A partial rough path space for rough volatility

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Abstract

We develop a variant of rough path theory tailor-made for the analysis of a class of financial asset price models, the so-called rough volatility models. As an application, we prove a pathwise large deviation principle (LDP) for a certain class of rough volatility models, which in turn describes the limiting behavior of implied volatility for short maturity under those models. First we introduce a partial rough path space and an integration map on it, and then investigate several fundamental properties including local Lipschitz continuity of the integration map from the partial rough path space to a rough path space. Second we construct a rough path lift of a rough volatility model. Finally, an LDP on the partial rough path space is proved and the LDP for rough volatility then follows by the continuity of the solution map of rough differential equations (RDE).

1 Introduction

A rough volatility model is a stochastic volatility model for an asset price process with volatility being rough, meaning that the Hölder regularity of the volatility path is less than half. Recently such a model has been attracted attention in mathematical finance because of its unique consistency to market data. The rough volatility models are indeed the only class of continuous price models that are consistent to a power law of implied volatility term structure typically observed in equity option markets, as shown by [13]. Not only to derive a power law but also to seek a precise approximation formula, an LDP under rough volatility models has been extensively investigated by many authors [7, 3, 2, 9, 10, 22, 23, 24, 27, 25, 28].

Under rough volatility models, the volatility of an asset price has a lower Hölder regularity than the asset price process. The stochastic integrands are therefore not controlled by the stochastic integrators in the sense of [20]. Hence, a rough volatility model is beyond the scope of the rough path theory, which motivated [2] to develop a regularity structure for rough volatility. For classical stochastic differential equations, the Freidlin-Wentzell LDP can be obtained as a consequence of the continuity of the solution map (the

Lyons-Itô map) that is the core of the rough path theory. In [2], the LDP for rough volatility models is obtained using the continuity of Hairer's reconstruction map. We takes a similar to [2] in spirit but different approach in this paper. In stead of embedding a rough volatility model into the abstract framework of regularity structure, we develop a minimal extension of the rough path theory to incorporate rough volatility models. The advantage of our approach is, besides its relatively elementary construction, is that we can prove the continuity of the integration map between rough path spaces. This enables us to treat a more general model than in [2] using the simple fact that the composition of continuous maps is continuous.

We focus on a model of the following form:

$$\mathrm{d}S_t = \sigma(S_t, t) f(\hat{X}_t) \mathrm{d}X_t, \quad S_0 \in \mathbb{R}, \tag{1.1}$$

where X is a d-dimensional Brownian motion and $\hat{X}$ is a one dimensional Riemann-Liouville fractional Brownian motion with the Hurst parameter $H \in (0, 1/2]$. The stochastic integration is in the Itô sense. From empirical evidences, we are particularly interested in the case where $\hat{X}$ is correlated with X and $H < 1/4$ [18, 4, 17, 5]. The rough Bergomi model introduced by [1] and its SABR-type extension (the rough SABR model) [15, 29, 16, 14] are of the form (1.1) with f being an exponential function. Notice that $\hat{X}$ is not controlled by X due to its lower regularity. Note also that a rough path lift of $(X, \hat{X})$ requires to consider iterated integrals of $\hat{X}$, which is problematic when $H < 1/4$ as is well-known in the rough path literature. Our idea, inspired by [2], is to consider a partial rough path space where we lack of the iterated integrals of $\hat{X}$ but are still able to treat (1.1).

Here we argue how such a partial rough path space should be. Suppose that $x : [0, T] \to \mathbb{R}^d$ ($d \geqq 1$), $\hat{x} : [0, T] \to \mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$ are good enough. By the Taylor expansion, for $s < t$ (which are close enough),

$$\int_s^t f(\hat{x}_r) \mathrm{d}x_r \approx f(\hat{x}_s)(x_t - x_s) + \sum_{i=1}^n \frac{1}{i!} \nabla^i f(\hat{x}_s) \left[\int_s^t (\hat{x}_r - \hat{x}_s)^i \mathrm{d}x_r \right],$$

and

$$\int_s^t \left(\int_s^r \mathrm{d}y_u \right) \otimes \mathrm{d}y_r$$
$$\approx \sum_{0 \leqq j+k \leqq n} \frac{1}{j!k!} \nabla^j f(\hat{x}_s) \nabla^k f(\hat{x}_s) \left[\int_s^t (\hat{x}_r - \hat{x}_s)^k \left(\int_s^r (\hat{x}_u - \hat{x}_s)^j \mathrm{d}x_u \right) \otimes \mathrm{d}x_r \right],$$

where $y_t := \int_0^t f(\hat{x}_r) \mathrm{d}x_r$. Therefore, if we could define

$$X_{st}^{(i)} := \frac{1}{i!} \int_s^t (\hat{x}_r - \hat{x}_s)^i \mathrm{d}x_r, \quad \mathbf{X}_{st}^{(jk)} := \frac{1}{k!} \int_s^t (\hat{x}_r - \hat{x}_s)^k X_{sr}^{(j)} \otimes \mathrm{d}x_r,$$

we would be able to define a rough path integral $\int f(\hat{x}_r)\mathrm{d}x_r$. By the linearity of the integration and the binomial theorem, $X^{(i)}$ and $\mathbf{X}^{(jk)}$ satisfy the following formula respectively: for any $i, j, k \geqq 0$, and $s \leqq u \leqq t$,

$$X^{(i)}_{st} = X^{(i)}_{su} + \sum_{p=0}^{i} \frac{1}{(i-p)!} (\hat{X}_{su})^{i-p} X^{(p)}_{ut}, \tag{1.2}$$

and

$$\begin{aligned} \mathbf{X}^{(jk)}_{st} = \mathbf{X}^{(jk)}_{su} &+ \sum_{q=0}^{k} \frac{1}{(k-q)!} (\hat{X}_{su})^{k-q} X^{(j)}_{su} \otimes X^{(q)}_{ut} \\ &+ \sum_{p=0}^{j} \sum_{q=0}^{k} \frac{1}{(j-p)!(k-q)!} (\hat{X}_{su})^{j+k-p-q} \mathbf{X}^{(pq)}_{ut}. \end{aligned} \tag{1.3}$$

These should play the role of Chen's identity for our partial rough path space. We remark that essential in avoiding iterated integrals of $\hat{X}$ is the assumption that $\hat{X}$ is one-dimensional.

In Section 2, we formulate such a partial rough path space and state some fundamental properties including the continuity of the integration map. In Section 3, we construct a rough path lift of our rough volatility model and state an LDP. Proofs are relegated to Section 4.

2 A partial rough path space

2.1 Definition

Throughout this article, we fix $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, $\beta \in (0, \frac{1}{2})$ and denote

$$\Delta_T := \{(s,t) | 0 \leqq s \leqq t \leqq T\}, \quad I := \{i \in \mathbb{Z}_+ | i\beta + \alpha \leqq 1\},$$

and

$$J := \{(j,k) \in \mathbb{Z}_+ \times \mathbb{Z}_+ | (j+k)\beta + 2\alpha \leqq 1\},$$

where $\mathbb{Z}_+$ is the set of the nonnegative integers. Extending the notion of α-Hölder rough path in the rough path theory, here we define an (α, β) rough path.

Definition 2.1. An (α, β) rough path $\mathbb{X} = \left(\hat{X}, X^{(i)}, \mathbf{X}^{(jk)}\right)_{i \in I, (j,k) \in J}$ is a triplet of functions on Δ_T satisfying the following conditions; for any $i \in I$, $(j,k) \in J$ and $s \leqq u \leqq t$,

(i) $\hat{X}$ is $\mathbb{R}$-valued, $X^{(i)}$ is $\mathbb{R}^d$-valued, and $\mathbf{X}^{(jk)}$ is $\mathbb{R}^d \otimes \mathbb{R}^d$-valued.

(ii) *Modified Chen's relation*: $\hat{X}_{st} = \hat{X}_{su} + \hat{X}_{ut}$, and $X^{(i)}$ and $\mathbf{X}^{(jk)}$ satisfy (1.2) and (1.3) respectively.

(iii) *Hölder regularity*:

$$|\hat{X}_{st}| \lesssim |t-s|^{\beta}, \quad |X^{(i)}_{st}| \lesssim |t-s|^{i\beta+\alpha}, \quad |\mathbf{X}^{(jk)}_{st}| \lesssim |t-s|^{(j+k)\beta+2\alpha}.$$

Let $\Omega_{(\alpha,\beta)\text{-Hld}}$ denote the set of the (α,β) rough paths. We define a metric function $d_{(\alpha,\beta)}$ on $\Omega_{(\alpha,\beta)\text{-Hld}}$ and a homogeneous norm $|||\mathbb{X}|||_{(\alpha,\beta)}$ respectively by

$$\begin{aligned}&d_{(\alpha,\beta)}(\mathbb{X},\mathbb{Y})\\ &:= \|\hat{X}-\hat{Y}\|_{\beta\text{-Hld}} + \sum_{i\in I,(j,k)\in J} \|X^{(i)}-Y^{(i)}\|_{i\beta+\alpha\text{-Hld}} + \|\mathbf{X}^{(jk)}-\mathbf{Y}^{(jk)}\|_{(j+k)\beta+2\alpha\text{-Hld}},\end{aligned}$$

and

$$\begin{aligned}&|||\mathbb{X}|||_{(\alpha,\beta)}\\ &:= \|\hat{X}\|_{\beta\text{-Hld}} + \sum_{i\in I,(j,k)\in J} \left(\|X^{(i)}\|_{i\beta+\alpha\text{-Hld}}\right)^{1/(i+1)} + \left(\|\mathbf{X}^{(jk)}\|_{(j+k)\beta+2\alpha\text{-Hld}}\right)^{1/(j+k+2)},\end{aligned}$$

where $\|\cdot\|_{\gamma\text{-Hld}}$ is the usual γ-Hölder norm for $\gamma \in (0,1]$.

Remark 2.2. Note that the modified Chen's relation and the Hölder regularity of $X^{(i)}$ and $\mathbf{X}^{(jk)}$ are, as explained in Introduction, from the following correspondence:

$$X^{(i)}_{st} \leftrightarrow \frac{1}{i!}\int_s^t \left(\hat{X}_{sr}\right)^i dX^{(0)}_r, \quad \mathbf{X}^{(jk)}_{st} \leftrightarrow \frac{1}{k!}\int_s^t \left(\hat{X}_{sr}\right)^k X^{(j)}_{sr} \otimes dX^{(0)}_r$$

when $X^{(0)}$ and $\hat{X}$ have the Hölder regularity α and β respectively. Note also that $\left(X^{(0)}, \mathbf{X}^{(00)}\right)$ is an α-Hölder rough path with the first level $X^{(0)}$ and the second level $\mathbf{X}^{(00)}$ in the usual rough path terminology. An (α,β) rough path has two first level paths: $X^{(0)}$ and $\hat{X}$.

Remark 2.3. Our modified Chen's relation is a particular form of the algebraic structure of branched rough paths studied in [21]. However, since $\hat{X}$ is not a controlled path of X, the novel framework of (α,β) rough path is essential to establish the rough path integral stated introduction.

2.2 (α,β) rough path integration

Extending the rough path integration, here we introduce an integration with respect to an (α,β) rough path.

Definition 2.4. Fix $\mathbb{X} \in \Omega_{(\alpha,\beta)\text{-Hld}}$. We define $Y^{(1)}$ and $Y^{(2)}$ as follows if exist;

$$Y^{(1)}_{st} := \lim_{|\mathcal{P}|\searrow 0} \sum_{p=1}^{N} \sum_{i\in I} \nabla^i f(\hat{x}_{t_{p-1}}) X^{(i)}_{t_{p-1}t_p},$$

$$Y^{(2)}_{st} := \lim_{|\mathcal{P}|\searrow 0} \sum_{p=1}^{N} \left(Y^{(1)}_{t_0 t_{p-1}} \otimes Y^{(1)}_{t_{p-1}t_p} + \sum_{(j,k)\in J} \nabla^j f(\hat{x}_{t_{p-1}}) \nabla^k f(\hat{x}_{t_{p-1}}) \mathbf{X}^{(jk)}_{t_{p-1}t_p} \right),$$

where $\hat{x}_s := \hat{X}_{0s}$, and $\mathcal{P} = \{s = t_0 < t_1 < \ldots < t_N = t\}$ is a partition of the interval $[s,t]$. The mesh size $|\mathcal{P}|$ is defined by $|\mathcal{P}| = \max_j |t_j - t_{j-1}|$. If they exist on Δ_T, we denote $(Y^{(1)}, Y^{(2)})$ by $\int f(\hat{\mathbb{X}})\mathrm{d}\mathbb{X}$, and call it the (α,β) rough path integral of f.

Denote by $\Omega_{\alpha\text{-Hld}}$ the α-Hölder rough path space, and denote by d_α the metric function on $\Omega_{\alpha\text{-Hld}}$; see e.g., [11]. Here we state our first main result. The proof is given in Section 4.1.

Theorem 2.5. Let $n := \max I$ and assume that $f : \mathbb{R} \to \mathbb{R}$ is C^{n+1}.

(i) For any $\mathbb{X} \in \Omega_{(\alpha,\beta)\text{-Hld}}$, the (α,β) rough path integral $\int f(\hat{\mathbb{X}})\mathrm{d}\mathbb{X}$ is well-defined, and $\int f(\hat{\mathbb{X}})\mathrm{d}\mathbb{X} \in \Omega_{\alpha\text{-Hld}}$.

(ii) The integration map $\int : \Omega_{(\alpha,\beta)\text{-Hld}} \to \Omega_{\alpha\text{-Hld}}$ is locally Lipschitz continuous. More precisely, for any $M > 0$, the map $\int|_{\mathcal{E}_M}$, restricted on the set

$$\mathcal{E}_M := \left\{ \mathbb{X} \in \Omega_{(\alpha,\beta)\text{-Hld}} \middle| \ |||\mathbb{X}|||_{(\alpha,\beta)} \leqq M \right\}$$

is Lipschitz continuous, that is, there exists a positive constant $C > 0$ such that,

$$d_\alpha\left(\int f(\hat{\mathbb{V}})\mathrm{d}\mathbb{V}, \int f(\hat{\mathbb{W}})\mathrm{d}\mathbb{W} \right) \leqq C d_{(\alpha,\beta)}(\mathbb{V},\mathbb{W}), \quad \mathbb{V},\mathbb{W} \in \mathcal{E}_M.$$

3 Large Deviation

3.1 A lift to the partial rough path space

We now construct an (α,β) rough path, which plays an important role in this paper. The proof is defered to Section 4.2.

Proposition 3.1. Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geqq 0})$ be a filtered probability space, and let $H \in (0, 1/2)$. Suppose that $X = (X^1, \ldots, X^d)$ is a d-dimensional (possibly correlated) Brownian motion, and W is a one-dimensional Brownian motion possiblly correlated to X. Using the Itô integration, define $\hat{X}$, $X^{(i)}$, and $\mathbf{X}^{(jk)}$ as

$$\hat{X}_{st} := \int_0^t k_H(t-r)\mathrm{d}W_r - \int_0^s k_H(s-r)\mathrm{d}W_r, \quad X^{(i)}_{st} := \frac{1}{i!}\int_s^t \left(\hat{X}_{sr}\right)^i \mathrm{d}X_r,$$

$$\mathbf{X}^{(jk)}_{st} := \frac{1}{k!}\int_s^t \left(\hat{X}_{sr}\right)^k X^{(j)}_{sr} \otimes \mathrm{d}X_r, \quad k_H(r) := \frac{1}{\Gamma(H+1/2)}\, r^{H-1/2}$$

for $(s,t) \in \Delta_T$. Then we have the following.

(i) For a.s. $\omega \in \Omega$, $\mathbb{X}(\omega) := \left(\hat{X}(\omega), X^{(i)}(\omega), \mathbf{X}^{(jk)}(\omega)\right)_{i\in I,(j,k)\in J}$ is an (α,β) rough path for any $\beta < H$.

(ii) It holds

$$\left(\int f(\hat{\mathbb{X}})\mathrm{d}\mathbb{X}\right)^{(1)}_{0t} = \int_0^t f(\hat{X}_{0r})\mathrm{d}X_r, \quad a.s.$$

where the left-hand-side is the first level of the (α,β) rough path integral and the right-hand-side is the Itô integral.

3.2 The large deviation principle on $\Omega_{(\alpha,\beta)\text{-Hld}}$

We now discuss about the LDP on $\Omega_{(\alpha,\beta)\text{-Hld}}$. Following [27, 25], we use Garcia's theorem [19]. Let $(W, W^{\perp})$ be a two dimensional standard Brownian motion and $X := \rho W + \sqrt{1-\rho^2}W^{\perp}$, $\rho \in [-1,1]$. Define $\hat{X}, X^{(i)}, \mathbf{X}^{(jk)}$ as in Proposition 3.1 with $d = 1$. We state our second main result. The proof is given in Section 4.3.

Theorem 3.2. Let $\mathbb{X} = (\hat{X}, X^{(i)}, \mathbf{X}^{(jk)})$ be the random variable taking value on $(\Omega_{(\alpha,\beta)\text{-Hld}}, d_{(\alpha,\beta)})$ defined as above. Then, the sequence of triplets:

$$\mathbb{X}^{\epsilon} := \left(\epsilon^H \hat{X}, \epsilon^{(i+1)H} X^{(i)}, \epsilon^{(j+k+2)H}\mathbf{X}^{(jk)}\right)$$

satisfies the LDP on $(\Omega_{(\alpha,\beta)\text{-Hld}}, d_{(\alpha,\beta)})$ with speed ϵ^{-2H} with good rate function

$$\begin{aligned} &I^{\#\#}(\hat{x}, x^{(i)}, \mathbf{x}^{(jk)}) \\ &:= \inf\left\{I^{\#}(u,v) \middle| u,v \in C_{[0,T]}, v \in \mathrm{BV},\ (\hat{x}, x^{(i)}, \mathbf{x}^{(jk)}) = \mathbb{L}(u,v)\right\}, \end{aligned}$$

where BV is the set of the functions of bounded variation on $[0,T]$, and

$$\mathbb{L}(u,v) := (\delta u, u \cdot v, u * v), \quad u,v \in C_{[0,T]},\ v \in \mathrm{BV},$$

$u \cdot v = (u \cdot_i v)$, $u * v = (u *_{jk} v)$, $(\delta u)_{st} := u_t - u_s$, and

$$(u \cdot_i v)_{st} := \int_s^t (u_r - u_s)^i \mathrm{d}v_r, \quad (u *_{jk} v)_{st} := \int_s^t (u \cdot_j v)_{sr}(u_r - u_s)^k \mathrm{d}v_r.$$

Here, $I^{\#}$ is the same as in [25]:

$$I^{\#}(u,v) := \begin{cases} \frac{1}{2}||(u,v)||^2_{\mathcal{H}^\Psi}, & (u,v) \in \mathcal{H}^\Psi, \\ \infty, & \text{otherwise,} \end{cases}$$

where $\mathcal{H}^\Psi := \left\{\mathcal{I}^\Psi g; g \in L^2([0,T],\mathbb{R}^2)\right\}$ with inner product

$$\langle \mathcal{I}^\Psi g_1, \mathcal{I}^\Psi g_2 \rangle := \langle g_1, g_2 \rangle_{L^2},$$

and $\mathcal{I}^\Psi : L^2([0,T],\mathbb{R}^2) \to L^2([0,T],\mathbb{R}^2)$ is defined by

$$\mathcal{I}^\Psi g := \int_0^{\cdot} \Psi(\cdot - u) g(u) \mathrm{d}u, \quad g \in L^2([0,T],\mathbb{R}^2),$$

with $\Psi : \mathbb{R}^+ \to \mathbb{R}^{2\times 2}$ defined by

$$\Psi := \begin{pmatrix} k_H & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}.$$

Theorem 3.3. The sequence of the processes $\left\{Y^\epsilon := \int f(\hat{\mathbb{X}}^\epsilon) \mathrm{d}\mathbb{X}^\epsilon\right\}_{\epsilon \geqq 0}$ satisfies the LDP on $(\Omega_{\alpha\text{-Hld}}, d_\alpha)$ with speed ϵ^{-2H} with good rate function

$$\begin{aligned} I^{\#\#\#}(y) &:= \inf\left\{ I^{\#\#}(\mathbb{X}) \,\middle|\, \mathbb{X} \in \Omega_{(\alpha,\beta)\text{-Hld}},\ y = \int f(\hat{\mathbb{X}}) \mathrm{d}\mathbb{X} \right\} \\ &= \inf\left\{ I^{\#}(u,v) \,\middle|\, u,v \in C_{[0,T]},\ v \in \mathrm{BV},\ y = \int f(\hat{\mathbb{L}}(u,v)) \mathrm{d}\mathbb{L}(u,v) \right\}. \end{aligned}$$

where $I^{\#\#}$ is defined in Theorem 3.2.

Proof. By Theorems 2.5 and 3.2 together with the contraction principle, we have the claim. □

3.3 RDE driven by an (α,β) rough path integral and the Freidlin-Wentzell LDP

We now discuss about the following type of RDE (in Lyons' sense; see e.g., Section 8.8 of [11]):

$$\bar{S}_t = \int_0^t \bar{\sigma}(\bar{S}_u, u) \mathrm{d}Y_u, \quad Y := \int f(\hat{\mathbb{X}}) \mathrm{d}\mathbb{X} \in \Omega_{\alpha\text{-Hld}}([0,T],\mathbb{R}^d), \tag{3.1}$$

where $\bar{S}_t = S_t - S_0$, $\bar{\sigma}(s,t) = \sigma(S_0 + s, t)$.

Theorem 3.4. Let $\sigma \in C_b^3$.

(i) RDE (3.1) driven by $Y = \int f(\hat{\mathbb{X}})d\mathbb{X}$ has the unique solution. Moreover, the solution map Φ;

$$\Phi : \Omega_{\alpha\text{-Hld}}([0,T],\mathbb{R}^d) \times \mathbb{R} \to \Omega_{\alpha\text{-Hld}}([0,T],\mathbb{R}^{d+1})$$

is locally Lipschitz continuous with respect to d_α.

(ii) The first level of the solution to RDE (3.1) is the solution to the Itô SDE (1.1).

Proof. See Appendix B. □

Theorem 3.5. Let $\sigma \in C_b^3$ and $\bar{S}^\epsilon := \Phi(Y^\epsilon)$, where Φ is the solution map of Theorem 3.4. Then the sequence of the processes $\{\bar{S}^\epsilon\}_{\epsilon \geqq 0}$ satisfies the LDP on $\Omega_{\alpha\text{-Hld}}$ with speed ϵ^{-2H} with the good rate function I

$$\begin{aligned} I(\bar{s}) &:= \inf \left\{ I^{\#\#\#}(Y) \middle| \, Y \in \Omega_{\alpha\text{-Hld}},\ \bar{s} = \Phi(Y) \right\} \\ &= \inf \left\{ I^{\#}(u,v) \middle| \, u, v \in C_{[0,T]}, v \in \mathrm{BV},\ \bar{s} = \int \bar{\sigma}(\bar{s},\cdot) f(\hat{\mathbb{L}}(u,v)) d\mathbb{L}(u,v) \right\}. \end{aligned}$$

Proof. Since the solution map Φ is continuous, Theorem 3.4 and the contraction theorem imply the claim. □

3.4 Short time asymptotics

By the scaling property of the Riemann-Liouville fractional Brownian motion $\hat{X}$ and the standard Brownian motion X, we have

$$\hat{X}_{\epsilon t} \sim \epsilon^H \hat{X}_t, \quad X_{\epsilon t} \sim \epsilon^{1/2} X_t.$$

This implies

$$\tilde{Y}_t^\epsilon := \epsilon^{H-1/2} \int_0^{\epsilon t} f(\hat{X}_u) dX_u \sim \int_0^t f(\hat{X}_u^\epsilon) dX_u^\epsilon,$$

where $(\hat{X}^\epsilon, X^\epsilon) = \epsilon^H(\hat{X}, X)$, of which the rough path lift is $\mathbb{X}^\epsilon$ of Theorem 3.2. Let

$$\tilde{S}_t^\epsilon = \frac{S_{\epsilon t} - S_0}{\epsilon^{1/2-H}}, \quad \tilde{\sigma}^\epsilon(s,t) = \sigma(S_0 + \epsilon^{1/2-H} s, \epsilon t).$$

Then,

$$\tilde{S}_t^\epsilon = \int_0^t \tilde{\sigma}^\epsilon(\tilde{S}_u^\epsilon, u) d\tilde{Y}_u^\epsilon$$

and we can derive an LDP for $\tilde{S}^\epsilon$ by an extended contraction principle [30].

Theorem 3.6. Let $\sigma \in C_b^3$. Then, $\{\tilde{S}^\epsilon\}_{0<\epsilon\leqq 1}$ satisfies the LDP on $\Omega_{\alpha\text{-Hld}}$ as $\epsilon \to 0$ with speed ϵ^{-2H} with good rate function

$$J(\tilde{s}) := \inf\left\{ I^{\#}(u,v) \middle| u, v \in C_{[0,T]}, v \in \mathrm{BV},\ \tilde{s} = \sigma(S_0, 0)\int f(\hat{\mathbb{L}}(u,v))\mathrm{d}\mathbb{L}(u,v) \right\}.$$

Proof. Denote by Φ_ϵ the solution map of the RDE (3.1) with $\bar{\sigma} = \tilde{\sigma}^\epsilon$. We are going to show that Φ_ϵ is locally equicontinuous. Since

$$||\nabla^i \tilde{\sigma}^\epsilon||_\infty \leqq (1+\epsilon)^i ||\nabla^i \sigma||_\infty \leqq 2^i ||\nabla^i \sigma||_\infty,$$

the local Lipschitz constants of Φ_ϵ can be taken uniformly in ϵ (as clearly seen from the proof of Theorem 3.4 in Appendix B). Therefore Φ_ϵ is equicontinuous on bounded sets, and we conclude $\Phi_\epsilon(Y_\epsilon) \to \Phi_0(Y)$ for any converging sequence $Y_\epsilon \to Y$ for any Y with $I^{\#\#\#}(Y) < \infty$. Then by Theorem 3.3 and an extended contraction principle [30][Theorem 2.1], we have the desired results. □

Remark 3.7. By using usual arguments, adding the drift term to above RDE is straightforward. Theorem 3.6 generalizes the preceding LDP results for the case $\sigma = 1$ called rough Bergomi model in [7, 2, 22, 27, 25]:

$$\mathrm{d}X_t = -\frac{1}{2} f^2(\hat{X}_t)\mathrm{d}t + f(\hat{X}_t)\mathrm{d}X_t.$$

Indeed, an LDP for the marginal distribution $\tilde{S}_1^\epsilon$ follows from the contraction principle, and the corresponding one-dimensional rate function extends the one first obtained by [7] as follows.

Theorem 3.8. Assume $\sigma \in C_b^3$ and $|\rho| < 1$. Then, $t^{H-1/2}\bar{S}_t$ satisfies the LDP as $t \to 0$ with speed t^{-2H} with good rate function

$$\bar{J}(z) := \inf_{g \in L^2([0,1])} \left[\frac{1}{2}\int_0^1 |g_r|^2 \mathrm{d}r + \frac{\left\{ z - \rho\sigma(S_0,0)\int_0^1 f\left(K_H g(r)\right) g_r \mathrm{d}r \right\}^2}{2(1-\rho^2)\sigma(S_0,0)^2 \int_0^1 f(K_H g(r))^2 \mathrm{d}r} \right],$$

where $K_H g(t) = \int_0^t k_H(t-r) g_r \mathrm{d}r$.

Proof. See Appendix C. □

4 Proofs of Main Theorems

4.1 Proof of Theorem 2.5

Proof. By a localizing argument, we can assume without loss of generality that the derivatives of f are bounded. To shorten, let $K := ||f||_{C_b^{n+1}}$, and

$M := |||\mathbb{X}|||_{(\alpha,\beta)}$. Let

$$J^{(1)}_{st} := \sum_{i\in I} \nabla^i f(\hat{x}_s) X^{(i)}_{st}, \quad J^{(2)}_{st} := \sum_{(j,k)\in J} \nabla^j f(\hat{x}_s) \nabla^k f(\hat{x}_s) \mathbf{X}^{(jk)}_{st}.$$

Below we follow the standard argument of the rough path theory with Chen's identity replaced by our modified version (1.2), (1.3).

(Claim 1) The first level of (α,β) rough path integral $Y^{(1)}_{st}$ is well-defined, and has the following inequality;

$$|Y^{(1)}_{st}| \leqq KC_1 |t-s|^{\alpha}, \tag{4.1}$$

where

$$C_1 := \left\{1 + 2^{(n+1)\beta+\alpha} \zeta((n+1)\beta+\alpha)\right\} (n+1)(1+M)^{n+1}(1+T)^{n+1}.$$

and $\zeta(r) := \sum_{p=1}^{\infty} \frac{1}{p^r}$.

Proof. By Taylor expansion, we have

$$\begin{aligned} \sum_{i\in I} \nabla^i f(\hat{x}_u) X^{(i)}_{ut} &= \sum_{i\in I} \left\{ \sum_{p=0}^{n-i} \frac{1}{p!} \nabla^{i+p} f(\hat{x}_s) \left(\hat{X}_{su}\right)^p X^{(i)}_{ut} + R_i \right\} \\ &= \sum_{i\in I} \nabla^i f(\hat{x}_s) \left\{ \sum_{p=0}^{i} \frac{1}{(i-p)!} \left(\hat{X}_{su}\right)^{i-p} X^{(p)}_{ut} \right\} + \sum_{i\in I} R_i. \end{aligned} \tag{4.2}$$

where

$$R_i := \left(\int_0^1 \frac{(1-\theta)^{n-i}}{(n-i)!} \nabla^{n+1} f(\hat{x}_s + \theta \hat{X}_{su}) \mathrm{d}\theta \right) (\hat{X}_{su})^{n+1-i} X^{(i)}_{ut}. \tag{4.3}$$

By the modified Chen's relation (1.2) and (4.2), for any $s \leqq u \leqq t$,

$$\begin{aligned} & J^{(1)}_{su} + J^{(1)}_{ut} - J^{(1)}_{st} \\ &= \sum_{i\in I} \nabla^i f(\hat{x}_s) \left(X^{(i)}_{su} - X^{(i)}_{st} \right) + \sum_{i\in I} \nabla^i f(\hat{x}_u) X^{(i)}_{ut} \\ &= -\sum_{i\in I} \nabla^i f(\hat{x}_s) \left\{ \sum_{p=0}^{i} \frac{1}{(i-p)!} \left(\hat{X}_{su}\right)^{i-p} X^{(p)}_{ut} \right\} + \sum_{i\in I} \nabla^i f(\hat{x}_u) X^{(i)}_{ut} \\ &= \sum_{i\in I} R_i. \end{aligned} \tag{4.4}$$

Since for all $i \in I$

$$|R_i| \leqq K \left|(\hat{X}_{su})^{n+1-i} X_{ut}^{(i)}\right| \leqq KM^{n+1-i}|t-s|^{(n+1)\beta+\alpha},$$

we have

$$\left|J_{su}^{(1)} + J_{ut}^{(1)} - J_{st}^{(1)}\right| \leqq K(n+1)(1+M)^{n+1}|t-s|^{(n+1)\beta+\alpha}.$$

For any partition $\mathcal{P} = \{s = t_0 < t_1 < \ldots < t_N = t\}$, let $J_{st}^{(1)}(\mathcal{P}) := \sum_{p=1}^{N} J_{t_{p-1}t_p}^{(1)}$, then for some integer p which satisfies the condition of Lemma A.1, we have

$$\begin{aligned}
&\left|J_{st}^{(1)}(\mathcal{P}) - J_{st}^{(1)}(\mathcal{P}\setminus\{t_p\})\right| \\
&= \left|J_{t_{p-1}t_p}^{(1)} + J_{t_pt_{p+1}}^{(1)} - J_{t_{p-1}t_{p+1}}^{(1)}\right| \\
&\leqq K(n+1)(1+M)^{n+1}|t_{p+1} - t_{p-1}|^{(n+1)\beta+\alpha} \\
&\leqq K(n+1)(1+M)^{n+1}\left(\frac{2}{N-1}\right)^{(n+1)\beta+\alpha}|t-s|^{(n+1)\beta+\alpha},
\end{aligned}$$

and this implies (note that $(n+1)\beta + \alpha > 1$),

$$\begin{aligned}
&\left|J_{st}^{(1)}(\mathcal{P}) - J_{st}^{(1)}\right| \\
&\leqq K(n+1)(1+M)^{n+1}2^{(n+1)\beta+\alpha}\zeta\left((n+1)\beta+\alpha\right)|t-s|^{(n+1)\beta+\alpha}
\end{aligned} \tag{4.5}$$

This inequality shows that $\{J_{st}^{(1)}(\mathcal{P})\}$ is a Cauchy sequence with $|\mathcal{P}| \searrow 0$. Therefore $Y_{st}^{(1)}$ is well-defined. Furthermore, by (4.5), we have

$$|Y_{st}^{(1)}| \leqq |J_{st}^{(1)}| + |Y_{st}^{(1)} - J_{st}^{(1)}| \leqq KC_1|t-s|^{\alpha}.$$

We finish to prove the statement of (Claim 1). □

(Claim 2) Let $m := \max_{(j,k)\in J} |j+k|$. Then the second level of (α,β) rough path integral $Y_{st}^{(2)}$ is well-defined, and has the following inequality;

$$|Y_{st}^{(2)}| \leqq K^2C_2|t-s|^{2\alpha},$$

where

$$C_2 := m^2M + \left(\tilde{C}_2 + C_1^2T^{n-m}\right)2^{(m+1)\beta+2\alpha}\zeta\left((m+1)\beta+2\alpha\right),$$

and

$$\tilde{C}_2 := (n^2+2m)(1+M)^{2m+3}(1+T)^{m+1}.$$

In particular, we have $\int f(\hat{\mathbb{X}})d\mathbb{X} \in \Omega_{\alpha\text{-Hld}}$.

Proof. By the modified Chen's relation (1.3), for all $s \leqq u \leqq t$,

$$
\begin{aligned}
& J_{su}^{(2)} + J_{ut}^{(2)} + J_{su}^{(1)} \otimes J_{ut}^{(1)} - J_{st}^{(2)} \\
&= J_{su}^{(1)} \otimes J_{ut}^{(1)} \\
&\quad + \sum_{(j,k)\in J} \left[\nabla^j f(\hat{x}_s) \nabla^k f(\hat{x}_s) \left(\mathbf{X}_{su}^{(jk)} - \mathbf{X}_{st}^{(jk)} \right) + \nabla^j f(\hat{x}_u) \nabla^k f(\hat{x}_u) \mathbf{X}_{ut}^{(jk)} \right] \\
&= S_1 + S_2,
\end{aligned}
$$

where

$$
S_1 := J_{su}^{(1)} \otimes J_{ut}^{(1)} - \sum_{(j,k)\in J} \nabla^j f(\hat{x}_s) \nabla^k f(\hat{x}_s) \left(\sum_{q=0}^{k} \frac{1}{(k-q)!} \left(\hat{X}_{su} \right)^{k-q} X_{su}^{(j)} \otimes X_{ut}^{(q)} \right),
$$

and

$$
\begin{aligned}
S_2 := & \sum_{(j,k)\in J} \nabla^j f(\hat{x}_u) \nabla^k f(\hat{x}_u) \mathbf{X}_{ut}^{(jk)} \\
& - \sum_{(j,k)\in J} \nabla^j f(\hat{x}_s) \nabla^k f(\hat{x}_s) \left(\sum_{p=0}^{j} \sum_{q=0}^{k} \frac{1}{(j-p)!(k-q)!} \left(\hat{X}_{su} \right)^{j+k-p-q} \mathbf{X}_{ut}^{(pq)} \right).
\end{aligned}
$$

By Taylor expansion, we have

$$
\begin{aligned}
& \sum_{(j,k)\in J} \nabla^j f(\hat{x}_s) \nabla^k f(\hat{x}_u) X_{su}^{(j)} \otimes X_{ut}^{(k)} \\
&= \sum_{(j,k)\in J} \left\{ \sum_{p=0}^{m-j-k} \frac{1}{p!} \nabla^j f(\hat{x}_s) \nabla^{k+p} f(\hat{x}_s) (\hat{X}_{su})^p X_{su}^{(j)} \otimes X_{ut}^{(k)} + R_{jk}^{(1)} \right\} \\
&= \sum_{(j,k)\in J} \left\{ \nabla^j f(\hat{x}_s) \nabla^k f(\hat{x}_s) \left(\sum_{q=0}^{k} \frac{1}{(k-q)!} \left(\hat{X}_{su} \right)^{k-q} X_{su}^{(j)} \otimes X_{ut}^{(q)} \right) + R_{jk}^{(1)} \right\},
\end{aligned}
$$

where

$$
\begin{aligned}
R_{jk}^{(1)} := & \left(\int_0^1 \frac{(1-\theta)^{m-j-k}}{(m-j-k)!} \nabla^j f(\hat{x}_s) \nabla^{m+1-j} f(\hat{x}_s + \theta \hat{X}_{su}) \mathrm{d}\theta \right) \\
& \times (\hat{X}_{su})^{m+1-j-k} X_{su}^{(j)} \otimes X_{ut}^{(k)}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&J_{su}^{(1)} \otimes J_{ut}^{(1)} \\
&= \left(\sum_{j\in I} \nabla^j f(\hat{x}_s) X_{su}^{(j)} \right) \otimes \left(\sum_{k\in I} \nabla^k f(\hat{x}_u) X_{ut}^{(k)} \right) \\
&= \sum_{j\in I} \sum_{k\in I} \nabla^j f(\hat{x}_s) \nabla^k f(\hat{x}_u) X_{su}^{(j)} \otimes X_{ut}^{(k)} \\
&= \sum_{(j,k)\in J} \nabla^j f(\hat{x}_s) \nabla^k f(\hat{x}_u) X_{su}^{(j)} \otimes X_{ut}^{(k)} + \sum_{(j,k)\in I\times I\setminus J} \nabla^j f(\hat{x}_s) \nabla^k f(\hat{x}_u) X_{su}^{(j)} \otimes X_{ut}^{(k)},
\end{aligned}
$$

implies that

$$
\begin{aligned}
S_1 &= J_{su}^{(1)} \otimes J_{ut}^{(1)} - \sum_{(j,k)\in J} \nabla^j f(\hat{x}_s) \nabla^k f(\hat{x}_s) \left(\sum_{q=0}^{k} \frac{1}{(k-q)!} \left(\hat{X}_{su} \right)^{k-q} X_{su}^{(j)} \otimes X_{ut}^{(q)} \right) \\
&= \sum_{(j,k)\in J} R_{jk}^{(1)} + \sum_{(j,k)\in I\times I\setminus J} \nabla^j f(\hat{x}_s) \nabla^k f(\hat{x}_u) X_{su}^{(j)} \otimes X_{ut}^{(k)}, \qquad (4.6)
\end{aligned}
$$

and

$$
\begin{aligned}
|S_1| &\leqq \sum_{(j,k)\in J} \left| R_{jk}^{(1)} \right| + \sum_{(j,k)\in I\times I\setminus J} \left| \nabla^j f(\hat{x}_s) \nabla^k f(\hat{x}_u) X_{su}^{(j)} \otimes X_{ut}^{(k)} \right| \\
&\leqq n^2 K^2 (1+M)^{m+3} (1+T)^m |t-s|^{(m+1)\beta+2\alpha}. \qquad (4.7)
\end{aligned}
$$

By Taylor expansion again, we have

$$
\begin{aligned}
\sum_{(j,k)\in J} \nabla^j f(\hat{x}_u) \nabla^k f(\hat{x}_u) \mathbf{X}_{ut}^{(jk)} = \sum_{(j,k)\in J} &\left\{ \sum_{p=0}^{m-j-k} \nabla^{j+p} f(\hat{x}_s) (\hat{X}_{su})^p + R_{jk}^{(2)} \right\} \\
&\times \left\{ \sum_{q=0}^{m-j-k} \nabla^{k+q} f(\hat{x}_s) (\hat{X}_{su})^q + R_{kj}^{(2)} \right\} \mathbf{X}_{ut}^{(jk)},
\end{aligned}
$$

where

$$
R_{jk}^{(2)} := \left(\int_0^1 \frac{(1-\theta)^{m-j-k}}{(m-j-k)!} \nabla^{m+1-k} f(\hat{x}_s + \theta \hat{X}_{su}) \mathrm{d}\theta \right) (\hat{X}_{su})^{m+1-j-k}.
$$

Therefore we have

$$
S_2 = \sum_{(j,k)\in J} R_{jk}^{(3)} \mathbf{X}_{ut}^{(jk)}, \qquad (4.8)
$$

where

$$R^{(3)}_{jk} := \left(\sum_{p=0}^{m-j-k} \nabla^{j+p} f(\hat{x}_s)(\hat{X}_{su})^p R^{(2)}_{kj}\right) + \left(\sum_{q=0}^{m-j-k} \nabla^{k+q} f(\hat{x}_s)(\hat{X}_{su})^q R^{(2)}_{jk}\right)$$
$$+ R^{(2)}_{jk} R^{(2)}_{kj}.$$

Since for all $p = 0, 1, 2 \ldots, m - j - k$,

$$\left|\nabla^{j+p} f(\hat{x}_s)(\hat{X}_{su})^p R^{(2)}_{kj} \mathbf{X}^{(jk)}_{ut}\right| \leqq K^2(1 + M)^{2m+2}(1 + T)^{m+1}|t - s|^{(m+1)\beta+2\alpha},$$

and

$$\left|R^{(2)}_{kj} R^{(2)}_{jk} \mathbf{X}^{(jk)}_{ut}\right| \leqq K^2(1 + M)^{2m+3}(1 + T)^{m+1}|t - s|^{(m+1)\beta+2\alpha},$$

we have

$$|S_2| \leqq 2mK^2(1 + M)^{2m+3}(1 + T)^{m+1}|t - s|^{(m+1)\beta+2\alpha}. \tag{4.9}$$

By (4.7) and (4.9), we have

$$\begin{aligned}\left|J^{(2)}_{su} + J^{(2)}_{ut} + J^{(1)}_{su} \otimes J^{(1)}_{ut} - J^{(2)}_{st}\right| &\leqq |S_1| + |S_2| \\ &\leqq K^2\tilde{C}_2|t - s|^{(m+1)\beta+2\alpha}.\end{aligned}$$

where $\tilde{C}_2 = (n^2 + 2m)(1 + M)^{2m+3}(1 + T)^{m+1}$. Moreover, by (4.1) and (4.5), we have

$$\begin{aligned}\left|Y^{(1)}_{su} \otimes Y^{(1)}_{ut} - J^{(1)}_{su} \otimes J^{(1)}_{ut}\right| &\leqq \left|Y^{(1)}_{su}\right|\left|Y^{(1)}_{ut} - J^{(1)}_{ut}\right| + \left|Y^{(1)}_{su} - J^{(1)}_{su}\right|\left|Y^{(1)}_{ut}\right| \\ &\leqq K^2C_1^2|t - s|^{(n+1)\beta+2\alpha}.\end{aligned}$$

Let $J^{(2)}_{st}(\mathcal{P}) := \sum_{p=1}^{n} Y^{(1)}_{t_0 t_{p-1}} \otimes Y^{(1)}_{t_{p-1}t_p} + J^{(2)}_{t_{p-1}t_p}$. Then, above discussions and Lemma A.1 imply that (note that since $(n+1)\beta+\alpha > 1$, $(n+1)\beta+2\alpha > 1$ and the minimality of m, we have $m \leqq n$)

$$\begin{aligned}&\left|J^{(2)}_{st}(\mathcal{P}) - J^{(2)}_{st}(\mathcal{P}\backslash\{t_p\})\right| \leqq \left|J^{(2)}_{t_{p-1}t_p} + J^{(2)}_{t_p t_{p+1}} + Y^{(1)}_{t_{p-1}t_p} \otimes Y^{(1)}_{t_p,t_{p+1}} - J^{(2)}_{t_{p-1}t_{p+1}}\right| \\ &\leqq \left|J^{(2)}_{t_{p-1},t_p} + J^{(2)}_{t_p t_{p+1}} + J^{(1)}_{t_{p-1}t_p} \otimes J^{(1)}_{t_p t_{p+1}} - J^{(2)}_{t_{p-1}t_{p+1}}\right| \\ &\quad + \left|Y^{(1)}_{t_{p-1}t_p} \otimes Y^{(1)}_{t_p t_{p+1}} - J^{(1)}_{t_{p-1}t_p} \otimes J^{(1)}_{t_p t_{p+1}}\right| \\ &\leqq K^2\tilde{C}_2|t_{p+1} - t_{p-1}|^{(m+1)\beta+2\alpha} + K^2C_1^2|t_{p+1} - t_{p-1}|^{(n+1)\beta+2\alpha} \\ &\leqq K^2\left(\tilde{C}_2 + C_1^2T^{n-m}\right)\left(\frac{2}{N-1}\right)^{(m+1)\beta+2\alpha}|t - s|^{(m+1)\beta+2\alpha}.\end{aligned}$$

This implies that (note that $(m+1)\beta + 2\alpha > 1$)

$$
\begin{aligned}
&|J^{(2)}_{st}(\mathcal{P}) - J^{(2)}_{st}| \\
&\leqq K^2 \left(\tilde{C}_2 + C_1^2 T^{n-m}\right) 2^{(m+1)\beta+2\alpha} \zeta\left((m+1)\beta + 2\alpha\right) |t-s|^{(m+1)\beta+2\alpha}.
\end{aligned}
$$

This shows that $\{J^{(2)}_{st}(\mathcal{P})\}_{\mathcal{P}}$ is a Cauchy sequence when $|\mathcal{P}| \searrow 0$. Hence $Y^{(2)}_{st}$ is well-defined. Also take C_2 big enough, we have

$$
\begin{aligned}
|Y^{(2)}_{st}| &\leqq |J^{(2)}_{st}| + |Y^{(2)}_{st} - J^{(2)}_{st}| \\
&\leqq m^2 K^2 M |t-s|^{2\alpha} \\
&\quad + K^2 \left(\tilde{C}_2 + C_1^2 T^{n-m}\right) 2^{(m+1)\beta+2\alpha} \zeta\left((m+1)\beta + 2\alpha\right) |t-s|^{(m+1)\beta+2\alpha} \\
&\leqq K^2 C_2 |t-s|^{2\alpha}.
\end{aligned}
$$

Next we prove that $\int f(\hat{\mathbb{X}}) \mathrm{d}\mathbb{X}$ satisfies the Chen's relation. Fix $\epsilon > 0$ and $s < u < t$. By taking a partition $\mathcal{P} = \{s = t_0 < t_1 < \ldots < t_N = t\}$ of $[s,t]$ small enough (which has the point $t_M = u$), we have

$$
\begin{aligned}
&\left| Y^{(1)}_{st} - Y^{(1)}_{su} - Y^{(1)}_{ut} \right| \\
&\leqq \left| Y^{(1)}_{st} - \sum_{p=1}^{N} J^{(1)}_{t_{p-1}t_p} \right| + \left| Y^{(1)}_{su} - \sum_{p=1}^{M} J^{(1)}_{t_{p-1}t_p} \right| + \left| Y^{(1)}_{ut} - \sum_{p=M+1}^{N} J^{(1)}_{t_{p-1}t_p} \right| \\
&\leqq 3\epsilon
\end{aligned}
$$

and so the first level of $\int f(\hat{\mathbb{X}}) \mathrm{d}\mathbb{X}$ satisfies Chen's relation. This result implies that

$$
\sum_{q=1}^{N} Y^{(1)}_{t_0 t_{q-1}} \otimes Y^{(1)}_{t_{q-1}t_q} = \sum_{0<p<q\leqq N} Y^{(1)}_{t_{p-1}t_p} \otimes Y^{(1)}_{t_{q-1}t_q}.
$$

Also,

$$
\begin{aligned}
&\left(\sum_{p=1}^{M} J^{(1)}_{t_{p-1}t_p} \right) \otimes \left(\sum_{q=M+1}^{N} J^{(1)}_{t_{q-1}t_q} \right) \\
&= \sum_{0<p<q\leqq N} J^{(1)}_{t_{p-1}t_p} \otimes J^{(1)}_{t_{q-1}t_q} - \sum_{0<p<q\leqq M} J^{(1)}_{t_{p-1}t_p} \otimes J^{(1)}_{t_{q-1}t_q} - \sum_{M<p<q\leqq N} J^{(1)}_{t_{p-1}t_p} \otimes J^{(1)}_{t_{q-1}t_q},
\end{aligned}
$$

and so we have

$$
\begin{aligned}
&\left| Y^{(2)}_{st} - Y^{(2)}_{su} - Y^{(2)}_{ut} - Y^{(1)}_{su} \otimes Y^{(1)}_{ut} \right| \\
&\leqq \left| Y^{(2)}_{st} - \mathcal{S}_{st} \right| + \left| Y^{(2)}_{su} - \mathcal{S}_{su} \right| + \left| Y^{(2)}_{ut} - \mathcal{S}_{ut} \right| \\
&\leqq 3\epsilon,
\end{aligned}
$$

where

$$\mathcal{S}_{st} := \sum_{p=1}^{N}\left(Y^{(1)}_{t_0 t_{p-1}} \otimes Y^{(1)}_{t_{p-1}t_p} + \sum_{(j,k)\in J} \nabla^j f(\hat{x}_{t_{p-1}})\nabla^k f(\hat{x}_{t_{p-1}})\mathbf{X}^{(jk)}_{t_{p-1}t_p}\right),$$

$$\mathcal{S}_{su} := \sum_{p=1}^{M}\left(Y^{(1)}_{t_0 t_{p-1}} \otimes Y^{(1)}_{t_{p-1}t_p} + \sum_{(j,k)\in J} \nabla^j f(\hat{x}_{t_{p-1}})\nabla^k f(\hat{x}_{t_{p-1}})\mathbf{X}^{(jk)}_{t_{p-1}t_p}\right),$$

$$\mathcal{S}_{ut} := \sum_{p=M+1}^{N}\left(Y^{(1)}_{t_M t_{p-1}} \otimes Y^{(1)}_{t_{p-1}t_p} + \sum_{(j,k)\in J} \nabla^j f(\hat{x}_{t_{p-1}})\nabla^k f(\hat{x}_{t_{p-1}})\mathbf{X}^{(jk)}_{t_{p-1}t_p}\right).$$

Therefore the second level of $\int f(\hat{\mathbb{X}})\mathrm{d}\mathbb{X}$ also satisfies Chen's relation. Above argument prove that the statement (1) of Theorem 2.5. □

(Claim 3) Suppose that there exist $M > 0$ and $\epsilon > 0$ such that

$$|\hat{V}_{st}| \vee |\hat{W}_{st}| \leqq M|t-s|^{\beta}, \quad |V^{(i)}_{st}| \vee |W^{(i)}_{st}| \leqq M|t-s|^{i\beta+\alpha},$$

$$|\mathbf{V}^{(jk)}_{st}| \vee |\mathbf{W}^{(jk)}_{st}| \leqq M|t-s|^{(j+k)\beta+2\alpha}, \quad |\hat{V}_{st} - \hat{W}_{st}| \leqq \epsilon|t-s|^{\beta},$$

and

$$|V^{(i)}_{st} - W^{(i)}_{st}| \leqq \epsilon|t-s|^{i\beta+\alpha}, \quad |\mathbf{V}^{(jk)}_{st} - \mathbf{W}^{(jk)}_{st}| \leqq \epsilon|t-s|^{(j+k)\beta+2\alpha}.$$

Then,

$$\left|\left(\int f(\hat{\mathbb{V}})\mathrm{d}\mathbb{V}\right)^{(1)}_{st} - \left(\int f(\hat{\mathbb{W}})\mathrm{d}\mathbb{W}\right)^{(1)}_{st}\right| \leqq K\epsilon C_3|t-s|^{\alpha}, \tag{4.10}$$

where

$$\begin{aligned} C_3 := &(n+1)M(1+T)^n \\ &+ 2(n+1)(1+M)^{n+2}(1+T)2^{(n+1)\beta+\alpha}\zeta((n+1)\beta+\alpha)T^{(n+1)\beta}. \end{aligned}$$

Proof. By the assumption and the mean value theorem, we have

$$\begin{aligned} &\left|J^{(1)}(\mathbb{V})_{st} - J^{(1)}(\mathbb{W})_{st}\right| \\ &= \left|\sum_{i\in I} \nabla^i f(\hat{v}_s)V^{(i)}_{st} - \sum_{i\in I} \nabla^i f(\hat{w}_s)W^{(i)}_{st}\right| \\ &\leqq \sum_{i\in I}\left\{|\nabla^i f(\hat{v}_s) - \nabla^i f(\hat{w}_s)||V^{(i)}_{st}| + |\nabla^i f(\hat{w}_s)||V^{(i)}_{st} - W^{(i)}_{st}|\right\} \\ &\leqq K\epsilon(n+1)M(1+T)^n|t-s|^{\alpha}. \end{aligned} \tag{4.11}$$

By (4.3), (4.4) and the mean value theorem, for all $s \leqq u \leqq t$,

$$\begin{aligned}
&\left|J^{(1)}(\mathbb{V})_{su} + J^{(1)}(\mathbb{V})_{ut} - J^{(1)}(\mathbb{V})_{st} - \left\{J^{(1)}(\mathbb{W})_{su} + J^{(1)}(\mathbb{W})_{ut} - J^{(1)}(\mathbb{W})_{st}\right\}\right| \\
&\leqq \sum_{i \in I} |R_i(\mathbb{V}) - R_i(\mathbb{W})| \\
&\leqq K\epsilon(1+T)(1+M)^{n+2}|t-s|^{(n+1)\beta+\alpha} \\
&\quad + K\epsilon 2(n+1)(1+M)^{n+1}|t-s|^{(n+1)\beta+\alpha} \\
&\leqq K\epsilon 2(n+1)(1+M)^{n+2}(1+T)|t-s|^{(n+1)\beta+\alpha}.
\end{aligned}$$

By this result and Lemma A.1, we have

$$\begin{aligned}
&\left|J^{(1)}(\mathbb{V})_{st}(\mathcal{P}) - J^{(1)}(\mathbb{V})_{st}(\mathcal{P}\backslash\{t_p\}) - \left\{J^{(1)}(\mathbb{W})_{st}(\mathcal{P}) - J^{(1)}(\mathbb{W})_{st}\left(\mathcal{P}\backslash\{t_p\}\right)\right\}\right| \\
&= \Bigg|J^{(1)}(\mathbb{V})_{t_{p-1}t_p} + J^{(1)}(\mathbb{V})_{t_p t_{p+1}} - J^{(1)}(\mathbb{V})_{t_{p-1}t_{p+1}} \\
&\qquad\qquad - \left\{J^{(1)}(\mathbb{W})_{t_{p-1}t_p} + J^{(1)}(\mathbb{W})_{t_p t_{p+1}} - J^{(1)}(\mathbb{W})_{t_{p-1}t_{p+1}}\right\}\Bigg| \\
&\leqq K\epsilon 2(n+1)(1+M)^{n+2}(1+T)|t_{p+1} - t_{p-1}|^{(n+1)\beta+\alpha} \\
&\leqq K\epsilon 2(n+1)(1+M)^{n+2}(1+T)\left(\frac{2}{N-1}\right)^{(n+1)\beta+\alpha} |t-s|^{(n+1)\beta+\alpha}.
\end{aligned}$$

This implies that (note that $(n+1)\beta + \alpha > 1$)

$$\begin{aligned}
&\left|J^{(1)}(\mathbb{V})_{st}(\mathcal{P}) - J^{(1)}(\mathbb{V})_{st} - \{J^{(1)}(\mathbb{W})_{st}(\mathcal{P}) - J^{(1)}(\mathbb{W})_{st}\}\right| \\
&\leqq K\epsilon 2(n+1)(1+M)^{n+2}(1+T)2^{(n+1)\beta+\alpha}\zeta((n+1)\beta+\alpha)|t-s|^{(n+1)\beta+\alpha}.
\end{aligned} \tag{4.12}$$

Therefore by (4.11) and (4.12), we conclude that

$$\begin{aligned}
&\left|J^{(1)}(\mathbb{V})_{st}(\mathcal{P}) - J^{(1)}(\mathbb{W})_{st}(\mathcal{P})\right| \\
&\leqq \left|J^{(1)}(\mathbb{V})_{st} - J^{(1)}(\mathbb{W})_{st}\right| \\
&\quad + \left|J^{(1)}(\mathbb{V})_{st}(\mathcal{P}) - J^{(1)}(\mathbb{V})_{st} - \left\{J^{(1)}(\mathbb{W})_{st}(\mathcal{P}) - J^{(1)}(\mathbb{W})_{st}\right\}\right| \\
&\leqq \epsilon(n+1)M(1+T)^n|t-s|^\alpha \\
&\quad + K\epsilon 2(n+1)(1+M)^{n+2}(1+T)2^{(n+1)\beta+\alpha}\zeta((n+1)\beta+\alpha)|t-s|^{(n+1)\beta+\alpha} \\
&\leqq K\epsilon C_3|t-s|^\alpha.
\end{aligned}$$

Take $|\mathcal{P}| \searrow 0$, we prove (4.10). □

(Claim 4) Suppose that there exist $M > 0$ and $\epsilon > 0$ such that

$$|\hat{V}_{st}| \vee |\hat{W}_{st}| \leqq M|t-s|^\beta, \quad |V^{(i)}_{st}| \vee |W^{(i)}_{st}| \leqq M|t-s|^{i\beta+\alpha},$$

$$|\mathbf{V}^{(jk)}_{st}| \vee |\mathbf{W}^{(jk)}_{st}| \leqq M|t-s|^{(j+k)\beta+2\alpha}, \quad |\hat{V}_{st} - \hat{W}_{st}| \leqq \epsilon|t-s|^{\beta},$$

and

$$|V^{(i)}_{st} - W^{(i)}_{st}| \leqq \epsilon|t-s|^{i\beta+\alpha}, \quad |\mathbf{V}^{(jk)}_{st} - \mathbf{W}^{(jk)}_{st}| \leqq \epsilon|t-s|^{(j+k)\beta+2\alpha}.$$

Then

$$\left|\left(\int f(\hat{\mathbb{V}})\mathrm{d}\mathbb{V}\right)^{(2)}_{st} - \left(\int f(\hat{\mathbb{W}})\mathrm{d}\mathbb{W}\right)^{(2)}_{st}\right| \leqq K^2\epsilon C_4|t-s|^{2\alpha}, \tag{4.13}$$

where

$$C_4 := M(1+T^{2\beta}) + (1+T^{n-m})(\tilde{C}_4 + 4C_1C_3)2^{(m+1)\beta+2\alpha}\zeta((m+1)\beta+2\alpha).$$

In particular, the integration map is Lipschitz continuous.

Proof. By the assumption and mean value theorem imply that

$$\begin{aligned} &\left|J^{(2)}(\mathbb{V})_{st} - J^{(2)}(\mathbb{W})_{st}\right| \\ &\leqq \sum_{(j,k)\in J} \left|\nabla^j f(\hat{v}_s)\nabla^k f(\hat{v}_s)\mathbf{V}^{(jk)}_{st} - \nabla^j f(\hat{w}_s)\nabla^k f(\hat{w}_s)\mathbf{W}^{(jk)}_{st}\right| \\ &\leqq K^2\epsilon M(1+T^{2\beta})|t-s|^{(j+k)\beta+2\alpha}. \end{aligned} \tag{4.14}$$

On the other hand, by (4.6) and (4.8), we can calculate

$$|S_1(\mathbb{V}) - S_1(\mathbb{W})| \leqq K^2\epsilon(2n+3)(1+M)^{m+3}(1+T)^{2n}|t-s|^{(m+1)\beta+2\alpha},$$

and

$$\begin{aligned} &|S_2(\mathbb{V}) - S_2(\mathbb{W})| \\ &\leqq K^2\epsilon(m^2+4m+2)(1+M)^{2m+3}(1+T)^{m+2}(1+2T)|t-s|^{(m+1)\beta+2\alpha}. \end{aligned}$$

Therefore we have

$$\begin{aligned} |\Sigma(\mathbb{V})_{sut} - \Sigma(\mathbb{W})_{sut}| &\leqq |S_1(\mathbb{V}) - S_1(\mathbb{W})| + |S_2(\mathbb{V}) - S_2(\mathbb{W})| \\ &\leqq K^2\epsilon\tilde{C}_4|t-s|^{(m+1)\beta+2\alpha}. \end{aligned}$$

where

$$\Sigma_{sut} := J^{(2)}_{su} + J^{(2)}_{ut} + J^{(1)}_{su} \otimes J^{(1)}_{ut} - J^{(2)}_{st}, \quad s \leqq u \leqq t,$$

and

$$\begin{aligned} \tilde{C}_4 = &(2n+3)(1+M)^{m+3}(1+T)^{2n} \\ &+ (m^2+4m+2)(1+M)^{2m+3}(1+T)^{m+2}(1+2T). \end{aligned}$$

Let

$$\Gamma_{sut} := Y^{(1)}_{su} \otimes Y^{(1)}_{ut} - J^{(1)}_{su} \otimes J^{(1)}_{ut}, \quad s \leqq u \leqq t.$$

Then by (4.1),(4.5), (4.10) and (4.12), we have

$$\begin{aligned}
&|\Gamma(\mathbb{V})_{sut} - \Gamma(\mathbb{W})_{sut}| \\
&\leqq \Big|Y^{(1)}(\mathbb{V})_{su} \otimes (Y^{(1)}(\mathbb{V})_{ut} - J^{(1)}(\mathbb{V})_{ut}) \\
&\qquad - Y^{(1)}(\mathbb{W})_{su} \otimes (Y^{(1)}(\mathbb{W})_{ut} - J^{(1)}(\mathbb{W})_{ut})\Big| \\
&\quad + \Big|(Y^{(1)}(\mathbb{V})_{su} - J^{(1)}(\mathbb{V})_{su}) \otimes J^{(1)}(\mathbb{V})_{ut} \\
&\qquad - (Y^{(1)}(\mathbb{W})_{su} - Y^{(1)}(\mathbb{W})_{su}) \otimes J^{(1)}(\mathbb{W})_{ut}\Big| \\
&\leqq \Big|Y^{(1)}(\mathbb{V})_{su}\Big| \Big|Y^{(1)}(\mathbb{V})_{ut} - J^{(1)}(\mathbb{V})_{ut} - Y^{(1)}(\mathbb{W})_{ut} + J^{(1)}(\mathbb{W})_{ut}\Big| \\
&\quad + \Big|Y^{(1)}(\mathbb{V})_{su} - Y^{(1)}(\mathbb{W})_{su}\Big| \Big|Y^{(1)}(\mathbb{W})_{ut} - J^{(1)}(\mathbb{W})_{ut}\Big| \\
&\quad + \Big|Y^{(1)}(\mathbb{V})_{ut} - J^{(1)}(\mathbb{V})_{ut} - Y^{(1)}(\mathbb{W})_{ut} + J^{(1)}(\mathbb{W})_{ut}\Big| \Big|J^{(1)}(\mathbb{V})_{ut}\Big| \\
&\quad + \Big|Y^{(1)}(\mathbb{W})_{ut} - J^{(1)}(\mathbb{W})_{ut}\Big| \Big|J^{(1)}(\mathbb{V})_{ut} - J^{(1)}(\mathbb{W})_{ut}\Big| \\
&\leqq K^2 \epsilon 4 C_1 C_3 |t-s|^{(n+1)\beta + 2\alpha}.
\end{aligned}$$

Therefore by Lemma A.1, we have

$$\begin{aligned}
&\Big|J^{(2)}(\mathbb{V})_{st}(\mathcal{P}) - J^{(2)}(\mathbb{V})_{st}(\mathcal{P}\backslash\{t_p\}) - \Big\{J^{(2)}(\mathbb{W})_{st}(\mathcal{P}) - J^{(2)}(\mathbb{W})_{st}(\mathcal{P}\backslash\{t_p\})\Big\}\Big| \\
&\leqq \Big|\Sigma(\mathbb{V})_{t_{p-1}t_p t_{p+1}} - \Sigma(\mathbb{W})_{t_{p-1}t_p t_{p+1}}\Big| + \Big|\Gamma(\mathbb{V})_{t_{p-1}t_p t_{p+1}} - \Gamma(\mathbb{V})_{t_{p-1}t_p t_{p+1}}\Big| \\
&\leqq K^2 \epsilon \tilde{C}_4 |t_{i+1} - t_{i-1}|^{(m+1)\beta + 2\alpha} + K^2 \epsilon 4 C_1 C_3 |t_{i+1} - t_{i-1}|^{(n+1)\beta + 2\alpha} \\
&\leqq K^2 \epsilon (1 + T^{n-m})(\tilde{C}_4 + 4 C_1 C_3) \left(\frac{2}{N-1}\right)^{(m+1)\beta + 2\alpha} |t-s|^{(m+1)\beta + 2\alpha}.
\end{aligned}$$

This implies that (note that $(m+1)\beta + 2\alpha > 1$)

$$\begin{aligned}
&\Big|J^{(2)}(\mathbb{V})_{st}(\mathcal{P}) - J^{(2)}(\mathbb{V})_{st} - \Big\{J^{(2)}(\mathbb{W})_{st}(\mathcal{P}) - J^{(2)}(\mathbb{W})_{st}\Big\}\Big| \\
&\leqq K^2 \epsilon (1 + T^{n-m})(\tilde{C}_4 + 4 C_1 C_3) 2^{(m+1)\beta + 2\alpha} \zeta((m+1)\beta + 2\alpha) |t-s|^{(m+1)\beta + 2\alpha}.
\end{aligned} \tag{4.15}$$

Therefore by (4.14) and (4.15), we conclude that

$$\begin{aligned}
&\Big|J^{(2)}(\mathbb{V})_{st}(\mathcal{P}) - J^{(2)}(\mathbb{W})_{st}(\mathcal{P})\Big| \\
&\leqq \Big|J^{(2)}(\mathbb{V})_{st} - J^{(2)}(\mathbb{W})_{st}\Big| \\
&\quad + \Big|J^{(2)}(\mathbb{V})_{st}(\mathcal{P}) - J^{(2)}(\mathbb{V})_{st} - \{J^{(2)}(\mathbb{W})_{st}(\mathcal{P}) - J^{(2)}(\mathbb{W})_{st}\}\Big| \\
&\leqq K^2 \epsilon C_4 |t-s|^{2\alpha}.
\end{aligned}$$

Take $|\mathcal{P}| \searrow 0$, we have (4.13).

For any $\mathbb{V}, \mathbb{W} \in \mathcal{E}_M$, take $\epsilon := d_{(\alpha,\beta)}(\mathbb{V}, \mathbb{W})$. Then we have

$$|\hat{V}_{st}| \vee |\hat{W}_{st}| \leqq M|t-s|^{\beta}, \quad |V^{(i)}_{st}| \vee |W^{(i)}_{st}| \leqq M|t-s|^{i\beta+\alpha},$$

$$|\mathbf{V}^{(jk)}_{st}| \vee |\mathbf{W}^{(jk)}_{st}| \leqq M|t-s|^{(j+k)\beta+2\alpha}, \quad |\hat{V}_{st}-\hat{W}_{st}| \leqq \epsilon|t-s|^{\beta},$$

and

$$|V^{(i)}_{st}-W^{(i)}_{st}| \leqq \epsilon|t-s|^{i\beta+\alpha}, \quad |\mathbf{V}^{(jk)}_{st}-\mathbf{W}^{(jk)}_{st}| \leqq \epsilon|t-s|^{(j+k)\beta+2\alpha}.$$

Therefore by (4.10), (4.13), we conclude that

$$\begin{aligned} d_\alpha\left(\int f(\hat{\mathbb{V}})\mathrm{d}\mathbb{V}, \int f(\hat{\mathbb{W}})\mathrm{d}\mathbb{W}\right) &\leqq KC_3\epsilon + K^2C_4\epsilon \\ &\leqq K(C_3+KC_4)d_{(\alpha,\beta)}(\mathbb{V},\mathbb{W}), \quad \mathbb{V},\mathbb{W}\in\mathcal{E}_M. \end{aligned}$$

□

(Claim1) to (Claim4) complete the proof of Theorem 2.5. □

4.2 Proof of Proposition 3.1

Lemma 4.1 ([26] Corollary 9.7)**.** Let Y belong to the m-th Wiener chaos and $p \geqq 2$. Then we have

$$\|Y\|_p \leqq \sqrt{m+1}(p-1)^{m/2}\|Y\|_2.$$

proof of Proposition 3.1. (i) The modified Chen's relation follows from the binomial theorem as illustrated in Introduction. For the Hölder property, by Proposition A.2 (a version of Kolmogorov's continuity theorem), it is sufficient to prove the following inequalities;

$$\|X^{(i)}_{st}\|_p \leqq C|t-s|^{iH+1/2}, \quad \|\mathbf{X}^{(jk)}_{st}\|_p \leqq C|t-s|^{(j+k)H+1}, \quad p\geqq 2, \quad (s,t)\in\Delta_T.$$

Fix $r > s$ and let $Z^r_{su} := \int_0^u k_H(r-t_1)-k_H(s-t_1)1_{[0,s]}(t_1)\mathrm{d}W_{t_1}$, $u\in[0,r)$. By Itô formula, we have

$$(Z^r_{su})^i = i\int_0^u (Z^r_{sv})^{i-1}\mathrm{d}Z^r_{sv} + \frac{i(i-1)}{2}\int_0^u (Z^r_{sv})^{i-2}\mathrm{d}\left\langle Z^r_{s\cdot}\right\rangle_v.$$

Also by Itô formula, we have

$$\left(\int_0^w F_v\mathrm{d}Z^r_{sv}\right)\left(\int_0^w G_u\mathrm{d}\left\langle Z^r_{s\cdot}\right\rangle_u\right) = \int_0^w \left\{\int_0^u G_v\mathrm{d}\left\langle Z^r_{s\cdot}\right\rangle_v\right\}F_u\mathrm{d}Z^r_{su} + \int_0^w\left\{\int_0^u F_v\mathrm{d}Z^r_{sv}\right\}G_u\mathrm{d}\left\langle Z^r_{s\cdot}\right\rangle_u,$$

Using these formulae inductively, we have

$$(Z^r_{su})^i = \sum_{0\leqq q\leqq\lfloor i/2\rfloor}\frac{i!}{2^q}F^{(i-2q)}_u G^{(q)}_u,$$

where

$$F_u^{(p)} := \int_0^u \int_0^{t_p} \dots \left(\int_0^{t_2} \mathrm{d}Z_{st_1}^r \right) \dots \mathrm{d}Z_{st_p}^r, \quad G_u^{(q)} := \int_0^u \int_0^{\tau_q} \dots \left(\int_0^{\tau_2} \mathrm{d}\left\langle Z_{s\cdot}^r \right\rangle_{\tau_1} \right) \dots \mathrm{d}\left\langle Z_{s\cdot}^r \right\rangle_{\tau_q},$$

since

$$\begin{aligned}
(Z_{su}^r)^i &= i \int_0^u (Z_{sv}^r)^{i-1} \mathrm{d}Z_{sv}^r + \frac{i(i-1)}{2} \int_0^u (Z_{sv}^r)^{i-2} \mathrm{d}\left\langle Z_{s\cdot}^r \right\rangle_v \\
&= \sum_{0 \leqq q \leqq \lfloor (i-1)/2 \rfloor} \frac{i!}{2^q} \int_0^u F_v^{(i-1-2q)} G_v^{(q)} \mathrm{d}Z_{sv}^r + \sum_{0 \leqq q \leqq \lfloor (i-2)/2 \rfloor} \frac{i!}{2^{q+1}} \int_0^u F_v^{(i-2-2q)} G_v^{(q)} \mathrm{d}\left\langle Z_{s\cdot}^r \right\rangle_v \\
&= i! \int_0^u F_v^{(i)} \mathrm{d}Z_{sv}^r \\
&\quad + \sum_{1 \leqq q \leqq \lfloor (i-1)/2 \rfloor} \frac{i!}{2^q} \int_0^u F_v^{(i-1-2q)} G_v^{(q)} \mathrm{d}Z_{sv}^r + \sum_{1 \leqq q \leqq \lfloor i/2 \rfloor} \frac{i!}{2^q} \int_0^u F_v^{(i-2q)} G_v^{(q-1)} \mathrm{d}\left\langle Z_{s\cdot}^r \right\rangle_v \\
&= \sum_{0 \leqq q \leqq \lfloor i/2 \rfloor} \frac{i!}{2^q} F_u^{(i-2q)} G_u^{(q)}.
\end{aligned}$$

Take $u \to r$, we have

$$(\hat{X}_{sr})^i = (Z_{sr}^r)^i = \sum_{0 \leqq q \leqq \lfloor i/2 \rfloor} \frac{i!}{2^q} F_r^{(i-2q)} G_r^{(q)},$$

and so

$$X_{st}^{(i)} = \sum_{0 \leqq q \leqq \lfloor i/2 \rfloor} \frac{i!}{2^q} \int_s^t F_r^{(i-2q)} G_r^{(q)} \mathrm{d}X_r.$$

Therefore by Lemma 4.1, we have

$$||X_{st}^{(i)}||_p \leqq \sum_{0 \leqq q \leqq \lfloor i/2 \rfloor} \frac{i!}{2^q} \left\| \int_s^t F_r^{(i-2q)} G_r^{(q)} \mathrm{d}X_r \right\|_p \leqq \left(\sum_{0 \leqq q \leqq \lfloor i/2 \rfloor} \frac{i!}{2^q} \right) p^{(i+1)/2} |t-s|^{iH+1/2}. \tag{4.16}$$

By the same argument, we have

$$||\mathbf{X}_{st}^{(jk)}||_p \leqq C p^{(j+k+2)/2} |t-s|^{(j+k)H+1}. \tag{4.17}$$

(ii) By (i) and Theorem 2.5, for a.s. ω, the limit

$$\left(\int f(\hat{\mathbb{X}}) \mathrm{d}\mathbb{X} \right)_{st}^{(1)} = \lim_{N \to \infty} \sum_{q=1}^{N} \sum_{i \in I} \nabla^i f(\hat{X}_{t_{q-1}}) X_{t_{q-1} t_q}^{(i)}$$

exists. Since

$$\int_s^t f(\hat{X}_r) \mathrm{d}X_r = \lim_{N \to \infty} \sum_{q=1}^{N} f(\hat{X}_{t_{q-1}}) X_{t_{q-1} t_q}^{(0)}$$

in the sense of the convergence in probability, it is sufficient to prove for all $i \in I \backslash \{0\}$,

$$\lim_{N\to\infty} \sum_{q=1}^{N} \nabla^i f(\hat{X}_{t_{q-1}}) X^{(i)}_{t_{q-1}t_q} = 0,$$

in probability. Fix $i \in I\backslash\{0\}$. We can assume $f \in C_b^{n+1}$ without loss of generality. By the result (i), we have

$$\mathsf{E}\left[\left(X^{(i)}_{st}\right)^2\right] = C|t-s|^{2iH+1} < \infty,$$

and so take $K := ||f||_{C^n}$, we conclude that

$$\begin{aligned}
\mathsf{E}\left[\left(\sum_{q=1}^{N} \nabla^i f(\hat{x}_{t_q}) X^{(i)}_{t_{q-1},t_q}\right)^2\right] &= \sum_{q=1}^{N} \mathsf{E}\left[\left(\nabla^i f(\hat{x}_{t_q}) X^{(i)}_{t_{q-1}t_q}\right)^2\right] \\
&\leqq K^2 \sum_{q=1}^{N} |t_q - t_{q-1}|^{2iH+1} \\
&= K^2 \left(\sup_{|t-s|\leqq|\mathcal{P}|} |t-s|\right)^{2iH} T \\
&\to 0 \quad (as\ |\mathcal{P}| \to 0),
\end{aligned}$$

and this indicates the L^2 convergence. □

4.3 Proof of Theorem 3.2

Denote by $C_{[0,T]}$ the set of the $\mathbb{R}$-valued continuous functions on $[0,T]$ equipped with the uniform topology. Let C_{Δ_T} be the set of continuous functions on Δ_T, taking values in $\mathbb{R}^D$, equipped with the uniform topology for the metric

$$d(X,Y) := \sup_{(s,t)\in\Delta_T} |X_{st} - Y_{st}|, \quad X, Y \in C_{\Delta_T}.$$

We use the same notation C_{Δ_T} for different dimensions D. More specifically, any one of $D = 1$, $D = \max I$ or $D = \max\{j+k; (j,k) \in J\}$. Let $\mathcal{S}_0$ be the set of the $\mathbb{R}$-valued $\{\mathcal{F}_t\}$-adapted simple processes on $[0,T] \times \Omega$ and

$$\mathcal{S} := \left\{ Z \in \mathcal{S}_0 \middle| \sup_{t\in[0,T]} |Z_t| \leqq 1 \right\}.$$

Definition 4.2 ([19]). Let $\{Y^n\}$ be a sequence of $\mathbb{R}$-valued semimartingales on $[0,T]\times\Omega$. We say that the sequence is Uniformly Exponentially Tight (UET) if for every $T>0$ and every $a>0$ there is $K_{T,a}$ such that

$$\limsup_{n\to\infty}\frac{1}{n}\log\sup_{Z\in\mathcal{S}^n}\mathsf{P}\left[\sup_{t\in[0,T]}|(Z_-\cdot Y^n)_t|\geqq K_{T,a}\right]\leqq -a, \tag{4.18}$$

where $Z_-\cdot Y$ is the Itô integral of Z with respect to Y;

$$(Z_-\cdot Y)_t := \int_0^t Z_{r-}\mathrm{d}Y_r.$$

For a one-dimensional Brownian motion W, $Y^n = n^{-1/2}W$ is an example of UET sequences; see Lemma 2.4 of [19].

Theorem 4.3. Let $\{Y^n\}$ be a UET sequence of $\mathbb{R}$-valued semimartingales and $\{X^n\}$ a sequence of $\mathbb{R}$-valued continuous adapted processes. Assume that the sequence $\{(X^n,Y^n)\}$ satisfies the large deviation principle (LDP) on $C_{[0,T]}\times C_{[0,T]}$ with speed n^{-1} and good rate function $I^{\#}$. Then the sequence $\{(X^n,Y^n,(X^n\odot_i Y^n)_{i\in I})\}$ satisfies the large deviation principle on $C_{[0,T]}\times C_{[0,T]}\times C_{\Delta_T}$ with speed n^{-1} and good rate function

$$\begin{aligned} I^{\#\#}(u,v,x) &:= \begin{cases} I^{\#}(u,v), & \forall i\in I, x^{(i)} = u\odot_i v,\ v\in \mathrm{BV},\\ \infty, & \exists i\in I \text{ s.t. } x^{(i)}\neq u\odot_i v,\ v\in\mathrm{BV},\\ \infty, & v\notin \mathrm{BV}. \end{cases}\\ &= \inf\left\{I^{\#}(u,v)\middle|\ \forall i\in I, x^{(i)} = u\odot_i v,\ u,v\in C_{[0,T]}, v\in\mathrm{BV}\right\}, \end{aligned} \tag{4.19}$$

where $x=(x^{(i)})_{i\in I}\in C_{\Delta_T}$ and

$$(u\odot_i v)_{st} := \int_s^t (u_r-u_s)^i \mathrm{d}v_r.$$

Proof. By the assumption and contraction principle, $\{(X^n,Y^n,((X^n)^i)_{i\in I})\}$ satisfies the LDP with good rate function

$$\Lambda_1(u,v,\varphi) = \inf\left\{I^{\#}(u,v)\middle|\ \forall i\in I, \varphi^{(i)} = u^i\right\}.$$

Therefore, by [19][Theorem1.2], we have $\{(X^n,Y^n,((X^n)^i,X^n\cdot_i Y^n)_{i\in I})\}$ satisfies the LDP with good rate function

$$\begin{aligned}&\Lambda_2(u,v,\varphi,x)\\ &=\inf\left\{I^{\#}(u,v)\middle|(\varphi^{(i)},x^{(i)}) = (u^i, u\cdot_i v),\ u,v\in C_{[0,T]},\ v\in\mathrm{BV}\right\},\end{aligned}$$

where $(u \cdot_i v)_t = (u \odot_i v)_{0t}$. Note that by the modified Chen's relation (1.2), we have

$$(x \odot_i y)_{st} = (x \cdot_i y)_t - (x \cdot_i y)_s - \sum_{p=0}^{i-1} \frac{1}{(i-p)!}(x_s - x_0)^{i-p}(x \odot_p y)_{st}.$$

Hence by the contraction principle again with the aid of induction, we see that $\{(X^n, Y^n, (X^n \odot_i Y^n)_{i\in I})\}$ satisfies the LDP with good rate function (4.19). □

Theorem 4.4. Under the same conditions as in Theorem 4.3, the sequence $\{(\delta X^n, \delta Y^n, (X^n \odot_i Y^n)_{i\in I}, (X^n \circledast_{jk} Y^n)_{(j,k)\in J})\}$ satisfies the large deviation principle on $C_{\Delta_T} \times C_{\Delta_T} \times C_{\Delta_T}$ with speed n^{-1} with good rate function

$$\begin{aligned} &I(\hat{x}, x, \mathbf{x}) \\ &= \inf\left\{ I^{\#}(u,v) \middle| \begin{array}{c} \forall i \in I, \forall (j,k) \in J, (\hat{x}, x^{(i)}, \mathbf{x}^{(jk)}) = (\delta u, u \odot_i v, u \circledast_{jk} v) \\ u, v \in C_{[0,T]},\ v \in \mathrm{BV} \end{array} \right\}, \end{aligned} \tag{4.20}$$

where $(\delta u)_{st} := u_t - u_s$ and

$$(u \circledast_{jk} v)_{st} := \int_s^t (u \odot_j v)_{sr}(u_r - u_s)^k \mathrm{d}v_r,$$

Proof. By the modified Chen's relation (1.3), we have

$$\begin{aligned} (X^n \circledast_{jk} Y^n)_{st} &= (X^n *_{jk} Y^n)_t - (X^n *_{jk} Y^n)_{0s} \\ &\quad - \sum_{q=0}^{k} \frac{1}{(k-q)!}(X^n_{0s})^{k-q}(X^n \odot_j Y^n)_{0s} \otimes (X^n \odot_q Y^n)_{st} \\ &\quad - \sum_{p+q<j+k} \frac{1}{(j-p)!(k-q)!}(X^n_{0s})^{j+k-p-q}(X^n \circledast_{pq} Y^n)_{st}, \end{aligned}$$

where $(X *_{jk} Y)_t = (X \circledast_{jk} Y)_{0t}$.

First Step. By Theorem 4.3 and the contraction principle, the sequence

$$\left\{\left(X^n, Y^n, (X^n \odot_i Y^n)_{i\in I}, ((X^n \cdot_j Y^n)(X^n)^k)_{(j,k)\in J}\right)\right\}$$

satisfies the LDP with good rate function

$$\begin{aligned} &\Lambda_1(u, v, x, \varphi) \\ &= \inf\left\{ I^{\#}(u,v) \middle| (x^{(i)}, \varphi^{(jk)}) = (u \odot_i v, (u \cdot_j v)v^k), u, v \in C_{[0,T]},\ v \in \mathrm{BV} \right\}. \end{aligned}$$

Therefore, by [19][Theorem1.2], we have

$$\left\{\left(X^n, Y^n, (X^n \odot_i Y^n)_{i\in I}, (X^n *_{jk} Y^n)_{(j,k)\in J}\right)\right\}$$

satisfies the LDP with good rate function

$$\begin{aligned}&\Lambda_2(u,v,x,\varphi)\\&= \inf\left\{I^\#(u,v)\middle|(x^{(i)},\varphi^{(jk)}) = (u\odot_i v, u *_{jk} v),\ u,v\in C_{[0,T]},\ v\in \mathrm{BV}\right\}.\end{aligned}$$

By the contraction principle, we conclude that $\{M_n := (X^n, Y^n, X^n \odot_i Y^n, (X^n *_{jk} Y^n), S(X^n, Y^n))\}$ satisfies the LDP with good rate function

$$\begin{aligned}&\Lambda_3(u,v,x,\varphi,\psi)\\&= \inf\left\{I^\#(u,v)\middle|\begin{matrix}(x^{(i)},\varphi^{(jk)},\psi^{(jk)}) = \left(u\odot_i v, (u *_{jk} v)_t, S(u,v)\right)\\ u,v\in C_{[0,T]},\ v\in \mathrm{BV}\end{matrix}\right\},\end{aligned}$$

where

$$S(u,v)_{st} := \sum_{q=0}^{k} \frac{1}{(k-q)!} u_s^{k-q} (u\odot_j v)_{0s} \otimes (u \odot_q v)_{st}.$$

Second Step: To shorten, set

$$\text{condition A} := \left[\begin{matrix}(x^{(i)},\varphi^{(jk)},\psi^{(jk)}) = \left(u\odot_i v, (u *_{jk} v)_t, S(u,v)\right),\\ u,v\in C_{[0,T]},\ v\in \mathrm{BV}\end{matrix}\right].$$

By the modified Chen's relation (1.3), we have $(u \circledast_{00} v)_{st} = (u \circledast_{00} v)_t - (u \circledast_{00} v)_{0s} - v_s \otimes (v_t - v_s)$, and similar for $u \circledast_{01} v$ and $u \circledast_{10} v$. Therefore, by the contraction principle, we conclude that $\{(M_n, X^n \circledast_{00} Y^n, X^n \circledast_{01} Y^n, X^n \circledast_{10} Y^n)\}$ satisfies the LDP with good rate function

$$\inf\left\{I^\#(u,v)\middle|\ \text{cond. A},\ (\mathbf{x}^{(00)},\mathbf{x}^{(01)},\mathbf{x}^{(10)}) = (u\circledast_{00} v, u\circledast_{01} v, u\circledast_{10} v)\right\}.$$

In particular, $\{(X^n, Y^n, X^n \odot_i Y^n, X^n \circledast_{00} Y^n, X^n \circledast_{01} Y^n, X^n \circledast_{10} Y^n)\}$ satisfies the LDP with good rate function

$$\begin{aligned}&\Lambda_4(u,v,x,\mathbf{x}^{(00)},\mathbf{x}^{(01)},\mathbf{x}^{(10)})\\&= \inf\left\{I^\#(u,v)\middle|\begin{matrix}(x^{(i)},\mathbf{x}^{(00)},\mathbf{x}^{(01)},\mathbf{x}^{(10)})\\ = (u\odot_i v, u\circledast_{00} v, u\circledast_{01} v, u\circledast_{10} v)\end{matrix}\right\}.\end{aligned}$$

Hence, by the contraction principle and mathematical induction, we conclude that $\{(\delta X^n, \delta Y^n, X^n \odot_i Y^n, X^n \circledast_{jk} Y^n)\}$ satisfies the LDP with good rate function (4.20).

□

Lemma 4.5 ([12] Lemma A.18)**.** Let Z be a real valued non-negative variables such that there exists $c_1 > 0$ such that for all $p \in [1, \infty)$,

$$||Z||_p \leqq \frac{1}{c_1}\sqrt{p} < \infty.$$

Then there exists $c_2 > 0$ such that

$$\mathbb{E}[\exp(c_2 Z^2)] < \infty.$$

Lemma 4.6. (i) The (α, β) rough path $\mathbb{X}$ of Theorem 3.2 has exponential integrability, i.e., there exists $c > 0$ such that

$$\mathsf{E}\left[\exp\left\{c|||\mathbb{X}|||^2_{(\alpha,\beta)}\right\}\right] < \infty.$$

(ii) Assume that the family of random variables

$$\mathbb{X}^\epsilon = (\epsilon^H \hat{X}, \epsilon^{(i+1)H} X^{(i)}, \epsilon^{(j+k+2)H} \mathbf{X}^{(jk)})$$

taking values in $\Omega_{(\alpha,\beta)\text{-Hld}}$ satisfies the LDP on $C_{\Delta_T} \times C_{\Delta_T} \times C_{\Delta_T}$ (with the uniform topology). Then, $\mathbb{X}^\epsilon$ satisfies the LDP on $\Omega_{(\alpha,\beta)\text{-Hld}}$ (in the $d_{(\alpha,\beta)}$ topology) with the same good rate function.

Proof. (i) Let $Z := |||\mathbb{X}|||_{(\alpha,\beta)}$. By Lemma 4.5, it is enough to prove that there exists $c > 0$ such that

$$||Z||_p \leqq \frac{1}{c}\sqrt{p}, \quad p \in [1, \infty). \tag{4.21}$$

By the inequality (4.16), (4.17), we have

$$||X^{(i)}_{st}||_p \leqq p^{(i+1)/2}|t-s|^{iH+1/2}, \quad ||\mathbf{X}^{(jk)}_{st}||_p \leqq p^{(j+k+2)/2}|t-s|^{(j+k)H+1},$$

and this inequality and Proposition A.2 imply that for all $p \in [1, \infty)$,

$$\left\|||\hat{X}||_{\beta\text{-Hld}}\right\|_p \leqq c\sqrt{p}, \quad \left\|||X^{(i)}||_{i\beta+\alpha\text{-Hld}}\right\|_p \leqq cp^{(i+1)/2},$$

and

$$\left\|||\mathbf{X}^{(jk)}||_{(j+k)\beta+2\alpha\text{-Hld}}\right\|_p \leqq cp^{(j+k+2)/2},$$

and so the Hölder inequality implies

$$\begin{aligned}
\left\|\left(||X^{(i)}||_{i\beta+\alpha\text{-Hld}}\right)^{1/(i+1)}\right\|_p &\leqq \mathsf{E}\left[\left(||X^{(i)}||_{i\beta+\alpha\text{-Hld}}\right)^{p/(i+1)}\right]^{1/p} \\
&\leqq \mathsf{E}\left[\left(||X^{(i)}||_{i\beta+\alpha\text{-Hld}}\right)^{p}\right]^{1/p(i+1)} \\
&\leqq \left\|||X^{(i)}||_{i\beta+\alpha\text{-Hld}}\right\|_p^{1/(i+1)} \\
&\leqq c\sqrt{p},
\end{aligned}$$

and similarly,

$$\left\| \left(||\mathbf{X}^{(jk)}||_{(j+k)\beta+2\alpha\text{-Hld}} \right)^{1/(j+k+2)} \right\|_p \leqq c\sqrt{p}.$$

Therefore we conclude (4.21) by the definition of the homogeneous norm.

(ii) We adapt the argument of [12][Proposition 13.43]. By the inverse contraction principle (see Theorem 4.2.4 of [6]), it is sufficient to prove that $\mathbb{X}^\epsilon$ is exponentially tight on $\Omega_{(\alpha,\beta)\text{-Hld}}$. By (i), there exists $c > 0$ such that

$$\mathsf{P}\left[|||\mathbb{X}|||_{(\alpha',\beta')} > l\right] \leqq \exp(-cl^2),$$

for any $\alpha' \in (\alpha, 1/2)$ and $\beta' \in (\beta, H)$, and this implies that for all $M > 0$, there exists a precompact set

$$K_M = \left\{ X \in \Omega_{(\alpha,\beta)\text{-Hld}} \middle| \; |||\mathbb{X}|||_{(\alpha',\beta')} \leqq \sqrt{M/c} \right\}$$

on $\Omega_{(\alpha,\beta)\text{-Hld}}$ such that

$$\begin{aligned} \epsilon^{2H} \log \mathsf{P}\left[\mathbb{X}^\epsilon \in K_M^c\right] &= \epsilon^{2H} \log \mathsf{P}\left[|||\mathbb{X}^\epsilon|||_{(\alpha',\beta')} > \sqrt{\frac{M}{c}} \right] \\ &= \epsilon^{2H} \log \mathsf{P}\left[|||\mathbb{X}|||_{(\alpha',\beta')} > \sqrt{\frac{M}{c\epsilon^{2H}}} \right] \leqq -M, \end{aligned}$$

from which we conclude. □

As mentioned earlier, $X^\epsilon = \epsilon^H X$ is UET by Lemma 2.4 of [19] with $n = \epsilon^{-2H}$. The LDP for $\epsilon^H(\hat{X}, X)$ has been proved in [7] with speed ϵ^{-2H} and good rate function $I^\#$. Therefore by Lemma 4.6 (ii) and Theorem 4.4 (regarding as $n = \epsilon^{-2H}$), we have proved Theorem 3.2.

A Some Lemmas

Lemma A.1 ([26] Proposition1.6)**.** Let ω be a control function; i.e.

$$\omega(s,u) + \omega(u,t) \leqq \omega(s,t), \quad 0 \leqq s \leqq u \leqq t \leqq T,$$

and $\mathcal{P} = \{s = t_0 < t_1 < \dots < t_N = t\}$ be a partition on $[s,t]$ ($N \geqq 2$). Then there exists an integer $i (1 \leqq i \leqq N)$ such that;

$$\omega(t_{i-1}, t_{i+1}) \leqq \frac{2}{N-1}\omega(s,t)$$

Proof. By definition of ω, we have

$$\sum_{p=1}^{N-1} \omega(t_{i-1}, t_{i+1}) = \sum_{i:\text{odd}} \omega(t_{i-1}, t_{i+1}) + \sum_{i:\text{even}} \omega(t_{i-1}, t_{i+1}) \leqq 2\omega(s,t).$$

Therefore there exists such i that satisfies the desired inequality. □

Proposition A.2 (Kolmogorov's continuity theorem)**.** Let X be a process on Δ_T and assume that there exists $p \geqq 1, c, \epsilon > 0$ such that

$$||X_{st}||_p \leqq c|t-s|^{\epsilon+2/p}, \quad (s,t) \in \Delta_T$$

Then there exists a modification $\hat{X}$ of X such that for all $\gamma \in [0, \epsilon)$,

$$\left\| \sup_{(s,t)\in\Delta_T} \frac{|\hat{X}_{st}|}{|t-s|^\gamma} \right\|_p \leqq \frac{2c}{2^{-\gamma} - 2^{-\epsilon}} (6\sqrt{2})^{\epsilon+2/p}.$$

Proof. Let $D_m := \Delta_T \cap 2^{-m}\mathbb{Z}^2$, and $D := \cup_m D_m$. Let $\Delta_m := \{(s,t) \in D_m^2; |s-t| \leqq 2^{-m}\}$, then we have $\#\Delta_m \leqq (2^m+1)^2 3^2 \leqq 2^{2m}6^2$. Let

$$K_m := \sup_{(s,t)\in\Delta_m} |X_{st}|,$$

then by assumption, we have

$$\mathsf{E}\left[K_m^p\right] \leqq \sum_{(s,t)\in\Delta_m} \mathsf{E}\left[|X_{st}|^p\right] \leqq 2^{2m}6^2c^p(\sqrt{2}2^{-m})^{2+p\epsilon} = c^p 6^2 2^{(2+p\epsilon)/2} 2^{-mp\epsilon}.$$

Then for all $s, t \in D$, there exists a sequence $\{s_n\}$, $\{t_n\}$ such that

- $s_n \in D_n$, $s_n \leqq s_{n+1}$, and $s_n \longrightarrow s$ (same as t_n).
- $0 < t_n - s_n \leqq 2^{-n}$.

Therefore, for all $(s,t) \in D$ with $|t-s| \leqq 2^{-m}$, we conclude

$$|X_{st}| \leqq \sum_{i=m}^{\infty} \left|X_{s_{i+1}t_{i+1}} - X_{s_it_i}\right| + |X_{s_mt_m}| \leqq 3\sum_{i=m}^{\infty} \left|X_{s_it_i}\right| \leqq 3\sum_{i=m}^{\infty} |K_i|.$$

Let

$$M_\gamma := \sup_{(s,t)\in D\cap\Delta_T} \frac{|X_{st}|}{|t-s|^\gamma}.$$

Since for all $(s,t) \in D \cap \Delta_T$, there exists $m \in \mathbb{Z}_+$ such that $2^{-m-1} < |t-s| \leqq 2^{-m}$,

$$M_\gamma \leqq \sup_{m\in\mathbb{Z}_+} \left\{ 2^{(m+1)\gamma} \sup_{|s-t|\leqq 2^{-m}} |X_{st}| \right\} \leqq 3\cdot 2^\gamma \sup_{m\in\mathbb{Z}_+} \left\{ 2^{m\gamma} \sum_{i=m}^{\infty} K_i \right\} \leqq 3\cdot 2^\gamma \sum_{i=0}^{\infty} 2^{i\gamma} K_i,$$

and this implies that for all $\gamma \in [0, \epsilon)$, by Minkowski's inequality,

$$\begin{aligned} ||M_\gamma||_p &\leqq 3 \cdot 2^\gamma \sum_{i=0}^{\infty} 2^{i\gamma} ||K_i||_p \leqq 3 \cdot 2^\gamma c 6^{2/p} 2^{(\epsilon+2/p)/2} \sum_{i=0}^{\infty} 2^{i(\gamma-\epsilon)} \\ &= \frac{3 \cdot 2^\gamma c 6^{2/p} 2^{(\epsilon+2/p)/2}}{1 - 2^{\gamma-\epsilon}}. \end{aligned}$$

This implies that $M_\gamma < \infty$ a.e. and X is uniformly continuous on D. Then there exists a continuous process $\hat{X}$ such that $X = \hat{X}$ on D. By the continuity of $\hat{X}$, we have

$$\sup_{(s,t)\in\Delta_T} \frac{\left|\hat{X}_{st}\right|}{|t-s|^\gamma} = \sup_{(s,t)\in\Delta_T} \frac{|X_{st}|}{|t-s|^\gamma} = M_\gamma,$$

which completes the proof. □

B Proof of Proposition 3.4 (i)

The notation in this section is independent from the other parts of this paper and follows the standard in the rough path theory. Namely, we are going to show the continuity of the solution map $Y = \Phi(X)$ for the RDE

$$Y = \int \sigma_a(Y)\mathrm{d}X, \quad \sigma_a = \sigma(\cdot + a)$$

for $\sigma : \mathbb{R}^e \to \mathrm{Mat}(e, d)$ and $X \in \Omega_{\alpha\text{-Hld}}$, where $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. Therefore Y_{st} in Proposition 3.4 is replaced by X_{st} and $(\bar{S}_{st}, t-s)$ is replaced by Y_{st} with $a = (S_0, 0)$. The proof below originates from [26], where the corresponding result is proved under the framework of geometric rough path. Although X is not a geometric rough path, the argument remains valid, and here we give the details for the readers' convenience. Note that to simplify, we do not use a control function argument and this point is different from [26].

Lemma B.1 ([26] Lemma 1.16)**.** For $p \geqq 1$, $N \in \mathbb{N}$, and $a_1, a_2, \ldots, a_N \geqq 0$, we have

$$(a_1 + \ldots + a_N)^p \leqq N^{p-1}(a_1^p + \ldots + a_N^p).$$

Proof. The claim is equivalent to

$$\left(\frac{a_1 + \ldots + a_N}{N}\right)^p \leqq \frac{a_1^p + \ldots + a_N^p}{N},$$

and so by the convexity of the function $f(x) = x^p$, $x \in [0, \infty)$, we have the desired result. □

Lemma B.2 ([26] Lemma 4.11)**.** Let $W, \hat{W} \in \Omega_{\alpha\text{-Hld}}([0, T], \mathbb{R}^m)$ and $0 = \tau_0 < \tau_1 < \ldots < \tau_N = T$ be a partition of $[0, T]$.

(i) Assume that for all $j = 1, 2 \ldots, N$,

$$|W^{(i)}_{st}| \leqq |t-s|^{i\alpha}, \quad (s,t) \in \Delta_{[\tau_{j-1},\tau_j]},\ i = 1, 2, \tag{B.1}$$

where $\Delta_{[S,S']} := \{(s,t) | S \leqq s < t \leqq S'\}$. Then we have

$$|W^{(i)}_{st}| \leqq N^{i(1-\alpha)}|t-s|^{i\alpha}, \quad (s,t) \in \Delta_T,\ i = 1, 2.$$

(ii) Fix $\epsilon > 0$. Assume that for all $j = 1, 2, \ldots, N$, W and $\hat{W}$ satisfy ($B.1$) and

$$|W^{(i)}_{st} - \hat{W}^{(i)}_{st}| \leqq \epsilon |t-s|^{i\alpha}, \quad (s,t) \in \Delta_{[\tau_{j-1},\tau_j]},\ i = 1, 2.$$

Then we have

$$|W^{(i)}_{st} - \hat{W}^{(i)}_{st}| \leqq \epsilon N^{i(1-\alpha)}|t-s|^{i\alpha}, \quad (s,t) \in \Delta_T,\ i = 1, 2.$$

Proof. The most difficult case is when $s \in [\tau_0, \tau_1]$ and $t \in [\tau_{N-1}, \tau_N]$, so we prove under this condition. Let $t_0 = s$, $t_N = t$, $t_j = \tau_j$ $(1 \leqq j \leqq N-1)$. By Chen's relation, $W^{(1)}_{st} = \sum_{j=1}^{N} W^{(1)}_{t_{j-1}t_j}$, and so by Lemma B.1 (take $p = \frac{1}{\alpha}$), we have

$$|W^{(1)}_{st}| \leqq \sum_{j=1}^{N} |t_j - t_{j-1}|^{\alpha} \leqq N^{1-\alpha}|t-s|^{\alpha}.$$

By the same argument as above, we have the inequality of $|W^{(1)}_{st} - \hat{W}^{(1)}_{st}|$. By Chen's relation again, we have $W^{(2)}_{st} = \sum_{j=1}^{N} W^{(2)}_{t_{j-1}t_j} + \sum_{1 \leqq k < j \leqq N} W^{(1)}_{t_{k-1}t_k} \otimes W^{(1)}_{t_{j-1}t_j}$, and so by Lemma B.1, we have

$$\begin{aligned}
|W^{(2)}_{st}| &\leqq \sum_{j=1}^{N} |t_j - t_{j-1}|^{2\alpha} + \sum_{1 \leqq k < j \leqq N} |t_k - t_{k-1}|^{\alpha} |t_j - t_{j-1}|^{\alpha} \\
&\leqq \left\{ \sum_{j=1}^{N} |t_j - t_{j-1}|^{\alpha} \right\}^2 \leqq N^{2(1-\alpha)}|t-s|^{2\alpha}
\end{aligned}$$

By the same argument as above, we have the inequality of $|W^{(2)}_{st} - \hat{W}^{(2)}_{st}|$. □

Proof of Proposition 3.4. Fix $X \in \Omega_{\alpha\text{-Hld}}([0,T], \mathbb{R}^d)$ and let

$$\pi_1 : \Omega_{\alpha\text{-Hld}}([0,T], \mathbb{R}^{d+e}) \to \Omega_{\alpha\text{-Hld}}([0,T], \mathbb{R}^d)$$

$$\pi_2 : \Omega_{\alpha\text{-Hld}}([0,T], \mathbb{R}^{d+e}) \to \Omega_{\alpha\text{-Hld}}([0,T], \mathbb{R}^e)$$

be the projection map on the rough path space $\Omega_{\alpha\text{-Hld}}([0,T],\mathbb{R}^{d+e})$. For $Z \in \Omega_{\alpha\text{-Hld}}([0,T],\mathbb{R}^{d+e})$ satisfying with $\pi_1 Z = X$, we define $Y = \pi_2 Z$, $I[X,Y]$ and $I[Y,X]$ as follows;

$$Z^{(1)}_{st} = \begin{pmatrix} X^{(1)}_{st} \\ Y^{(1)}_{st} \end{pmatrix}, \quad Z^{(2)}_{st} = \begin{pmatrix} X^{(2)}_{st} & I[X,Y]_{st} \\ I[Y,X]_{st} & Y^{(2)}_{st} \end{pmatrix}.$$

For $R > 0$ and $[S,S'] \subset [0,T]$, we define

$$B_{R,[S,S']} := \left\{ Z \in \Omega_{\alpha\text{-Hld}}([S,S'],\mathbb{R}^{d+e}) \,\middle|\, \begin{array}{l} \pi_1 Z = X|_{\Delta_{[S,S']}}, \\ \text{and Z has the following inequalities (B.2)} \end{array} \right\},$$

$$|X^{(i)}_{st}| \leqq |t-s|^{i\alpha}, |Y^{(i)}_{st}| \leqq R^i |t-s|^{i\alpha}, \tag{B.2}$$
$$|I[X,Y]_{st}| \vee |I[Y,X]_{st}| \leqq R|t-s|^{2\alpha}, \quad (s,t) \in \Delta_{[S,S']} \quad i = 1,2.$$

Take a metric function on $B_{R,[S,S']}$ as

$$\begin{aligned} \tilde{d}(Z,\hat{Z}) := {} & R^{-1}||Y^{(1)} - \hat{Y}^{(1)}||_{\alpha\text{-Hld},[S,S']} \vee R^{-2}||Y^{(2)} - \hat{Y}^{(2)}||_{2\alpha\text{-Hld},[S,S']} \\ & \vee R^{-1}||I[X,Y]_{st} - I[X,\hat{Y}]_{st}||_{2\alpha\text{-Hld},[S,S']} \\ & \vee R^{-1}||I[Y,X]_{st} - I[\hat{Y},X]_{st}||_{2\alpha\text{-Hld},[S,S']}, \end{aligned}$$

then $B_{R,[S,S']}$ is a complete metric space in $\tilde{d}$. Let $F : \Omega_{\alpha\text{-Hld}}([0,T],\mathbb{R}^{d+e}) \to \Omega_{\alpha\text{-Hld}}([0,T],\mathbb{R}^{d+e})$ as

$$F(Z) := \int \tilde{\sigma}_a(Z)\mathrm{d}Z, \quad Z \in \Omega_{\alpha\text{-Hld}}$$

where $\tilde{\sigma}_a(z)\{z'\} := \begin{pmatrix} \mathrm{Id} & 0 \\ \sigma_a(y) & 0 \end{pmatrix}\{z'\}$, $z = \begin{pmatrix} x \\ y \end{pmatrix}$, $z' \in \mathbb{R}^{d+e}$ and Id is the identity matrix. Note that by assumption of Proposition, $\sigma : \mathbb{R}^e \to \mathrm{Mat}(e,d)$ is in C^3_b.

(Claim 1) RDE;

$$Y_t = \int_0^t \sigma_a(Y_u)\mathrm{d}X_u, \tag{B.3}$$

has a solution on some subinterval $[0,T_1]$.

Let $K := ||\sigma||_{C^3_b}$ and fix $R > K$. First we prove that there exists $S_1 \in (0,T]$ such that if $0 < S \leqq S_1$, then $F(B_{R,[0,S]}) \subset B_{R,[0,S]}$.

Fix $0 < r < 1$ and $R > K$. Take S such that $S^\alpha \leqq rR^{-1}$. Let c be a positive constant only depending on α. Let also $J^{(1)} := \tilde{\sigma}_a(z_s)Z^{(1)}_{st} +$

$\nabla\tilde{\sigma}_a(z_s)Z^{(2)}_{st}$, $J^{(2)}_{st} := \tilde{\sigma}_a(z_s) \otimes \tilde{\sigma}_a(z_s)Z^{(2)}_{st}$ and $J := (J^{(1)}, J^{(2)})$, where $z_s := Z^{(1)}_{0s}$. By Chen's relation and Taylor expansion, we have

$$J^{(1)}_{st} = \begin{pmatrix} X^{(1)}_{st} \\ \sigma_a(y_s)X^{(1)}_{st} + \nabla\sigma_a(y_s)\{I[Y,X]_{st}\} \end{pmatrix},$$

$$J^{(2)}_{st} = \begin{pmatrix} X^{(2)}_{st} & \mathrm{Id} \otimes \sigma_a(y_s)\{X^{(2)}_{st}\} \\ \sigma_a(y_s) \otimes \mathrm{Id}\{X^{(2)}_{st}\} & \sigma_a(y_s) \otimes \sigma_a(y_s)\{X^{(2)}_{st}\} \end{pmatrix},$$

where $y_s = Y^{(1)}_{0s}$. By the condition (B.2), for $Z \in B_{R,[0,S]}$ and $(s,t) \in \Delta_S$, we have

$$\begin{aligned} |\pi_2\{J^{(1)}_{st}\}| &\leqq |\sigma_a(y_s)X^{(1)}_{st}| + |\nabla\sigma_a(y_s)\{I[Y,X]_{st}\}| \\ &\leqq K|t-s|^{\alpha} + KR|t-s|^{2\alpha} \\ &\leqq (1+r)K|t-s|^{\alpha}. \end{aligned}$$

By the same argument as before, we have

$$|\pi_1 \otimes \pi_2\{J^{(2)}_{st}\}| = |\mathrm{Id} \otimes \sigma_a(y_s)\{X^{(2)}_{st}\}| \leqq K|t-s|^{2\alpha},$$

$$|\pi_1 \otimes \pi_2\{J^{(2)}_{st}\}| = |\sigma_a(y_s) \otimes \mathrm{Id}\{X^{(2)}_{st}\}| \leqq K|t-s|^{2\alpha},$$

$$|\pi_1 \otimes \pi_2\{J^{(2)}_{st}\}| = |\sigma_a(y_s) \otimes \sigma_a(y_s)\{X^{(2)}_{st}\}| \leqq K^2|t-s|^{2\alpha}.$$

By Chen's relation and Taylor expansion, we have

$$\begin{aligned} &\pi_2\{J^{(1)}_{su} + J^{(1)}_{ut} - J^{(1)}_{st}\} \\ &= \left(\int_0^1 (1-\theta)\nabla^2\sigma_a(y_s + \theta Y^{(1)}_{st})\mathrm{d}\theta\right)\left\{Y^{(1)}_{su} \otimes Y^{(1)}_{su} \otimes X^{(1)}_{ut}\right\} \\ &\quad + \left(\int_0^1 \nabla^2\sigma_a(y_s + \theta Y^{(1)}_{st})\mathrm{d}\theta\right)\left\{Y^{(1)}_{su} \otimes I[Y,X]_{ut}\right\}, \end{aligned} \tag{B.4}$$

and by using the condition (B.2), we have

$$\left|\pi_2\{J^{(1)}_{su} + J^{(1)}_{ut} - J^{(1)}_{st}\}\right| \leqq 2KR^2|t-s|^{3\alpha}.$$

Therefore, we have

$$\left|\pi_2\{F(Z)^{(1)}_{st} - J^{(1)}_{st}\}\right| \leqq 2^{3\alpha}\zeta(3\alpha)2KR^2|t-s|^{3\alpha},$$

and we conclude

$$\begin{aligned} |\pi_2\{F(Z)^{(1)}_{st}\}| &\leqq \left\{K + KRS^{\alpha} + 2^{3\alpha}\zeta(3\alpha)2KR^2S^{2\alpha}\right\}|t-s|^{\alpha} \\ &\leqq (1+cr)K|t-s|^{\alpha}, \quad (s,t) \in \Delta_S. \end{aligned}$$

Next we calculate the second level of $F(Z)$. Same argument as the first level path of J, by Taylor expansion and Chen's relation, for $(k,l) = (2,1),(1,2),(2,2)$, we have

$$|\pi_k \otimes \pi_l\{F(Z)^{(2)}_{st}\}| \leqq (1+cr)K^{k+l-2}|t-s|^{2\alpha}, \quad (s,t) \in \Delta_S.$$

Take r small enough, we can check that for $Z \in B_{R,[0,S]}$, $F(Z)$ satisfies the condition (B.2). Therefore we have $F(B_{R,[0,S]}) \subset B_{R,[0,S]}$.

Next we prove that there exists $T_1 \in (0, S_1]$ such that if $0 < S \leqq T_1$ and $Z, \hat{Z} \in B_{R,[0,S]}$, then

$$\tilde{d}(F(Z), F(\hat{Z})) \leqq \frac{1}{2}\tilde{d}(Z, \hat{Z}).$$

Take $\delta = \tilde{d}(Z, \hat{Z})$ and $r \in (0,1)$ small enough. By the definition of $\tilde{d}$, we have

$$|I[X,Y]_{st} - I[X,\hat{Y}]_{st}| \vee |I[Y,X]_{st} - I[\hat{Y},X]_{st}| \leqq R\delta|t-s|^{2\alpha}$$

$$|Y^{(i)}_{st} - \hat{Y}^{(i)}_{st}| \leqq R^i\delta|t-s|^{i\alpha}, (s,t) \in \Delta_S, i = 1,2.$$

First we calculate the first level path of J. For $(s,t) \in \Delta_S$ (note that $S^\alpha \leqq rR^{-1}$),

$$\begin{aligned}
|\pi_2\{J^{(1)}_{st} - \hat{J}^{(1)}_{st}\}| &\leqq |\sigma_a(y_s) - \sigma_a(\hat{y}_s)||X^{(1)}_{st}| + |\nabla\sigma_a(y_s)||I[Y,X]_{st} - I[\hat{Y},X]_{st}| \\
&\quad + |\nabla\sigma_a(y_s) - \nabla\sigma_a(\hat{y}_s)||I[\hat{Y},X]_{st}| \\
&\leqq KR\delta S^\alpha|t-s|^\alpha + KR\delta|t-s|^{2\alpha} + KR^2\delta S^\alpha|t-s|^{2\alpha} \\
&\leqq \delta\{2r + r^2\}K|t-s|^\alpha \\
&\leqq 3\delta rK|t-s|^\alpha.
\end{aligned}$$

Next, we calculate

$$J^{(1)}_{su} + J^{(1)}_{ut} - J^{(1)}_{st} - \{\hat{J}^{(1)}_{su} + \hat{J}^{(1)}_{ut} - \hat{J}^{(1)}_{st}\}.$$

By (B.4) and the boundedness of σ_a, we have

$$\begin{aligned}
&\left|\pi_2\left\{J^{(1)}_{su}+J^{(1)}_{ut}-J^{(1)}_{st}-(\hat{J}^{(1)}_{su}+\hat{J}^{(1)}_{ut}-\hat{J}^{(1)}_{st})\right\}\right| \\
&\leqq \int_0^1 (1-\theta)\Big|\nabla^2\sigma_a(y_s+\theta Y^{(1)}_{su})(Y^{(1)}_{su}\otimes Y^{(1)}_{su}\otimes X^{(1)}_{ut}) \\
&\qquad\qquad -\nabla^2\sigma_a(\hat{y}_s+\theta\hat{Y}^{(1)}_{su})(\hat{Y}^{(1)}_{su}\otimes\hat{Y}^{(1)}_{su}\otimes X^{(1)}_{ut})\Big|\mathrm{d}\theta \\
&\quad+\int_0^1\Big|\nabla^2\sigma_a(y_s+\theta Y^{(1)}_{su})(Y^{(1)}_{su}\otimes I[Y,X]_{ut}) \\
&\qquad\qquad -\nabla^2\sigma_a(\hat{y}_s+\theta\hat{Y}^{(1)}_{su})(\hat{Y}^{(1)}_{su}\otimes I[\hat{Y},X]_{ut})\Big|\mathrm{d}\theta \\
&\leqq \delta c\{KR^3S^\alpha+KR^2\}|t-s|^{3\alpha} \\
&\leqq \delta cKR^2|t-s|^{3\alpha}.
\end{aligned}$$

This implies

$$\left|\pi_2\left\{F(Z)^{(1)}_{st}-J^{(1)}_{st}-(F(\hat{Z})^{(1)}_{st}-\hat{J}^{(1)}_{st})\right\}\right| \leqq \delta cKR^2|t-s|^{3\alpha},$$

and so

$$\begin{aligned}
\left|\pi_2\left\{F(Z)^{(1)}_{st}-F(\hat{Z})^{(1)}_{st}\right\}\right| &\leqq \left|\pi_2\left\{F(Z)^{(1)}_{st}-J^{(1)}_{st}-(F(\hat{Z})^{(1)}_{st}-\hat{J}^{(1)}_{st})\right\}\right| \\
&\quad+|\pi_2\{J^{(1)}_{st}-\hat{J}^{(1)}_{st}\}| \\
&\leqq \delta\{3r+cR^2S^{2\alpha}\}K|t-s|^\alpha \\
&\leqq \delta crK|t-s|^\alpha \\
&\leqq crR|t-s|^\alpha.
\end{aligned}$$

Same argument as the first level path of $F(Z)$, for $(k,l)=(2,1),(1,2),(2,2)$, we have

$$\left|\pi_k\otimes\pi_l\left(F(Z)^{(2)}_{st}-F(\hat{Z})^{(2)}_{st}\right)\right| \leqq \delta crR^{k+l-2}|t-s|^{2\alpha}, \quad (s,t)\in\Delta_S.$$

Therefore by taking r small enough, we have $\tilde{d}(F(Z),F(\hat{Z}))\leqq\frac{1}{2}\tilde{d}(Z,\hat{Z})$. Therefore we conclude that there exists a solution of (B.3) on subinterval $[0,T_1]$.

(Claim 2) RDE solution has the unique time-global solution.

Let $0\leqq\tau_0<\tau_1<\tau_2\leqq T$ and

$$W(j):=(1,W(j)^{(1)},W(j)^{(2)})\in C(\Delta_{[\tau_{j-1},\tau_j]},\mathbb{R}\oplus\mathbb{R}^m\oplus\mathbb{R}^{m^2})$$

satisfies Chen's relation on $[\tau_{j-1}, \tau_j]$ $(j = 1, 2)$. Then we define W as a concatenation between $W(1)$ and $W(2)$ as follows; for $(s,t) \in [\tau_0, \tau_1] \times [\tau_1, \tau_2]$,

$$W^{(1)}_{st} := W(1)^{(1)}_{s\tau_1} + W(2)^{(1)}_{\tau_1 t},$$

$$W^{(2)}_{st} := W(1)^{(2)}_{s\tau_1} + W(2)^{(2)}_{\tau_1 t} + W(1)^{(1)}_{s\tau_1} \otimes W(2)^{(1)}_{\tau_1 t}.$$

Then if $W(j)$ is an α-Hölder rough path on $\Delta_{[\tau_{j-1},\tau_j]}$, then W is also an α-Hölder rough path on $\Delta_{[\tau_0,\tau_2]}$.

Let $f : \mathbb{R}^m \to \mathrm{Mat}(e,m)$ be C^3_b and $f_a = f(a + \cdot)$, $a \in \mathbb{R}^e$. For $W \in \Omega_{\alpha\text{-Hld}}([\tau_0, \tau_2], \mathbb{R}^m)$, let $W(j)$ be the restriction of $\Delta_{[\tau_{j-1},\tau_j]}$ and $V(1) := \int f_a(W(1))\mathrm{d}W(1)$ and $V(2) := \int f_{\hat{a}}(W(2))\mathrm{d}W(2)$, where $\hat{a} := a + V(1)^{(1)}_{\tau_0\tau_1}$. Then we have $V(j) \in \Omega_{\alpha\text{-Hld}}([\tau_{j-1}, \tau_j], \mathbb{R}^e)$ and we can prove that the concatenation between $V(1)$ and $V(2)$ is equal to $\int f_a(W)\mathrm{d}W$.

Let $U := (rR^{-1})^{\frac{1}{\alpha}}$ and take a partition $\{0 = T_0 < T_1 < \ldots < T_N = T\}$ of $[0, T]$ as follows;

$$U = |T_1| = |T_2 - T_1| = \ldots \geqq |T_N - T_{N-1}|.$$

Note that $(N-1)U = \sum_{i=1}^{N-1} |T_i - T_{i-1}| \leqq T$. By (Claim 1), we have a solution Z on $[0, T_1]$. On $[T_1, T_2]$, by changing an initial value a to $a + Y^{(1)}_{0T_1}$, we have a solution on $[T_1, T_2]$. By concatenating these rough paths by the above way, we have a rough path on $[0, T_2]$, and by the remark of the concatenating of rough path integral, this is a solution on $[0, T_2]$. By the same argument, we have a solution on subinterval $[T_{k-1}, T_k]$, and by concatenating these solutions, we have a time-global solution on $[0, T]$. Since i-th level of a solution on $[T_{k-1}, T_k]$ is $i\alpha-$Hölder continuous, so by Lemma B.2, we have

$$|Y^{(1)}_{st}| \leqq RN^{1-\alpha}|t-s|^{\alpha} \leqq R(1 + TU^{-1})^{1-\alpha}|t-s|^{\alpha}, \quad (s,t) \in \Delta_T$$

and for $(k, l) = (2, 1), (1, 2), (2, 2)$,

$$|\pi_k \otimes \pi_l\{Z^{(2)}_{st}\}| \leqq R^{k+l-2}(1 + TU^{-1})^{2(1-\alpha)}|t-s|^{2\alpha}, \quad (s,t) \in \Delta_T$$

and hence $Z^{(i)}$ is $i\alpha$-Hölder continuous $(i = 1, 2)$. To prove the uniqueness of the time-global solition, let Z and $\tilde{Z}$ be time-global solutions with respect to (a, X). By taking R large enough, we have $Z, \tilde{Z} \in B_{R,[0,S]}$. Therefore a solution of (B.3) is unique on $[0, T]$.

(Claim3) The solution map is locally Lipschitz continuous.

Let $Z, \hat{Z}$ be the unique solution with respect to $(X, a), (\hat{X}, \hat{a})$. Let

$\epsilon' := |a - \hat{a}|$ and $\epsilon := \max_{i=1,2} ||X^{(i)} - \hat{X}^{(i)}||_{i\alpha\text{-Hld}}$. For $R > 0$ and $[S, S'] \subset [0, T]$, let

$$\begin{aligned}\bar{d}(Z, \hat{Z}) := &R^{-1}||Y^{(1)} - \hat{Y}^{(1)}||_{\alpha\text{-Hld},[S,S']} \vee R^{-2}||Y^{(2)} - \hat{Y}^{(2)}||_{2\alpha\text{-Hld},[S,S']} \\ &\vee R^{-1}||I[X, Y]_{st} - I[\hat{X}, \hat{Y}]_{st}||_{\alpha\text{-Hld},[S,S']} \\ &\vee R^{-1}||I[Y, X]_{st} - I[\hat{Y}, \hat{X}]_{st}||_{\alpha\text{-Hld},[S,S']}.\end{aligned}$$

First we prove that there exists $U_1 \in (0, T_1]$ and $C > 0$ such that if $0 < U \leqq U_1$, then

$$|Y^{(1)}_{0U} - \hat{Y}^{(1)}_{0U}| \leqq \epsilon + \epsilon',$$

and also on subinterval $[0, U]$, the following inequality is true;

$$\bar{d}(Z, \hat{Z}) \leqq C(\epsilon + \epsilon').$$

Take $R > K$ and let $\delta = \bar{d}(Z, \hat{Z})$. Then

$$|I[X, Y]_{st} - I[\hat{X}, \hat{Y}]_{st}| \vee |I[Y, X]_{st} - I[\hat{Y}, \hat{X}]_{st}| \leqq R\delta|t - s|^{2\alpha},$$

$$|Y^{(i)}_{st} - \hat{Y}^{(i)}_{st}| \leqq R^i\delta|t - s|^{i\alpha}, \quad (s, t) \in \Delta_U, \quad i = 1, 2.$$

Note that for $j = 0, 1, 2$, $(s, t) \in \Delta_U$, and $\theta \in [0, 1]$, we have

$$\begin{aligned}&|\nabla^j\sigma_a(y_s + \theta Y^{(1)}_{su}) - \nabla^j\sigma_{\hat{a}}(\hat{y}_s + \theta\hat{Y}^{(1)}_{su})| \\ &\leqq ||\nabla^{j+1}\sigma||_\infty(|a - \hat{a}| + \sup_{0\leqq s\leqq U} |y_s - \hat{y}_s|) \\ &\leqq K(\epsilon' + R\delta U^\alpha).\end{aligned}$$

Then for all $(s, t) \in \Delta_U$,

$$\begin{aligned}|\pi_2\{J^{(1)}_{st} - \hat{J}^{(1)}_{st}\}| &\leqq |\sigma_a(y_s)X^{(1)}_{st} - \sigma_{\hat{a}}(\hat{y}_s)\hat{X}^{(1)}_{st}| \\ &\quad + |\nabla\sigma_a(y_s)I[Y, X]_{st} - \nabla\sigma_{\hat{a}}(\hat{y}_s)I[\hat{Y}, \hat{X}]_{st}| \\ &\leqq |\sigma_a(y_s)||X^{(1)}_{st} - \hat{X}^{(1)}_{st}| + |\sigma_a(y_s) - \sigma_{\hat{a}}(\hat{y}_s)||\hat{X}^{(1)}_{st}| \\ &\quad + |\nabla\sigma_a(y_s)||I[Y, X]_{st} - I[\hat{Y}, \hat{X}]_{st}| \\ &\quad + |\nabla\sigma_a(y_s) - \nabla\sigma_{\hat{a}}(\hat{y}_s)||I[\hat{Y}, \hat{X}]_{st}| \\ &\leqq \{K\epsilon + K(\epsilon' + R\delta U^\alpha)\}|t - s|^\alpha \\ &\quad + \{K\delta + K(\epsilon' + R\delta U^\alpha)\}R|t - s|^{2\alpha} \\ &\leqq \{\epsilon + 2\epsilon' + 3r\delta\}R|t - s|^\alpha.\end{aligned}$$

Also for $0 \leqq s \leqq u \leqq t \leqq U$, by (B.4), we have

$$\begin{aligned}
&|\pi_2\{J^{(1)}_{su} + J^{(1)}_{ut} - J^{(1)}_{st} - (\hat{J}^{(1)}_{su} + \hat{J}^{(1)}_{ut} - \hat{J}^{(1)}_{st})\}| \\
&\leqq \int_0^1 (1-\theta)|\nabla^2\sigma_a(y_s + \theta Y^{(1)}_{su})(Y^{(1)}_{su} \otimes Y^{(1)}_{su} \otimes X^{(1)}_{st}) \\
&\qquad\qquad - \nabla^2\sigma_{\hat{a}}(\hat{y}_s + \theta\hat{Y}^{(1)}_{su})(\hat{Y}^{(1)}_{su} \otimes \hat{Y}^{(1)}_{su} \otimes \hat{X}^{(1)}_{st})|\mathrm{d}\theta \\
&\quad + \int_0^1 |\nabla^2\sigma_a(y_s + \theta Y^{(1)}_{su})(Y^{(1)}_{su} \otimes I[Y, X]_{st}) \\
&\qquad\qquad - \nabla^2\sigma_{\hat{a}}(\hat{y}_s + \theta\hat{Y}^{(1)}_{su})(\hat{Y}^{(1)}_{su} \otimes I[\hat{Y}, \hat{X}]_{st})|\mathrm{d}\theta \\
&\leqq (\epsilon + 2\epsilon' + 6\delta)KR^2|t-s|^{3\alpha},
\end{aligned}$$

and so we have

$$|\pi_2(F(Z)^{(1)}_{st} - J^{(1)}_{st} - (F(\hat{Z})^{(1)}_{st} - \hat{J}^{(1)}_{st}))| \leqq c(\epsilon + 2\epsilon' + 6\delta)KR^2|t-s|^{3\alpha}.$$

Therefore we have

$$\begin{aligned}
|\pi_2\{F(Z)^{(1)}_{st} - F(\hat{Z})^{(1)}_{st}\}| &\leqq |\pi_2(F(Z)^{(1)}_{st} - J^{(1)}_{st} - (F(\hat{Z})^{(1)}_{st} - \hat{J}^{(1)}_{st}))| \\
&\quad + |\pi_2\{J^{(1)}_{st} - \hat{J}^{(1)}_{st}\}| \\
&\leqq c(\epsilon + \epsilon' + r\delta)R|t-s|^{\alpha}. \qquad \text{(B.5)}
\end{aligned}$$

By the same argument as the first level of path $F(Z)$, for $(k,l) = (2,1),(1,2),(2,2)$, we have

$$|\pi_k \otimes \pi_l(F(Z)^{(2)}_{st} - F(\hat{Z})^{(2)}_{st})| \leqq c(\epsilon + \epsilon' + r\delta)R^{k+l-2}|t-s|^{2\alpha} \quad (s,t) \in \Delta_U. \qquad \text{(B.6)}$$

Hence by (B.5) and (B.6), we have

$$\bar{d}(Z, \hat{Z}) = \delta \leqq c(\epsilon + \epsilon' + r\delta).$$

Fix $c > 0$ and take $r > 0$ small enough ($rc \leqq \frac{1}{3}$). Then we have $U^\alpha \leqq rR^{-1} \leqq (3Rc)^{-1}$ and so

$$|Y^{(1)}_{0U} - \hat{Y}^{(1)}_{0U}| \leqq \frac{3}{2}c(\epsilon + \epsilon')RU^\alpha \leqq \epsilon + \epsilon',$$

and we have the desired result.

Next take a partition $0 = U_0 < U_1 < \ldots < U_N = T$ of $[0,T]$ as follows;

$$(rR^{-1})^{1/\alpha} = U_1 = |U_2 - U_1| = \ldots = |U_N - U_{N-1}|.$$

Note that $N - 1 \leqq T(rR^{-1})^{-1/\alpha}$. For each subinterval $[U_i, U_{i+1}]$, take the unique solution Z and $\hat{Z}$ with respect to an initial value $a + Y^{(1)}_{0U_i}$ and $\hat{a} + \hat{Y}^{(1)}_{0U_i}$ respectively. Then we have

$$\left|a + Y^{(1)}_{0U_i} - (\hat{a} + \hat{Y}^{(1)}_{0U_i})\right| \leqq -\epsilon + 2^i(\epsilon + \epsilon'),$$

and hence on each subinterval $[U_i, U_{i+1}]$,

$$\bar{d}(Z, \hat{Z}) \leqq C2^i(\epsilon + \epsilon') \leqq C2^N(\epsilon + \epsilon').$$

Therefore by Lemma B.2, we have

$$|Y_{st}^{(1)} - \hat{Y}_{st}^{(1)}| \leqq RC2^N(\epsilon + \epsilon')N^{1-\alpha}|t-s|^\alpha, \quad (s,t) \in \Delta_T$$

and for $(k,l) = (2,1), (1,2), (2,2)$, we have

$$|\pi_k \otimes \pi_l(Z_{st}^{(2)} - \hat{Z}_{st}^{(2)})| \leqq (R)^{k+l-2}C2^N(\epsilon + \epsilon')N^{1-\alpha}|t-s|^{2\alpha}, \quad (s,t) \in \Delta_T.$$

Therefore since $N \leqq 1 + \frac{K^{1/\alpha}T}{r^{1/\alpha}}$, we have the claim.

□

C Proof of Theorem 3.8

Proof. To shorten, let $\sigma := \sigma(S_0, 0)$. Since $\tilde{S}_1^\epsilon \sim \tilde{S}_\epsilon^1$, by Theorem 3.6, $t^{H-1/2}\bar{S}_t$ satisfies the LDP with speed t^{-2H} with good rate function

$$J(\tilde{s}) := \inf\left\{ I^\#(u,v) \,\middle|\, u,v \in C_{[0,T]}, v \in \mathrm{BV},\ \tilde{s} = \left(\sigma \int f(\hat{\mathbb{L}}(u,v))\mathrm{d}\mathbb{L}(u,v)\right)_{01}^{(1)} \right\}.$$

Let $\mathcal{H}$ be the Cameron-Martin space. We can calculate that for $g = (h^1, h^2) \in \mathcal{H} \times \mathcal{H}$,

$$(\mathcal{I}^\Psi g)_t = \begin{pmatrix} \int_0^t k_H(t-r)\dot{h}_r^1 \mathrm{d}r \\ \rho h_t^1 + \sqrt{1-\rho^2}h_t^2 \end{pmatrix} = \begin{pmatrix} u_t \\ v_t \end{pmatrix},$$

and

$$\frac{1}{2}||\mathcal{I}^\Psi g||_{\mathcal{H}^\Psi}^2 = \frac{1}{2}\int_0^1 \left\{|\dot{h}_r^1|^2 + |\dot{h}_r^2|^2\right\} \mathrm{d}r.$$

Then

$$\begin{aligned}
\tilde{s} &= \left(\sigma \int f(\hat{\mathbb{L}}(u,v))\mathbb{L}(u,v)\right)_{01}^{(1)} \\
&= \sigma \int_0^1 f\left(\int_0^t k_H(t-r)\dot{h}_r^1 \mathrm{d}r\right) \mathrm{d}\left(\rho h_t^1 + \sqrt{1-\rho^2}h_t^2\right) \\
&= \rho\sigma \int_0^1 f\left(\int_0^t k_H(t-r)\dot{h}_r^1 \mathrm{d}r\right) \mathrm{d}h_t^1 \\
&\quad + \sqrt{1-\rho^2}\sigma \int_0^1 f\left(\int_0^t k_H(t-r)\dot{h}_r^1 \mathrm{d}r\right) \mathrm{d}h_t^2,
\end{aligned}$$

and so

$$\frac{\tilde{s} - \rho\sigma \int_0^1 f\left(\int_0^t k_H(t-r)\dot{h}^1_r \mathrm{d}r\right) \mathrm{d}h^1_t}{\sqrt{1-\rho^2}} = \sigma \int_0^1 f\left(\int_0^t k_H(t-r)\dot{h}^1_r \mathrm{d}r\right) \mathrm{d}h^2_t. \tag{C.1}$$

Fix $h_1 \in \mathcal{H}$, and minimize $\frac{1}{2}||\mathcal{I}^{\Psi} g||^2_{\mathcal{H}^{\Psi}}$ with respect to $h_2 \in \mathcal{H}$ under the condition (C.1). Let $\tilde{h}$ be the minimizer. Take $\epsilon > 0$ and $\hat{h} \in \mathcal{H}$, and consider $\tilde{h} + \epsilon\hat{h}$. Since $\tilde{h}$ satisfies the condition (C.1),

$$\int_0^1 f\left(\int_0^t k_H(t-r)\dot{h}^1_r \mathrm{d}r\right) \mathrm{d}\hat{h}_t = 0. \tag{C.2}$$

Since $\tilde{h}$ is the minimizer, we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \frac{1}{2}\int_0^1 (\dot{\tilde{h}}_r + \epsilon\dot{\hat{h}}_r)^2 \mathrm{d}r = 0,$$

we have

$$\int_0^1 \dot{\tilde{h}}_r \dot{\hat{h}}_r \mathrm{d}r = 0$$

for any $\hat{h}$ with (C.2). Therefore there exists $c \in \mathbb{R}$ such that

$$\dot{\tilde{h}} = cf\left(\int_0^{\cdot} k_H(\cdot - r)\dot{h}^1_r \mathrm{d}r\right).$$

Hence

$$\frac{\tilde{s} - \rho\sigma \int_0^1 f\left(\int_0^t k_H(t-r)\dot{h}^1_r \mathrm{d}r\right) \mathrm{d}h^1_t}{\sqrt{1-\rho^2}} = c\sigma \int_0^1 f^2\left(\int_0^t k_H(t-r)\dot{h}^1_r \mathrm{d}r\right) \mathrm{d}t.$$

and we calculate that

$$\begin{aligned} J(\tilde{s}) &= I^{\#}(u,v) = \frac{1}{2}||\mathcal{I}^{\Psi} g||^2_{\mathcal{H}^{\Psi}} \\ &= \frac{1}{2}\int_0^1 |\dot{h}^1_r|^2 \mathrm{d}s + \frac{\left\{\tilde{s} - \rho\sigma \int_0^1 f\left(\int_0^t k_H(t-r)\dot{h}^1_r \mathrm{d}r\right) \mathrm{d}h^1_t\right\}^2}{2(1-\rho^2)\sigma^2 \int_0^1 f^2(\int_0^t k_H(t-r)\dot{h}^1_r \mathrm{d}r)\mathrm{d}t}, \quad (u,v) = \mathcal{I}^{\Psi} g. \end{aligned}$$

□

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