commonly known that the yield curve has mainly three components, level, slope and convexity.

The paper by Litterman, Scheinkman and Weiss [20] was among the first to suggest that there is a direct link between interest rate volatility and the shape of the yield curve. In that paper they show that there is a high correlation between implied volatility from bond options and yield spreads on certain butterfly combinations. Using a simple binomial term-structure model, they trace the relationship between volatility and the butterfly spread to the convexity of the bond prices (as functions of yields).

In an empirical study, Brown and Schaefer [22] document that the term structure of long-term forward rates is downward sloping. Specifically, they consider the spread between the 25-year and the 15-year instantaneous forward rates. A theoretical explanation for this result is the downward convexity bias, which also explains the relationship between volatility and the curvature of the yield curve.

In a one-factor model, Brown and Schaefer [21] show that the second derivative of the forward rate with respect to a kind of duration function is equal to the short-rate volatility. Brown and Schaefer [22] use this result to calculate an implied volatility from the shape of the yield curve.

Even the yield curve can be flat, upward or downward (inverted), however, yield curve is generally concave. There is a lack of explanation of the concavity of the yield curve shape from economics theory. We offer in this article an explanation of the concavity shape of the yield curve from trading perspectives.

Our main argument is to construct an investment portfolio consisting fixed income instruments and demonstrate that if the yield curve is not concave, an arbitrage will emerge. Our results also depend on an assumption that yield curve moves up and down in parallel or proportional. This assumption is not precisely true but should be approximately acceptable in reality, in particular from statistical point of view.

1 Zero Coupon Bonds and Arbitrage Strategy

Our main results depend on the following well known results:

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Convexity Inequality The function $f(x), x \in \mathrm{R}$ is a convex function, then it should satisfy the following inequality. For any $\lambda_1 > 0, \lambda_2 > 0$ which $\lambda_1 + \lambda_2 = 1$ and $a, b \in \mathrm{R}$, we should have

$$f(\lambda_1 a + \lambda_2 b) \leq \lambda_1 f(a) + \lambda_2 f(b)$$

We now set up our securities. We have three zero coupon bonds, calling them B_1, B_2, B_3 corresponds to three maturities $T_1 < T_2 < T_3$. Their yields to maturity are y_1, y_2, y_3. So far we impose no conditions on these yields as long as the implied forward is positive. We now construct a trading portfolio by purchasing λ_1 dollar amount of B_1, λ_3 dollar amount of B_3 and short λ_2 dollar amount of B_2. We choose quantities $\lambda_1 > 0, \lambda_2 > 0, \lambda_3 > 0$ by the following rules

$$\lambda_1 + \lambda_3 = \lambda_2$$

and

$$\lambda_1 T_1 + \lambda_3 T_3 = \lambda_2 T_2.$$

We notice that by combining the two equations, we have

$$\lambda_1(T_2 - T_1) = \lambda_3(T_3 - T_2)$$

In fact by linear algebra, the solutions of λ_i is unique up to a scalar

$$\lambda_1 = T_3 - T_2, \lambda_2 = T_3 - T_1, \lambda_3 = T_2 - T_1$$

As a consequence

$$\frac{\lambda_1}{\lambda_2} + \frac{\lambda_3}{\lambda_2} = 1$$

We claim that portfolio we have constructed has zero cost. Zero cost is obvious given the rule that $\lambda_1 - \lambda_2 + \lambda_3 = 0$. After the yields move by amount a instantaneously, our portfolio value becomes

$$P(a) = \lambda_1 e^{-aT_1} + \lambda_3 e^{-aT_3} - \lambda_2 e^{-aT_2}$$

But we will show that this function $P(a)$ is always positive which is equivalent to

$$\lambda_1 e^{-aT_1} + \lambda_3 e^{-aT_3} \geq \lambda_2 e^{-aT_2}$$

But this is true by applying our convexity inequality $f(x) = e^x$ which is clearly a convex function.

$$\frac{\lambda_1}{\lambda_2} e^{-aT_1} + \frac{\lambda_3}{\lambda_2} e^{-aT_3} \geq e^{-a\frac{\lambda_1}{\lambda_2}T_1 - a\frac{\lambda_3}{\lambda_2}T_3} = e^{-aT_2}$$

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due to the fact that

$$\frac{T_3 - T_2}{T_3 - T_1} T_1 + \frac{T_2 - T_1}{T_3 - T_1} T_3 = T_2$$

This proves our statement.

However this is only true instantaneously. As time marches on, we need to deal with carry cost as well. It turns out we need to impose an additional sufficient condition: The yields y_1, y_2, y_3 as a function of time to maturity T_1, T_2, T_3 is convex, i.e.

$$(T_3 - T_2)y_1 + (T_2 - T_1)y_3 \geq (T_3 - T_1)y_2 \tag{1}$$

Theorem 1. *If yields y_1, y_2, y_3 as a function of time to maturity T_1, T_2, T_3 is convex, i.e.*

$$(T_3 - T_2)y_1 + (T_2 - T_1)y_3 \geq (T_3 - T_1)y_2 \tag{2}$$

the portfolio we constructed admits an arbitrage.

Proof. Now we assume yields move by the same amount a and time moves forward by t, therefore our portfolios new value becomes

$$P(a, t) = \lambda_1 e^{-a(T_1 - t) + y_1 t} + \lambda_3 e^{-a(T_3 - t) + y_3 t} - \lambda_2 e^{-a(T_2 - t) + y_2 t}$$

We want to show that this quantity is positive i.e.

$$\frac{\lambda_1}{\lambda_2} e^{-a(T_1 - t) + y_1 t} + \frac{\lambda_3}{\lambda_2} e^{-a(T_3 - t) + y_3 t} \geq e^{-a(T_2 - t) + y_2 t}$$

By the convexity inequality we have

$$\begin{aligned}
&\frac{\lambda_1}{\lambda_2} e^{-a(T_1 - t) + y_1 t} + \frac{\lambda_3}{\lambda_2} e^{-a(T_3 - t) + y_3 t} \\
&\geq e^{-a\frac{\lambda_1}{\lambda_2}(T_1 - t) - a\frac{\lambda_3}{\lambda_2}(T_3 - t) + \frac{\lambda_1}{\lambda_2} y_1 t + \frac{\lambda_3}{\lambda_2} y_3 t} \\
&= e^{-a(T_2 - t)} e^{\frac{\lambda_1}{\lambda_2} y_1 t + \frac{\lambda_3}{\lambda_2} y_3 t}
\end{aligned}$$

But if the yield y_i are convex, by definition we have

$$\frac{\lambda_1}{\lambda_2} y_1 + \frac{\lambda_3}{\lambda_2} y_3 \geq y_2$$

therefore

$$\lambda_1 e^{-a(T_1 - t) + y_1 t} + \lambda_3 e^{-a(T_3 - t) + y_3 t} \geq \lambda_2 e^{-a(T_2 - t) + y_2 t}$$

is true. □

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We have completed our argument that arbitrage exists by construction a zero cost portfolio consisting of three zero coupon bonds. The argument is valid for any three maturities as long as corresponding yields are convex and the yields move by the same amount. The entire argument is based on the convexity inequality. We have proved so far:

1. Under parallel movement in yields, we can construct zero cost portfolio and achieve positive profit instantaneously.

2. If Yields are convex with respect to time, we construct zero cost portfolio and achieve positive profit at any future time.

In fact if we consider our proof carefully, we will realize that duration free construction of the bond portfolio by longing one short-term and one long-term bond meanwhile shorting one intermediate term bond will achieve instantaneous profit. However, if the three bond yields are concave as well, the same position can give us positive carry as well. This is the essence of this proof.

2 Nonparallel Movement

We now extend the results in the previous sections to nonparallel movement. For this purposes, we assume that three yields move not necessarily in parallel, but the movements are proportional to three amounts l_i. When all l_i are identical, this is equivalent to parallel movements. The bond yields moves by $a_i = \lambda l_i$ with the same scalar λ. We now prove the following result.

Theorem 2. *If the three point*

$$(l_1 T_1, y_1), \quad (l_2 T_2, y_2), \quad (l_3 T_3, y_3)$$

are concave, we can construct an arbitrage zero coupon bond portfolio.

Proof. We set up the portfolio weights. We require zero cost

$$\lambda_1 + \lambda_3 = \lambda_2$$

and zero duration

$$\lambda_1 l_1 T_1 + \lambda_3 l_3 T_3 = \lambda_2 l_2 T_2.$$

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As a consequence we have up to a scalar

$$\lambda_1 = l_3T_3 - l_2T_2$$
$$\lambda_2 = l_3T_3 - l_1T_1$$
$$\lambda_3 = l_2T_2 - l_1T_1$$

As before, we first check the instantaneous result. The Portfolio value after the yield movement becomes

$$P(a_1, a_2, a_3) = \lambda_1 e^{-a_1T_1} + \lambda_3 e^{-a_3T_3} - \lambda_2 e^{-a_2T_2}$$

By convexity inequality

$$\frac{\lambda_1}{\lambda_2} e^{-a_1T_1} + \frac{\lambda_3}{\lambda_2} e^{-a_3T_3} \geq e^{-\frac{\lambda_1}{\lambda_2}a_1T_1 - \frac{\lambda_3}{\lambda_2}a_3T_3} = e^{-a_2T_2}$$

Secondly, we let time march forward by amount t, the new portfolio becomes

$$\lambda_1 e^{-a_1(T_1-t)+y_1t} + \lambda_3 e^{-a_3(T_3-t)+y_3t} - \lambda_2 e^{-a_2(T_2-t)+y_2t}$$

and we hope to demonstrate

$$\frac{\lambda_1}{\lambda_2} e^{-a_1(T_1-t)+y_1t} + \frac{\lambda_3}{\lambda_2} e^{-a_3(T_3-t)+y_3t} \geq e^{-a_2(T_2-t)+y_2t}$$

Given the assumption that (l_iT_i, y_i) are convex on the plane, for sufficiently small λ, the three points

$$(l_1T_1, y_1 + \lambda l_1), \quad (l_2T_2, y_2 + \lambda l_2), \quad (l_3T_3, y_3 + \lambda l_3)$$

must be convex as well. Again by applying the convex inequality,

$$\begin{aligned} & \frac{\lambda_1}{\lambda_2} e^{-a_1(T_1-t)+y_1t} + \frac{\lambda_3}{\lambda_2} e^{-a_3(T_3-t)+y_3t} \\ & \geq e^{-\frac{\lambda_1}{\lambda_2}a_1(T_1-t) + \frac{\lambda_1}{\lambda_2}y_1t - \frac{\lambda_3}{\lambda_2}a_3(T_3-t) + \frac{\lambda_3}{\lambda_2}y_3t} \\ & \geq e^{-a_2(T_2-t)+y_2t} \end{aligned}$$

which proved the theorem. □

3 Summary

In this paper, we have proved that under mild assumptions, the zero coupon bond yield curve should be concavely shaped. With much more careful derivation, we also reach the same conclusion for swap curves.

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