The VIX volatility index - A very thorough look at it

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Contents

1 Introduction **1**

2 Extreme price events in the VIX and its derivatives **4**

3 The CBOE Volatility Index **5**

3.1 Historical evolution of the VIX index 5

3.2 The VIX Market - How it works 6

3.3 The CBOE VIX formula explained 7

4 Literature Overview **9**

5 Data and replication methodology **11**

5.1 Data 11

5.2 Replication methodology and validation 11

6 Theoretical derivation of the VIX from first principles **14**

6.1 Empirical decomposition of the VIX 16

6.2 Theoretical Approximations in the VIX Calculation 20

6.3 Practical significance of the theoretical approximations in the VIX calculation 21

7 Can market participants influence the VIX formula? **25**

8 The VIX and its sub-optimal market micro structure and market mechanism **33**

9 Conclusion and summary **34**

10 Bibliography **35**

List of Figures **38**

List of Tables **39**

A Appendix **40**

Abstract

The CBOE Volatility Index, known by its ticker symbol VIX, is a popular measure of the stock market's expectation of volatility implied by S&P 500 index options, calculated and published by the Chicago Board Options Exchange (CBOE). It is colloquially referred to as the fear index or the fear gauge. The current VIX index value quotes the expected annualized change in the S&P 500 index over the following 30 days, as computed from options-based theory and current options-market data. Despite its theoretical foundation in option price theory, CBOE's Volatility Index is prone to inadvertent and deliberate errors. We shed light on many claims that have been brought up against the VIX in recent years. Replicating the VIX index by using empirical options data from CBOE we show which one of those claims are justified and lead to meaningful alterations of the VIX. Three main areas can be identified: Eight different alteration possibilities of the current VIX formula, three deficiencies in the theoretical derivation of the VIX and two major issues with the market microstructure of the VIX and its derivatives. First, eight theoretically possible ways of altering the current VIX formula are identified. We will thoroughly investigate those ways both from a theoretical point of view and using empirical data. The two major alteration possibilities are identified and their potential impact on the VIX calculation is shown using actual option quote data from the CBOE. Second, the theoretical derivation behind the VIX formula has three major flaws. We show their importance based on an empirical analysis of the VIX and the underlying options on the S&P 500. Our analysis helps to understand and put into context previous claims that were brought up against the VIX. Third, the market microstructure behind the pricing of VIX derivatives appears to be flawed. Very liquid derivatives are priced off an illiquid auction mechanism of out-of-the-money put and call options. To make matters worse, those derivatives are cash-settled and the settlement value of the VIX, as used for the pricing of its derivatives, is based on a different calculation than the regular VIX index. Finally, we also investigate claims that the settlement values can substantially deviate from the previous-day close. We show that those claims are based on a wide-spread misunderstanding of how the settlement values and the VIX index are calculated and that they are not justified in the way that they are brought up at the moment. Our empirical methodology is based on replicating the VIX based on the underlying quote data. Our conclusion is that the VIX needs a thorough review of both its theoretical foundations, its calculations and the market mechanism behind computing the value of its derivatives. Furthermore, some of the claims brought up against the VIX are more of a theoretical nature and not important, but some might have a dramatic impact on the value of the VIX as well as the pricing of its derivatives. To our knowledge, that is the first comprehensive study of the VIX index and its shortcomings that is both based on theoretical and empirical observations. We are confident that our analysis will lead to new ways of how to make the VIX index safer for investors and market participants as well as less prone to manipulation and errors.

1 Introduction

The VIX is a mathematical calculation which is considered the most important benchmark for volatility on the US stock market. This volatility index, which is published by the CBOE, is calculated using a weighted sum of the mid-quotes of out-of-the-money put and call options, of the S&P 500. The VIX cannot be traded directly, but there is a large amount of derivatives, including options and futures on the index. The calculation is updated by the CBOE every 15 seconds. Despite its solid theoretical foundations in option-price theory, accusations and assertions have been made against the VIX for many years and it is being closely examined by market participants, lawyers and supervisory authorities. The VIX is vulnerable to accidental and intentional errors. In this paper we shed light on the allegations of recent years.

By using options data for the S&P 500 from CBOE as well as the VIX Index we can show which one of those claims are justified and lead to meaningful alterations of the VIX and which ones are not.

Our approach consists of three main pillars: Eight different alteration possibilities of the current VIX value, based on its mathematical formulation, three deficiencies in the theoretical derivation of the VIX and three major issues with the market microstructure of the VIX and its derivatives are described, investigated and analysed.

Starting with possible ways of altering the current VIX formula, We will thoroughly investigate them both from a theoretical and empirical point of view. Two of those eight turn out to be major alteration possibilities and their potential impact on the VIX calculation is shown using actual option quote data from the CBOE. The two possibilities that turn out to be empirically relevant are: Extending the range of options that are included in the calculation of the VIX and directly changing the quotes of relevant options. Relevant options are those S&P 500 options that are used for the VIX at a given point in time.

Second, we theoretically derive the VIX formula as used by CBOE, based on option-pricing theory and stochastic calculus. We point out three major deficiencies of the CBOE calculation versus its theoretical grounding. CBOE has undertaken three major approximations to the theoretically correct formula. We enumerate specific adjustments of the VIX calculations to improve accuracy and reduce susceptibility to errors and manipulation and show, using actual option data, how relevant those approximations are, sheding more light on some of the allegations that were brought up against the VIX in recent years. The approximations show that the VIX systematically underestimates the volatility. Therefore investors should be aware of that fact if they invest in VIX derivatives. This is in addition to any difference between implied and historical volatility, which is well grounded in empirical observations. In normal times, the VIX usually over-estimates actual volatility, but in times of market crashes and crises VIX underestimates actual volatility Kownatzki (2016)

Third, the market microstructure behind the pricing of VIX derivatives appears to be flawed. The VIX calculation itself is based on a wide range of out-of-the-money call and put options which are relatively illiquid. In strong contrast to that, VIX derivatives are very liquid instruments nowadays. That is a major issue inherent in the different markets that cannot be solved easily. Very liquid derivatives are priced off an illiquid auction mechanism of out-of-the-money put and call options. To put it more directly: The prices of VIX futures and options, which have a notional amount outstanding of several billion dollars are derived from the quotes of illiquid S&P 500 options. Furthermore, any mispricings of those derivatives are accentuated by their cash-settlement feature. This feature has the immediate consequence that any mispricings that might occur during the settlement auction are immediately converted into profit and losses for the investors. The last flaw in the VIX calculation is the settlement value itself. It is determined during the auction and deviates from the VIX index in two ways: First, it is mostly based on open prices instead of mid-quotes. Second, only S&P 500 options with 30-days to expiry are used. This has two major consequences: First, when comparing the settlement value to the previous day close, large deviations naturally occur. On the previous day the VIX calculation includes S&P 500 options with 31 days and 24 days to expiry. The series with 24 days to expiry will all disappear on the next day in the settlement calculation. Furthermore, on the settlement day, the VIX calculation (after the auction) will include S&P 500 options with 30 days and 37 days to expiry. We do not say that the CBOE should change that, we rather point out that this leads to major discrepancies between the VIX settlement value and the VIX value on the previous day as well as on the settlement day. We consider those deviations to naturally occur, whereas many academics and practitioners have raised this observation as one indication for manipulation in the VIX Griffin and Shams (2017). In other words, the settlement value calculation is based on a completely different set of options than the VIX calculations. The settlement value for standard VIX derivatives is calculated using only standard SPX options, which are A.M.-settled and expire on the third Friday of each month. The settlement value for weekly VIX derivatives is calculated using only weekly SPX options, which are P.M.-settled and expire on all other Fridays Exchange (2018b)

In other words, the VIX index always includes weekly and monthly options whereas the settlement value is based on only either weekly or monthly options. Clearly, this is a wide-spread misunderstanding within the academic literature, in particular for any kind of analysis that compares the settlement value to the previous-day or same-day VIX index. Our conclusion is that the VIX needs a thorough review of both its theoretical foundations, its calculations and the market mechanism behind computing the value of its derivatives. Furthermore some of the claims brought up against the VIX are more of a theoretical nature and not important, but some might have a dramatic impact on the value of the VIX as well as the pricing of its derivatives.

The remainder of the paper is organised as follows: In Section 3 we show the historical evolution of the VIX from 1987 until 2018, explain how the VIX market and its derivatives work and review its calculation methodology. Furthermore, in Section 4 we give an overview of the related literature on our topics. Section 5 describes the data we have used and the methodology we use to replicate VIX, as well as our validation methods for the replication. In Section 6 we derive the current CBOE calculation method as an adaptation of the previous literature on variance swaps. Section 7 sheds light on eight possible ways of altering the VIX calculation. Two of them turn out to be practically relevant and could be used by market participants to manipulate the index. In Section 8, we look at the market mechanism for VIX derivatives, in particular at the settlement auction for obtaining the reference value of the VIX which is used in settling VIX derivatives. We identify three major flaws of the market mechanism. At the end, in Section 9, we will provide a brief conclusion of our findings.

2 Extreme price events in the VIX and its derivatives

On February 5, 2018, the VIX Index moved the most in a single day in the index's 25-year history. On this day the VIX closed with 37.32 points, an increase of 20.01 points over the previous day, that closed at 17.31. This corresponds to an increase of 115% in one day. The extraordinary move coincided with a steep sell-off in the equity markets with the S&P 500 index falling by 4.1%. This event shocked the financial world and there were accusations of market manipulation. On March 9, 2018, Atlantic Trading USA sued unknown "John Does" in a purported class action, alleging manipulation of the settlement price for VIX futures and options. The Chicago-based trading firm argues that the to-be-identified defendants "caused the monthly final settlement price of expiring VIX contracts to be artificial." They contend further that the defendants did so by "placing manipulative SPX options orders that were intended to cause, and at minimum recklessly caused, artificial VIX contract settlement prices in the expiring contracts." Singer (2018)
On April 18, 2018, shortly before the settlement auction at 8.30 a.m. in Chicago, the VIX increased by 10 percent on Wednesday in a few minutes. This unusual increase happened despite there being no significant increase in volatility in the S&P 500 equity market benchmark, although the VIX was supposed to reflect it

One noteworthy ETN is the VelocityShares Daily Inverse VIX Short Term ETN (ticker: XIV), which gives investors the inverse of the daily return on the S&P 500 VIX Short-Term Futures Index. The Short-Term Futures Index utilizes prices of the next two near-term VIX futures contracts to replicate a position that rolls the nearest month VIX futures to the next month on a daily basis in equal fractional amounts. This results in a constant one-month rolling long position in first and second month VIX futures contracts. When the VIX rose by 116% on February 5, 2018, the short-term futures contracts also rose significantly (by 113%, 87% and 64% for the front-, second- and third-months' contracts, respectively), essentially wiping out investors in XIV. Credit Suisse, XIV's sponsor, announced shortly afterwards that it would redeem the notes at large losses and shut down the product (Franck, 2018).

Various parties have warned of the potential for VIX manipulation. "The VIX has been suspect for at least seven years," former CFTC Commissioner Bart Chilton cautioned in a February 2018 interview on CNBC Belvedere (2018). Chilton commented in response to a question regarding a whistleblower letter that Zuckerman Law sent to the CFTC and SEC on behalf of an unidentified client. The letter alleges manipulation via the posting of bids and offers on SPX options to affect VIX levels Zuckerman (2018).

3 The CBOE Volatility Index

We start by looking at the CBOE Volatility Index. First, in Subsection 3.1, we give a short overview of the historical evolution of the Volatility Index on the U.S. equity market. The complex market mechanism requires a brief explanation of how it works. This is discussed in Subsection 3.2, where the the components of the VIX financial markets are presented in more detail. In Subsection 3.3, we explain the current CBOE methodology for computing the VIX.

3.1 Historical evolution of the VIX index

We provide a time line of some key events in the history of the VIX Index:
In 1987, Brenner and Galai introduced the Sigma Index in an academic paper.
(Brenner and Galai (1989)) wrote, ”Our volatility index, to be named Sigma Index, would be updated frequently and used as the underlying asset for futures and options... A volatility index would play the same role as the market index play for options and futures on the index.”. In 1992, The American Stock Exchange announced that it was conducting a feasibility study for a volatility index. This index was proposed as ”Sigma Index”. ”SI would be an underlying asset for futures and options that investors would use to hedge against the risk of volatility changes in the stock market.” (Whaley (1993)). On January 19, 1993, the Chicago Board Options Exchange (CBOE) introduced the VIX. The VIX was designed to measure the 30 days implied volatility of at-the-money (ATM) S&P 100 Index (OEX) option prices, and was developed by Robert Whaley (Whaley (1993)). 10 years later, in 2003, the CBOE together with Goldman Sachs developed further computational methodologies. The VIX is now no longer based on the S&P 100 (OEX) but on the S&P 500 (SPX). The new VIX is based on the S&P 500 registered Index (SPXSM), the core index for U.S. equities, and estimates expected volatility by averaging the weighted quotes of SPX put and call options over a wide range of strike prices. In 2004, the CBOE began to introduce futures and two years later, in 2006, presented its new product, VIX options. This product is so successful that 10 years later the number of contracts traded has risen to over 800,000 per day. In 2014 the VIX was improved by including the SPX weekly (SPXW) in the calculation. This inclusion in the calculation most precisely reflects the 30 days expected volatility of the S&P 500. Using SPX options with more than 23 days and less than 37 days to expiration ensures that the VIX Index will always reflect an interpolation of two points along the S&P 500 volatility term structure (Exchange (2009)).

3.2 The VIX Market - How it works

The VIX, in its current form and methodology, has been in existence since 2014. At first instance, it is just a mathematical calculation in the form of primarily a weighted sum of mid quotes of out-of-the money (OTM) put and call options on the S&P 500, which does not trade directly. However, derivatives - including futures and options - directly reference the VIX. Moreover, there are exchange-traded products (ETFs and ETNs) that offer investors exposure to VIX futures.
The most important VIX-based derivative instruments that are in existence, have been introduced as follows:

- 2004, Introducing VIX futures contracts
- 2006, Exchange-listed VIX options
- 2009, VIX futures based ETNs and ETFs, such as: S&P 500 VIX Short-Term Futures ETN (NYSE: VXX) and S&P 500 VIX Mid-Term Futures ETN (NYSE: VXZ)
- 2010, S&P 500 VIX ETF (LSE: VIXS)
- 2011, VIX Short-Term Futures ETF (NYSE: VIXY) and VIX Mid-Term Futures ETF (NYSE: VIXM)

About four billion USD have been invested in VIX ETP and ETFs in 2017 (RCMAlternatives (2017)) and 600 million USD of vega was invested in VIX futures in 2017. The average daily volume for options on the VIX has reached 722000 contracts in 2017 (Exchange (2018c)). The VIX is disseminated from 2:15 a.m. to 8:15 a.m. and from 8:30 a.m. until 3:15 p.m. Chicago time.

The final settlement value for VIX futures and options is a Special Opening Quotation (SOQ) of the VIX Index calculated using opening prices of constituent SPX or SPX Weekly options that expire 30 days after the relevant VIX expiration date. For example, the final settlement value for VIX derivatives expiring on November 21, 2018 will be calculated using SPX options that expire 30 days later on December 21, 2018[2]. Here one must differentiate between weekly derivatives and monthly derivatives, because different from the calculation of the VIX throughout the day, only one of those series is used for the calculation of the settlement value. There is always one minute each week when options with exactly one expiration date are used in the computation of the VIX Index. Notably, the settlement auction usually also uses actual traded prices in contrast to the intraday

[2]The cash VIX Index will use the previous expiration date available in the week for the calculation if Friday is a holiday. For example, Friday, March 30, 2018 was a holiday so SPXW options that expired on March 29 were used in the cash VIX Index calculation. Related to this, VIX derivatives settled on Tuesday, February 26 were based on the March 29 SPXW series.

calculations of the VIX that uses option quotes. Even though the SPXW expires at 3:00 p.m., the calculation for the derivatives takes place at the same time as the SPX options (8.30 a.m.). The opening prices for SPX options used in the SOQ are determined by an automatic auction mechanism on CBOE options, which matches locked or inverted buy and sell orders and quotes resting on the electronic order book at opening of trading. Option series used are not static throughout the trading day. At every 15 second snapshot all eligible options are evaluated for inclusion so the options used in the VIX calculation could change every 15 seconds. Exchange (2018d)

3.3 The CBOE VIX formula explained

CBOE uses the following formula for the calculation of the VIX

$$\sigma^2 = \frac{2}{T}\sum_{i=1}^{n}\frac{\Delta K_i}{K_i^2}e^{rT}Q(K_i) - \frac{1}{T}\left(\frac{F}{K_0}-1\right)^2 \tag{1}$$

Here r is the risk-free interest rate. We will use the terms price and mid-quote in the context of options interchangeably: The mid-point of the current best bid and best ask quote for a given option. For the calculation of F (forward price), we first look for the strike for which we obtain the smallest absolute difference between put and call mid-quotes. To this strike we add the compounded absolute difference. If the put and the call price are the same, F would be the strike price of the option itself. This leads to the following equation for F

$$F = \text{Strike Price} + e^{rT} \cdot (\text{Call Price} - \text{Put Price})$$

Here it should be pointed out that all calculations of the VIX are computed for the near- and next-term options. The CBOE distinguishes near-term options with a remaining time between 23 and 30 days and next-term options with a remaining term between 31 and 37 days. K_0 is the first strike below the forward index level F and K_i is the strike price of the $i-th$ OTM option. The next variable in Equation (1) is ΔK_i. It is half of the difference of the strike of the options above and below K_i.

$$\Delta K_i = \frac{K_{i+1} - K_{i-1}}{2}$$

There is an exception if K_i is the lowest or highest strike of all available OTM options. In that case one has to use the Equations $\Delta K_1 = K_2 - K_1$ and $\Delta K_n = K_n - K_{n-1}$. The value $Q(K_i)$ is the mid-point of the bid and ask quotes of the option with strike K_i. The time to expiration T is defined as follows:

$$T = (M_{\text{Current day}} + M_{\text{Settlement day}} + M_{\text{Other days}})/Minutesinayear \tag{2}$$

where $\text{M}_{\text{Current day}}$ denotes the minutes remaining until midnight of the current day, $\text{M}_{\text{Settlement day}}$ are the minutes from midnight until 8:30 a.m. for standard SPX options and minutes from midnight until 3:00 p.m. for SPXW expirations, $\text{M}_{\text{Other days}}$ are the number of minutes in the days between the current day and the expiration day of the options [3]. When selecting the OTM puts one goes successively from K_0 to the lower strikes and excludes all options with a zero-bid. If two consecutive zero bids occur, all options with lower strikes are no longer considered (Table 1). For the OTM calls we follow the same procedure from K_0 to higher strikes (Table 2). Knowing all of these rules and parameters one can calculate σ_1^2 and σ_2^2, which are the near- and next-term components of the VIX. To obtain the VIX value one then takes a weighted average of σ_1^2 and σ_2^2

$$\text{VIX} = 100 * \sqrt{\left[T_1\sigma_1^2\left(\frac{N_{T_2} - N_{30}}{N_{T_2} - N_{T_1}}\right) + T_2\sigma_2^2\left(\frac{N_{30} - N_{T_1}}{N_{T_2} - N_{T_1}}\right)\right] \cdot \frac{N_{365}}{N_{30}}} \tag{3}$$

where

1. T_1 = Time to expiry (as a fraction of the total number of minutes in a year) of the near-term options
2. T_2 = Time to expiry (as a fraction of the total number of minutes in a year) of the next-term options
3. N_{T_1} = number of minutes to settlement of the near-term options
4. N_{T_2} = number of minutes to settlement of the next-term options
5. N_{30} = number of minutes in 30 days $(43,200)$
6. N_{365} = number of minutes in a 365-day year $(525,600)$

[3] A day contains 1440 minutes which is 24 hours

4 Literature Overview

We give an overview of the academic literature on the VIX index and its derivatives. In (Whaley (2000)) the author describes the construction of the volatility index and examines its movements over the past fourteen years. He wants to understand how the VIX performs its role and therefore examines its history and its relation to stock market returns. In 2008, during weeks of market turmoil financial news services have begun routinely reporting the level of the CBOE Volatility Index. The author in (Whaley (2008)) describes the VIX, its history and purpose, and explains how it fits within the array of indexes that help describe where the economy stands relative to other points in recent decades.

In 2003, the CBOE published a white paper on the CBOE volatility index, which describes the precise approach they are taking to compute the VIX (Griffin and Shams have done a thorough analysis of manipulation opportunities of the VIX (Griffin and Shams (2017)). They show that at the settlement time of the VIX, volume spikes on S&P 500 Index options, but only in out-of-the-money options that are used to calculate the VIX, and more so for options with a higher and discontinuous influence on the VIX. Alternative explanations of hedging and coordinated liquidity trading are investigated. They undertake various tests of their hypothesis. Tests including those utilizing differences in put and call options, open interest around the settlement, and a similar volatility contract with an entirely different settlement procedure in Europe are inconsistent with these explanations but consistent with market manipulation.

(Demeterfi et al. (1999)) have done the first comprehensive analysis and derivation of the price of volatility and variance swaps. They explain the properties and the theory of both variance and volatility swaps. They show how a variance swap can be theoretically replicated by a hedged portfolio of standard options with suitably chosen strikes, as long as stock prices evolve without jumps. For volatility swaps they show that those can be replicated by dynamically trading the more straightforward variance swap.

Andersen et al. (2015) demonstrate that the VIX Index has deviations from true volatility due to the inclusion of illiquid options. Futures and options on the VIX have a relatively large volume. The settlement value of those derivatives is calculated from a wide range of OTM put and call options with different exercise prices. A manipulator would have to influence exactly those prices of the lower-level OTM SPX options to influence the expiring upper-level VIX derivatives. The authors also show that fluctuations of illiquid OTM options lead to undesired variations of the VIX value.

In 2017, (Li, 2017) shows that the CBOE VIX methodology underestimates implied variance in general. The under-estimation increases as the forward index value moves higher and away from a strike price, peaks at the next strike, and resets to zero when passing the strike. He points out that a significant under-estimation can show up in related VIX indices such as the CBOE VVIX (the VIX of VIX) where fewer strikes are quoted.

In (Pimbley and Phillips (2018)), Pimbley and Phillips point out several aspects which show that the CBOE Volatility Index is prone to inadvertent and deliberate errors. They indicate several measures that can be taken to improve the index's accuracy and curtail its susceptibility to misuses and falsifications.

5 Data and replication methodology

In this section we describe the data we have used in our analysis and the source of it. Additionally, we show how we have validated our replication of the VIX and what our findings are.

5.1 Data

Data is obtained directly from the CBOE historical market data shop. The data sets contain lower-level SPX options as well as upper-level VIX futures and options. We focus on lower-level SPX options, as those are needed to replicate the VIX. For the SPX options we use end-of-day (EOD) options quote and trade data, from January 2015 to March 2018, and intraday options quote and trade data with a one-minute granularity, in the interval from January 2018 to March 2018. The EOD data includes the close of trading, representing 3:15 p.m. Chicago time, as well as data at 2:45 p.m. Chicago time. The daily VIX spot data for the period from January 2004 to March 2018, which contains daily open, high, low and close values, is also sourced from the CBOE. In addition, we use the intraday VIX spot value, in 15 seconds granularity, from January 2017 to March 2018.
The daily risk-free rate is obtained from the US Department of the Treasury. It is the risk-free interest rate that is the bond-equivalent yield of the U.S. T-bill maturing closest to the expiration dates of relevant SPX options. Due to the small influence, a better comparison and easier understanding of the results, we are foregoing the inclusion of the yield curve in most of our analysis, and instead set the risk-free rate to zero. Nonetheless we have used the yield curve in certain situations to get an understanding for its importance in the context of the VIX.

5.2 Replication methodology and validation

Our replication methodology follows the description from the CBOE VIX white paper. Our focus is on replicating the VIX value at 3:15 p.m. Chicago time. For this, we use the end-of-day option quotes at 3:15 p.m. as well as the EOD VIX value.

We have validated the replication methodology in several ways. On the one hand, we have successfully replicated the theoretical example of the CBOE white paper with the example data set available there. This comparison leads to an exact match of our results with the CBOE computation. On the other hand, we compared the replicated VIX time series with the VIX close values from the CBOE data.

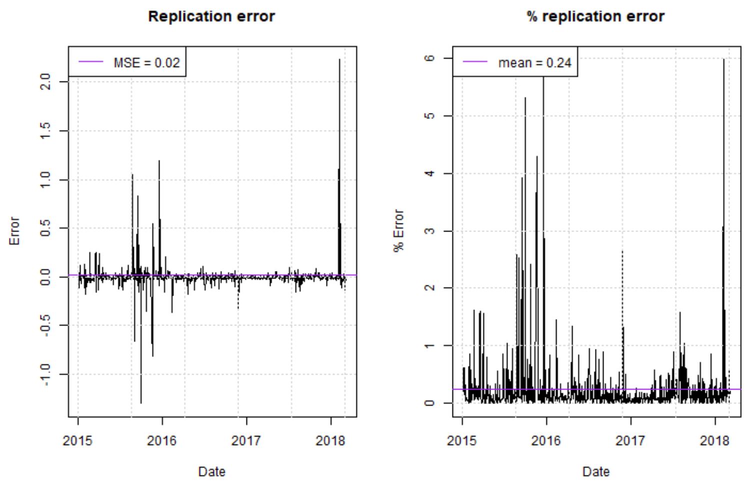

Figure 1: Replication error of the VIX for daily close values.

Despite the successful validation of our approach with the CBOE white paper example, we have small discrepancies between the replicated value and the actual value, when using actual data, see Figure 1. Even though the MSE is acceptable and the percentage error can partly be explained by setting the interest rates to zero, the outliers up to 6% are too large. As a consequence, we have conducted a more detailed analysis of the outliers and the replication errors in general.

One source of discrepancy are time-stamp difference. For the option data, we use the close prices at 3:15 p.m., but the VIX, being disseminated every 15 seconds, does not normally have its last print at 3:15 p.m., but up to 15 seconds earlier.

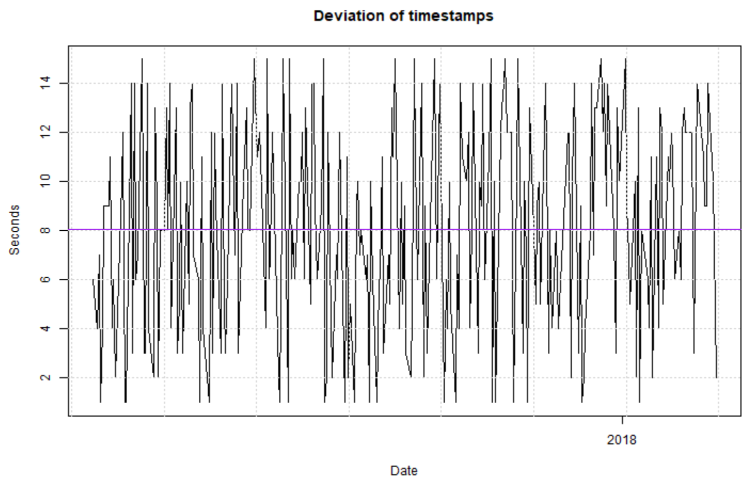

Figure 2: Deviation in seconds, of EOD intraday timestamp with option data timestamp (03:15 p.m.).

In Figure 2 the difference of the last timestamp of each day of intra-day data, with the time stamp 3:15:00 p.m. Chicago time is shown. 3:15:00 p.m. represents the time stamp of the daily options data. The purple line marks the arithmetic mean of the deviations. Since we checked that the close values of the daily VIX data matches the last daily value of the intra-day sample, these deviations give one explanation for the replication error that we observed. The replication error is correlated with periods of high volatility during which option prices can move substantially even during a short time period of just 15 seconds. It might well be worth a thought to ask CBOE to disseminate an official VIX closing value, that is based on the official closing values of the options.

6 Theoretical derivation of the VIX from first principles

The idea the of VIX was to create a financial instrument for investors representing the volatility of an underlying equity index. Using the Black-Scholes option pricing theory, we derive the VIX formula. The origin of its mathematical formulation is centered around the theory of pricing variance swaps (Demeterfi et al. (1999)). In the following section the VIX is theoretically and empirically disassembled into its components to get a better understanding of the mechanics of the formula. Some of the equations are based on (Xin (2011)). We explain the approximations that are made by CBOE when deriving the VIX formula based on option pricing theory. Several approximations are made when theoretically deriving the formula. We determine their importance empirically and visualize them.

We start with a filtered probability space $\left(\Omega,\ (\mathbb{F}_t)_{0\leq t\leq T}\,,\, P\right)$, satisfying the usual assumptions, i.e. the filtration is complete and right-continuous. On this space, we are given a Wiener process $(z_t)_{0\leq t\leq T}$ and the stock price process $(S_t)_{0\leq t\leq T}$ follows a geometric Brownian motion:

$$\frac{dS_t}{S_t} = rdt + \sigma dz, \quad 0 \leq t \leq T \tag{4}$$

with $r \geq 0$ being the risk-free rate and $\sigma \geq 0$ the volatility of the stock price process. Under the risk-neutral probability measure, we know that the forward price at time T is as follows

$$E\left(S_T\right) = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \frac{\sigma^2 T}{2}\right)$$

Solving for σ^2 yields

$$E(\sigma^2) = \frac{2}{T}\ln\frac{F}{S_0} - \frac{2}{T}E\left(\ln\left(\frac{S_T}{S_0}\right)\right) \tag{5a}$$

with F now being the forward price of the stock price, i.e. $F = E(S_T)$.

Now, using Itô's Lemma, combining with Equation (4), then integrating from time 0 to time T, one can show that

$$\frac{1}{2}\sigma^2 T = \int_0^T \frac{dS}{S} - \ln\frac{S_T}{S_0} \tag{5b}$$

Integrating the payout of European put and call options over the range of strike prices, weighted by the inverse of the square of the strike prices, one obtains the following formula

$$\int_0^{S^*} \frac{1}{K^2}\max(K - S_T, 0)\ dK + \int_{S^*}^{\infty} \frac{1}{K^2}\max(S_T - K, 0)\ dK = \ln\frac{S^*}{S_T} + \frac{S_T}{S^*} - 1 \tag{6}$$

with S^* being some value of S. We now assume that we have infinitely many call and put options with a continuum of strike prices K. Denote the call and put prices by $c(K)$ and $p(K)$, respectively. Taking expectations under the risk-neutral measure using the equations above, one can show that the expected value of the average variance from time

0 to time T is.

$$E(\sigma^2) = \frac{2}{T}\ln\frac{F}{S*} - \frac{2}{T}\left[\frac{F}{S*} - 1\right] + \frac{2}{T}\left[\int_0^{S*} \frac{1}{K^2}e^{rT}p(K)\ dK + \int_{S*}^{\infty} \frac{1}{K^2}e^{rT}c(K)\ dK\right] \tag{7}$$

where r is the risk-free interest rate for a maturity of T and F is the forward price for an options contract with maturity T.

Since in reality we have only a discrete, limited range of options, we now make two fundamental approximations

1. Assume that we have n discrete strike prices, $K_1, \ldots, K_n$ instead of a continuum of strikes. In addition we set S^* to K_0, which represents the first strike below the forward price F. This leads to

$$\int_0^{K_0} \frac{1}{K^2}e^{rT}p(K)\ dK + \int_{K_0}^{\infty} \frac{1}{K^2}e^{rT}c(K)\ dK = \sum_{i=1}^{n} \frac{\Delta K_i}{K_i^2}e^{rT}Q(K_i) \tag{8}$$

where

$$\Delta K_i = \frac{K_{i+1} - K_{i-1}}{2},\ \text{for } i = 2, \ldots, n-1$$

and $\Delta K_1 = K_2 - K_1$, $\Delta K_n = K_n - K_{n-1}$. $Q\left(K_i\right)_{i=1,\ldots,n}$ is defined as

$$Q\left(K_i\right) = \begin{cases} c\left(K_i\right), & K_i > K_0 \\ p\left(K_i\right), & K_i < K_0 \\ \frac{c(K_i)+p(K_i)}{2}, & K_i = K_0 \end{cases}$$

2. Approximating $\ln\left(N\right)$ using its Taylor polynomial, centred at the point of zero, of order 2,

$$\ln\left(N\right) = (N-1) - \frac{1}{2}(N-1)^2 + o((N-1)^2)$$

applying to $ln\frac{F}{S*}$, while setting $S*$ to K_0 and approximate leads to

$$ln\frac{F_0}{K_0} = \left(\frac{F_0}{K_0} - 1\right) - \frac{1}{2}\left(\frac{F_0}{K_0} - 1\right)^2$$

Rearranging terms gives

$$2\left(\frac{F - K_0}{K_0} - \ln\frac{F}{K_0}\right) \approx \left(\frac{F}{K_0} - 1\right)^2 \tag{9}$$

Using the approximations of Equation (8) and Equation (9) in Equation (7) leads to the final formula which is used in the VIX calculation

$$\sigma^2 = \frac{2}{T}\sum_{i}^{n} \frac{\Delta K_i}{K_i^2}e^{rT}Q(K_i) - \frac{1}{T}\left[\frac{F}{K_0} - 1\right]^2 \tag{10}$$

In summary, the derivation of the VIX formula starts in a Black-Scholes world, using Itô's Lemma and approximates an infinite number of option strikes by a finite sum. Using Taylor approximations, the formula is further simplified. One consequence of the Black-Scholes assumptions is, that the volatility σ is treated as if it is independent of the strike price. As a simple visualization of the implied volatility of options shows (Figure 26), we do know that the their volatility exhibits a skew and a smile. Obviously, it is easier to have a single, weighted-average volatility as an index, and that is the convention of the VIX. However, the authors in (Pimbley and Phillips (2018)) show one approach of how to solve that issue. They show how a linear volatility skew can be included in the calculation of deep OTM options.

6.1 Empirical decomposition of the VIX

In the previous section we have derived the VIX formula and looked for its weaknesses in a theoretical setting. The following section is about empirically examining the individual components of the formula. For this we use historical option data with daily granularity. The daily close value of the VIX since 2015 is given in (Figure 25), in the appendix, together with the S&P 500 index value (Figure 24). We start with the simulation of the variables F (Figure 22) and K_0 (Figure 23) from Equation (10). Since F represents the forward price of the underlying and K_0 is the first strike smaller than F, the two time series are very similar to the S&P 500, due to the low interest rate environment and the short time horizon.

In the following analysis we split Equation (10) into two separate parts. We want to understand the importance of each one of those additive terms. For the sake of simplicity we list the terms again here. In our paper, Formula (11a) is sometimes referred to as sum-term and Formula (11b) as correction-term.

$$\frac{2}{T}\sum_{i}^{n}\frac{\Delta K_i}{K_i^2}e^{rT}Q(K_i) \tag{11a}$$

$$\frac{1}{T}\left[\frac{F}{K_0}-1\right]^2 \tag{11b}$$

In the following empirical simulation and analysis only the near-term component of the VIX will be shown. The next-term component shows similar characteristics and will therefore be omitted in our analysis. In addition to the decomposition of the VIX formula, we will look at how many options have been included in the calculation of the VIX, which corresponds to the number of additive terms in Formula (11a).

In our first analysis (Figure 3) we simulated Formula (11b) from January 2015 to March 2018. At first glance, it is noticeable that the term has neither systematic fluctuations nor clusters: with a maximum value of $1 * 10^{-4}$, the term is relatively small and therefore the impact on VIX is minuscule.

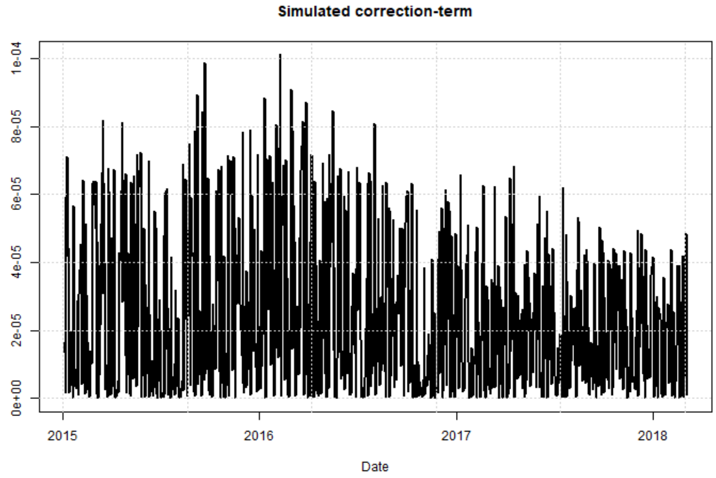

Figure 3: Simulated correction-term time-series of the near-term σ^2.

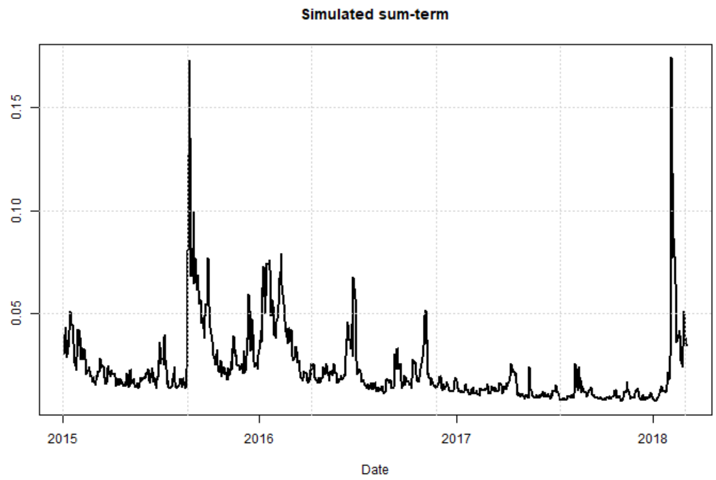

Figure 4: Simulated sum-term time-series of the near-term σ^2.

We implemented the sum-term of the VIX formula (Equation (10)) for the period between January 2015 and March 2018, which can be seen in Figure 4. Note that the entire formula represents σ^2 of the near-term options, and therefore the scaling cannot be directly compared to the final VIX. In contrast to the correction-term (11b), a clear temporal dependency can be identified here. It seems obvious, that one would have to manipulate components of the sum-term of the formula to achieve a significant impact on the VIX.

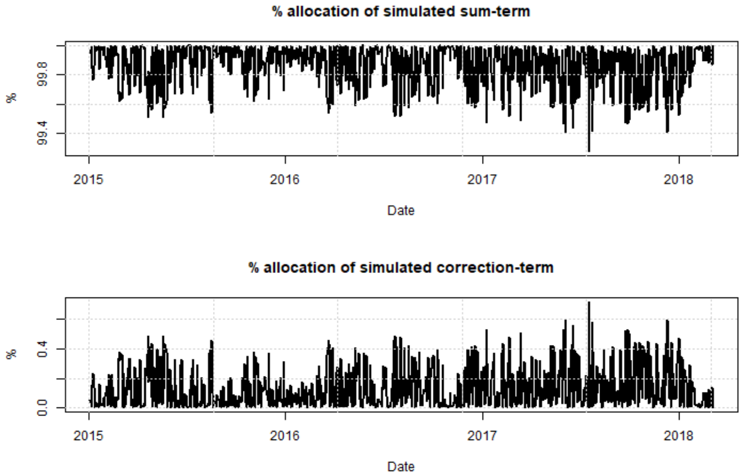

Figure 5: Percentage distribution of the simulated correction- and sum-term for near-term σ^2.

In order to get a better understanding of how both terms split the value of σ^2, we have mapped (in Figure 5) the percentage distribution of the first term (upper graph) and the second term (lower graph) of Equation (10) for σ^2. The correction term explains only a maximum of 0.6% of the total term σ^2 in the past three years and you see that its influence is reduced when the volatility, e.g. as observed in February 2018, is high. So we can summarize that Formula (11b) plays a minor role for the total value of the VIX value. For clarification, the gap in the time-series data is because of some missing data, which we could not obtain from CBOE.

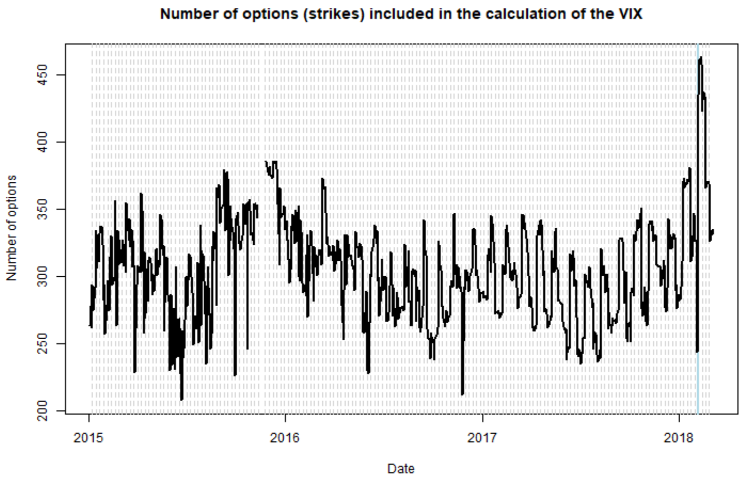

Figure 6: Number of put and call options which are included in our VIX replication.

If you compare the historical time-series of the VIX (Figure 25) with the number of call options, you can see that the number of options decreases at the same time as the VIX deflects. The vertical blue rectangle in Figure 6 shows the turbulent week for the VIX during the 5th and 9th of February in 2018. The dashed grey lines in the plot visualize the settlement days of weekly and monthly VIX derivatives. If we look at these numbers (Figure 6) we see that the number of options is low in the middle of the year and increases again towards the end of the year, indicating an annual periodic oscillation. The number of options rises nearly by 30% relative to previous years.
In the following Figure 7, we separated the number of options by puts and calls.

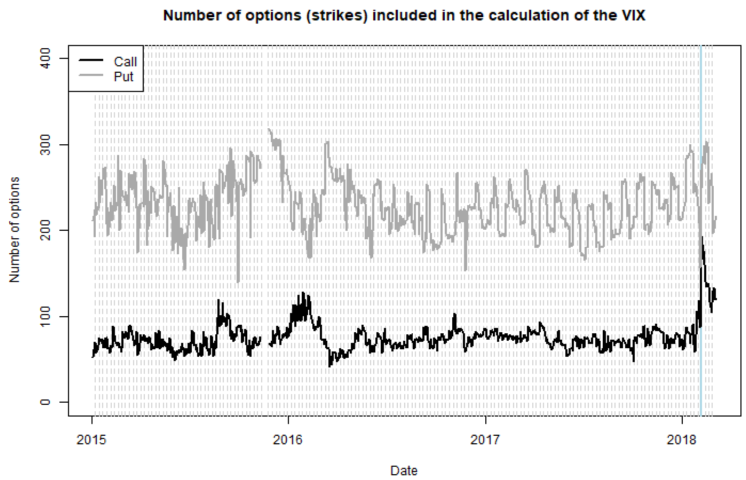

Figure 7: Number of put and call options for the VIX replication, separated by puts and calls.

It is not difficult to see that over the past two years, the number of put options has shown something of a seasonality which is much less apparent in call options. This seasonality is driven by the settlement days, so this represents the trading behaviour of market participants. If we now focus on the blue rectangle we can see that the number of call options doubles and those of put options decreases. It is straightforward that if the S&P 500 falls, the puts and calls which flow into the calculation are distributed differently, as the market updates the price of options with the change in the underlying. Now we are interested in whether the above findings behave similarly in an intraday setting, whether periodic or other conspicuous events occur. In addition, in the next analysis we will compare a „quiet“ day with the two days on which the VIX was suspected of manipulation and had those extreme jumps. As quiet days we define those days on which the index has fluctuated by a maximum of one VIX point.

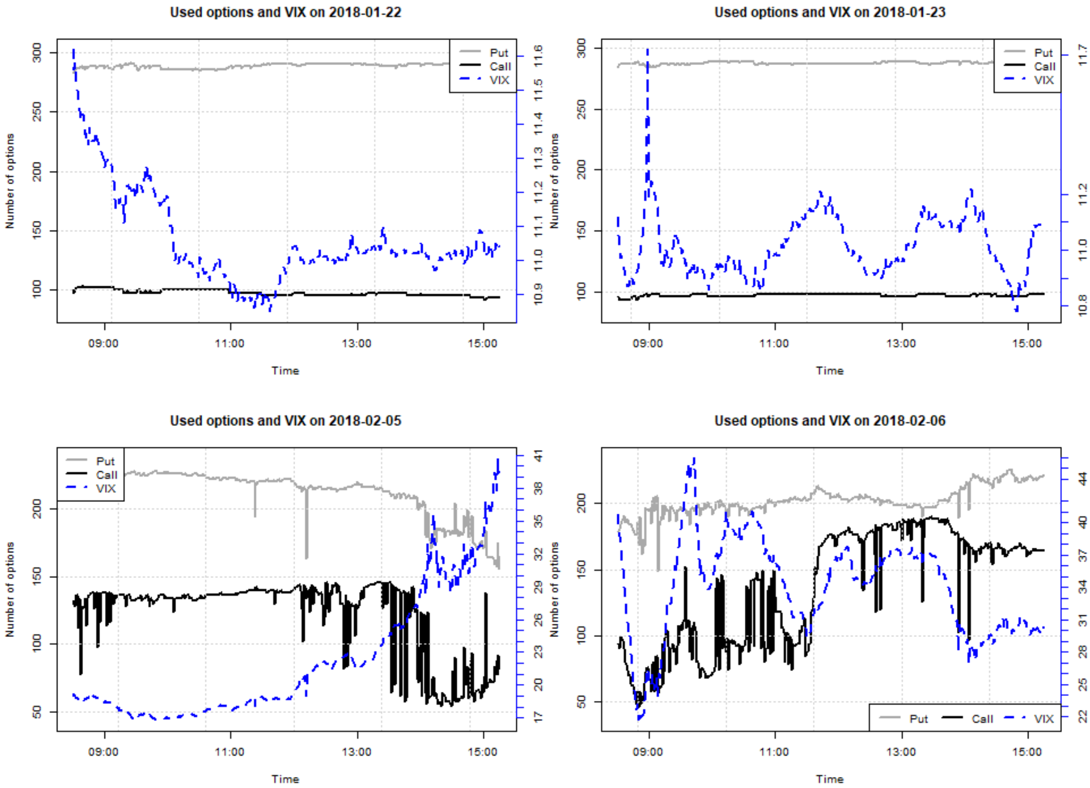

Figure 8: Simulated intraday number of options used in the VIX replication over four different days. Number of options separated by put and call. Associated replicated VIX on a separate y-axis.

The first observation based on Figure 8 is that there is a substantial difference in the number of options between quiet and highly volatile days. On days with little fluctuations in the VIX, almost the same number of options for each point in time are included in the calculation, while the number of options fluctuates sharply on highly volatile days. With the exception of periodicity, this shows similar characteristics to Figure (Figure 7).
In this section we have seen how the individual components of the VIX formula behave over time. The important part of the VIX formula is the first part, the weighted sum of option prices. This accounts for at least 99.4% of the entire VIX value. In addition, we looked at how the number of options included in the calculation of the VIX has behaved over time and discovered large fluctuations. The number of options is related to the S&P 500 (Figure 24), i.e. if the index falls sharply, K_0 falls, which means more calls and fewer puts are included in the calculation.

6.2 Theoretical Approximations in the VIX Calculation

We have identified two approximations, which are performed by the CBOE. On the one hand, the CBOE approximates the VIX (Equation (9)) and on the other hand, depending on the market situation, the CBOE excludes a varying number of options from the calculation. With the SPX options and VIX data we look at the individual influences of those approximations, as well as the combined impact on the VIX. We first look at the Taylor approximation of the second term of the VIX. To simplify matters, we repeat the two formulas here, first the theoretically correct one, second, the one used by CBOE.

$$\frac{2}{T}\left(\frac{F-K_0}{K_0}-\ln\frac{F}{K_0}\right) \tag{12a}$$

$$\frac{1}{T}\left[\frac{F}{K_0}-1\right]^2 \tag{12b}$$

This approximation (12b) only leads to errors if F deviates from K_0. Therefore it does not introduce major deviations, if the grid of option strikes is fine enough. However, it has no major advantages and is rather disadvantageous, as its error increases as the distance of F and K_0 increases. (Li (2017)) suggests a correction for this term.

As shown in the breakdown of the CBOE VIX formula, the calculation includes a weighted sum of the mid-quotes of all relevant options. Relevant means, all options which are included in the calculation of the VIX (cf. Exchange (2009)). Depending on the current market situation, the tail of relevant put options can be very long (Figure 21). Therefore this approximation method could have a considerable influence on the value of the VIX. The CBOE's approach of cutting off the tails after two zero bids, when calculating the VIX, has been criticized in several publications, see e.g. (Pimbley and Phillips (2018)).

6.3 Practical significance of the theoretical approximations in the VIX calculation

We use the historical data to show what influence the approximations made by the CBOE have on the VIX. In a first step, the VIX, as defined by CBOE, is simulated with the correction-term (12b) and compared to a modified formula with a modified correction-term (12a).

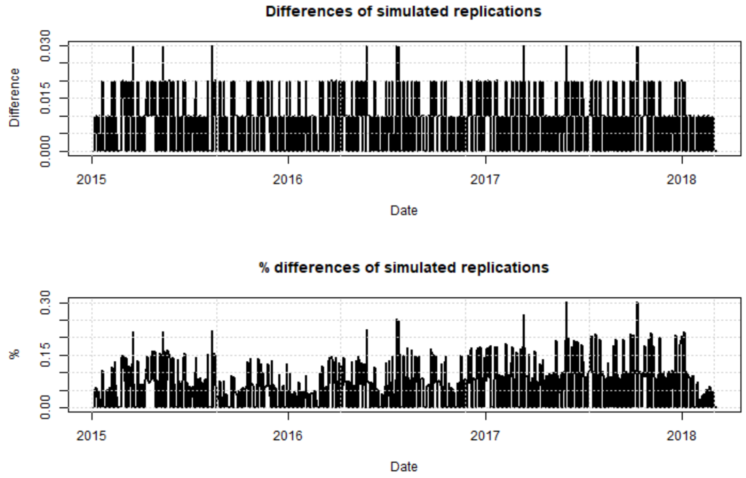

Figure 9: Absolute and percentage difference of two simulated VIX indices, one using the CBOE methodology and one using a modified version without a Tailor-approximation.

If one calculates the difference between the two values, one can see that we get a very constant difference with a maximum value of 0.3% (Figure 9). A 0.3% deviation through a mathematically „different“ approximation is remarkable. Since the alternative correction term is generally smaller than the version from CBOE, an adjustment of this term would imply a smaller underestimation of the true market volatility. We now focus on the second approximation of the CBOE calculation methodology. We are interested in how much influence the non-consideration of options after two consecutive zero-bid options has. To tackle this problem, we simulated both types of VIX.

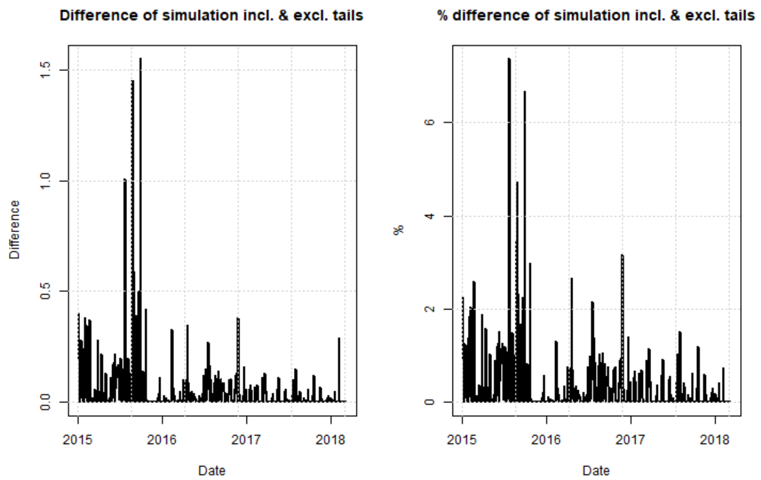

Figure 10: Absolute and percentage difference of two simulated VIX indices, one using the CBOE methodology and one using all options after two consecutive zero-bids.

Figure 10 shows the difference between the VIX, with the calculation used by the CBOE, and a simulated index in which in addition, all OTM options are included, irrespective of whether there have been two consecutive zero bids or not. The difference of both indices and their percentage deviations are plotted. Especially before 2016, very large differences between the two values can be seen. Interestingly, there are times when obviously no consecutive zero-bid options occur. But on the other days, the approximation changed the VIX by more than 7.5%.

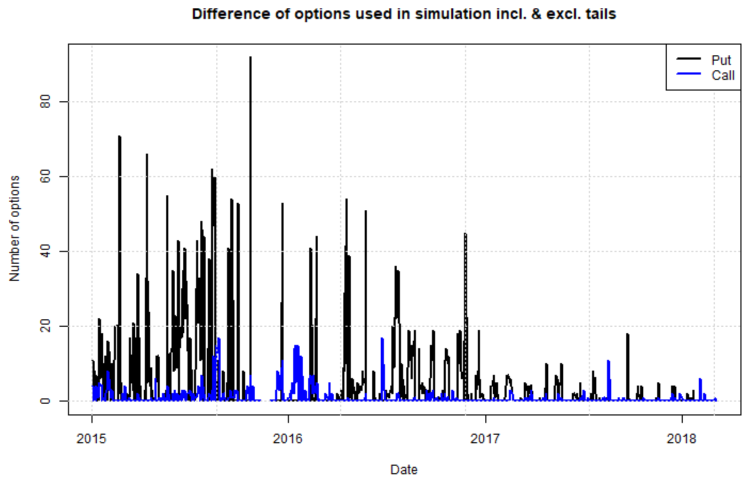

Figure 11: Difference between the number of put and call options used while simulating two VIX indices, one using the CBOE methodology and one including options after two consecutive zero-bids, separated by put and call options.

Similarly to the the first approximation, this approximation method causes an underestimation of the true market volatility. Additionally it is interesting to know how many options are truncated, this is shown in Figure 11. The difference between the number of options used in the CBOE calculation and the number of options without truncation is shown. In some cases up to 100 options were left out by the approximation. This fact illustrates how serious the consequences of such a regulation can be in the algorithm. Furthermore, it can be seen that many more put options than call options were excluded from the calculation. Since the distribution of the prices of OTM puts have a much wider tail than that of OTM calls (Figure 21), the probability of having two consecutive zero-bids for puts is more likely. VIX usually overestimates actual 30-day volatility because of an excess demand for put options because they can provide an insurance-like characteristic (Edwards and Preston (2017)).

In this section we have seen that the theoretical approximation in the VIX formula of the CBOE has a very constant but not extreme influence on the VIX value. On the other hand, the influence of cutting the tails is more severe. The fact that in the past up to 100 options were not included in the calculation is astonishing and worrying. Since the zero-bid rule can potentially have considerable effects, we will look at the impact of this rule in the next section.

7 Can market participants influence the VIX formula?

The VIX has come under suspicion of manipulation several times in the last few years. The turbulent VIX in February 2018 5^{th} made headlines again. For this reason, we want to use the theoretical VIX formula and empirical simulations in this paragraph to determine what possibilities market participants have to influence the index.
The VIX formula is composed of various variables. We want to go through them one by one and show if and how we can alter them and therefore change the VIX. For convenience, we repeat the VIX formula here:

$$\sigma^2 = \frac{2}{T}\sum_{i=1}^{n}\frac{\Delta K_i}{K_i^2}e^{rT}Q(K_i) - \frac{1}{T}\left(\frac{F}{K_0} - 1\right)^2 \tag{13}$$

We will use the term relevant option in the sequel. Relevant options are defined as OTM put and call options that are included in the CBOE Volatility Index calculation.

1. The only variable that appears in both terms of Equation (13) is the time to expiration T and cannot be changed by regular market participants.

2. Starting with the right-hand term, F, the forward index level, derived from index option prices of the S&P 500, is computed as

   $$F = \text{Strike Price} + e^{rT} \cdot (\text{Call Price} - \text{Put Price})$$

   Therefore, F will change if option quotes change, in particular those close to the forward price of the S&P 500.

3. Also in the right-hand term of Equation (13), K_0, the strike price immediately below the forward index level F, depends on F as well as the strikes of the quotes. K_0 will change if option quotes and available strikes close to the forward price of the S&P 500 change.

4. Switching over to the left-hand term of Equation (13), we start with the interest rate r. It is the risk-free interest rate that is the bond-equivalent yield of the U.S. T-bill maturing closest to the expiration dates of relevant SPX options. Since r depends on publicly traded U.S. T-bill, a change in their prices will directly impact the VIX. Even though, as latest investigations have shown, it is unfortunately not unthinkable, that this can be manipulated, the order of magnitude of changes in r are too small to impact our VIX value.

5. The remaining variables in the left-hand term of the formula are those related to call and put option quotes as well as their strike prices. K_i^2, for a given relevant option, cannot be altered by market participants directly, but are defined by a predetermined schedule.

6. ΔK_i, however, being the difference of two consecutive strikes, changes if an option with a given strike is included or removed from the calculation, notably in the case of excluding options after two consecutive zero-bid prices.

7. $Q(K_i)$ is the mid-point of the bid and ask price of each option with strike K_i. Being the mid-quote, the market can change those values by posting or removing orders for options at a given price level.

8. Having considered all variables in the formula, we now need to look at the sum. We are adding up the weighted quote prices over all relevant options.

As we know from Section 6, the variables in Equation (13) are strongly interdependent. So one would have to try to reduce the difference of the mid-prices of puts and calls with the same strike to be able to influence F (item 2). Since K_0 depends heavily on F and the available strikes, K_0 can only be influenced indirectly via F. Only the ratio of the two variables flows into the VIX using the right-hand term. We have shown the influence of this term in Figure 3 in Subsection 6.1. Thus, it makes little sense for market participants to influence option quotes close to the underlying spot value in order to change F. The interest rate set under item 4 has an influence on the left-hand term of Equation (13) and only a very small influence on the entire VIX. In Figure 20 in the Appendix we see that even if the interest rate r is increased by an absolute 5%, this only has an influence of at most 0.3% on the VIX. In the term at item 7, the mid-prices of the options are noted. From the derivation of the formula in Section 6 we can deduce that a change in the quotes results in a shift of the VIX. For example if you increase all quotes by 10%, the VIX would approximately increase by $\sqrt{10\%}$. Since a single market participant would have to post a large number of orders to increase each quote by 10%, it is not going to be a realistic example. Therefore, in the following analysis we consider how a certain quote, is weighted given its strike and which impact this has on the VIX. For better understanding we split the weighting of the quotes as follows

$$\Delta K_i \tag{14a}$$

$$\frac{1}{K_i^2} \tag{14b}$$

$$\frac{\Delta K_i}{K_i^2} \tag{14c}$$

The individual weights (14a) and (14b) depend on the strikes, as well as the distance between the strikes. To simulate the weights, we constructed a strike curve, which was created using OTM SPX options to get an example as close to reality as possible, see Figure 27 in the Appendix. In the first analysis we look at Formula (14a).

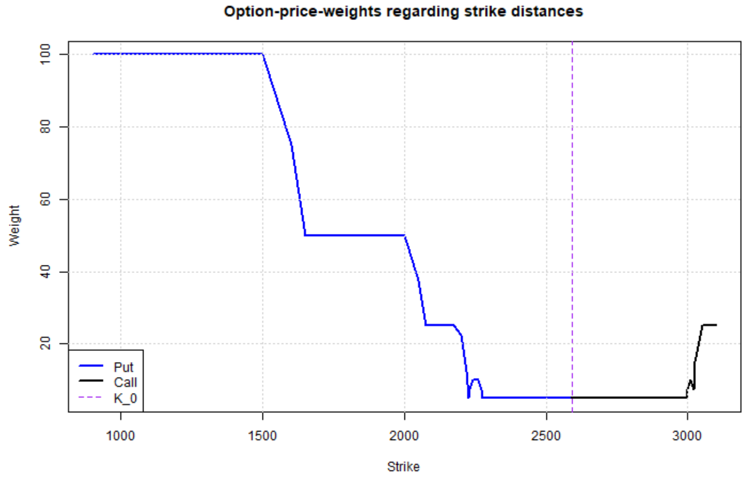

Figure 12: How much weight has the price of an option, given a certain strike, focusing only on the distances between each strike (for a specific day)

.

If we look at Formula (14a) as a function of the strike, we notice that deep OTM options have a much higher weight than options located in the middle of the strike spectrum. This is due to the fact that this weight measures the distances between the strikes and those are getting wider and wider for strikes further out in the tail. The two small drops in the weights of put and call options are related to non consecutive zero-bids which occur in the middle of the strike spectrum.

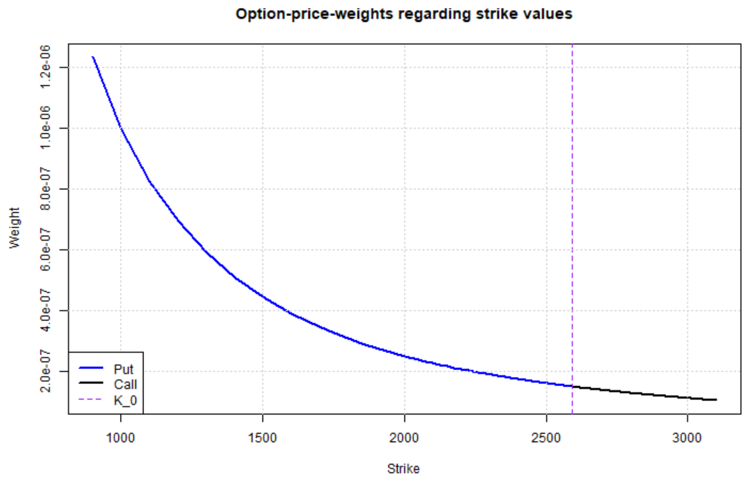

Figure 13: How much weight has the price of an option, given a certain strike, focusing only on the strike value itself (for a specific day).

Now we look at Formula (14b).

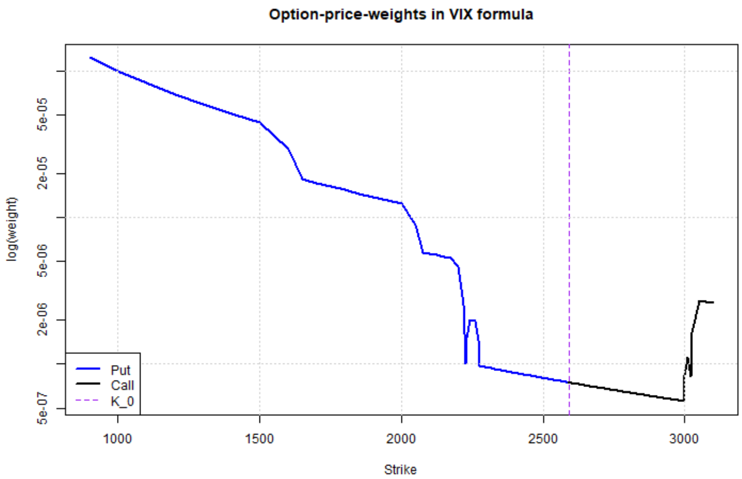

Figure 14: How much weight has the price of an option, given a certain strike, considering the strike value and the distance between strikes (for a specific day). This weight is used for calculating σ^2.

If we combine the two formulas (14c), we get the weight as it is used in the CBOE VIX formula. You can see that deep OTM puts have the highest weight. A market participant who intends to move the VIX in a certain direction achieves a much higher influence with orders in the tails. Similar findings were made by (Griffin and Shams (2017)), though using an entirely different logic .
In the following analysis, we look at the effects that a market participant who can change the strike range (n the sum of Equation (13)) can have on the VIX calculation. Due to the CBOE's two consecutive zero-bid rule, it might be possible for a market participant to influence the option range. Since quotes are based on posting orders, it is not necessary to execute an effective trade or use actual money. Thus the investor can place a large order on a deep OTM zero-bids with relatively little risk and generate a positive bid price. If he now closely monitors the two consecutive zero-bid rule, he is able to expand or even reduce the option. In Subsection 6.2 we have seen that, in recent years, the cutting of tails has amounted to a change of up to 7.5% in the value of the VIX. In the following analysis, however, we want to determine a rule from which we can derive the impact of the tails in general and the weight of options with quotes that are close to K_0. Therefore we have replicated the VIX in Figure 15 with different number of options. We always started with the actual K_0 for each time-point and only chose the number of options. It is important to say that we always include the same number of put and call options in the calculation. The selected range of options is between 10 (5 puts, 5 calls) and 330 options.

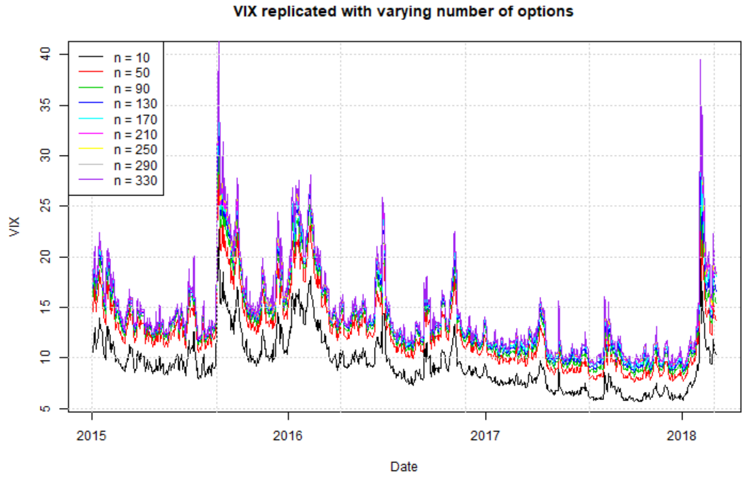

Figure 15: Influence of options included in the calculation of the VIX

Figure 15 shows the replicated VIX with a varying number of options. At first glance it is easy to see that a lot of information about the VIX is in the first 10 options (black line). Thus the additional information content of the simulations becomes smaller and smaller with an increasing number of options. The simulation with fewer options is only underestimating the VIX in percentage terms, but the time-dependence is not affected and is very similar to the VIX itself.

In the next step we have computed the deviations of our approximations tothe actual CBOE VIX..

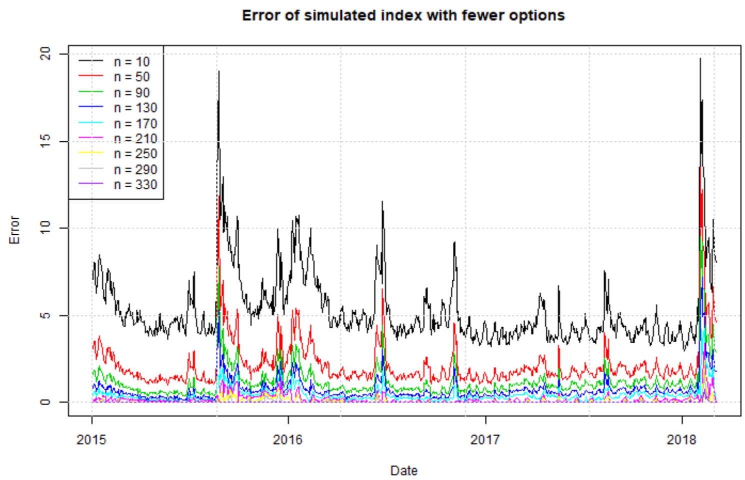

Figure 16: Differences between the replicated VIX, using the CBOE methodology and the simulated indices using fewer options in the calculation.

In Figure 16 we see that the approximation error of the simulation is time-dependent. The absolute errors are bigger on those days, when the VIX value is also high. Additionally, we see that the error does not decrease linearly with an increasing number of options.

In the following analysis, we consider the approximation errors as a percentage of the reference simulation, which contains all relevant options.

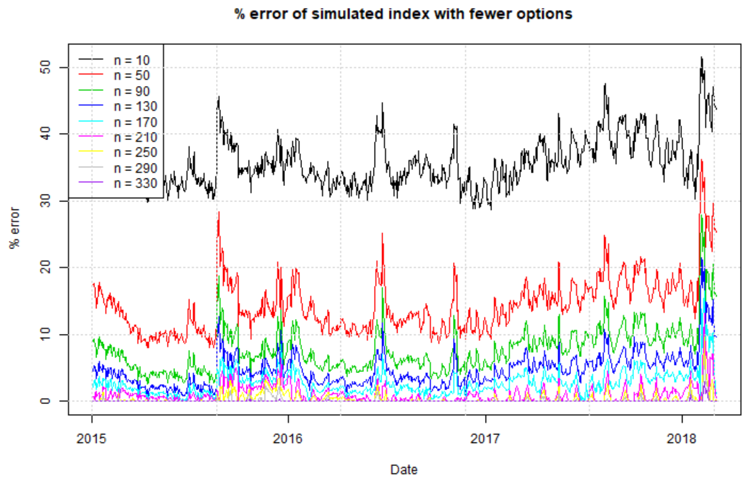

Figure 17: Percentage differences between the replicated VIX, using the CBOE methodology and the simulated indices using fewer options in the calculation.

One recognizes in Figure 17 that the more options are included in the calculation, the smaller the error becomes. This means that most of the information is in the first 10 options. These 10 have an error of about 40%, which means that they replicate the VIX to 60%.

Now that we know the percentage deviations, we want to find out how strongly the approximated simulations are related to the reference simulation, and how this correlation changes with a smaller number of options.

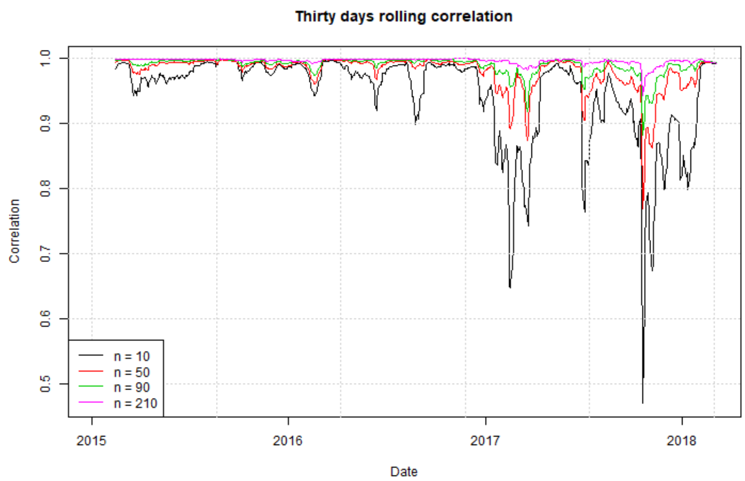

Figure 18: Thirty days rolling correlation between the replicated VIX, using the CBOE methodology and the simulated indices using fewer options in the calculation.

We have calculated the 30 day rolling correlation between the time series in Figure 15 and the simulated VIX with all options. Despite the omission of a large part of the options, the time series are still very strongly correlated with VIX. it can therefore be stated that the majority of the information for the effective VIX is in the put and call options, whose strikes are closest to K_0 (Figure 18). In addition, it can be assumed that as the number of options taken into account increases, the profit on information decreases.
We now want to derive from the previous analyses which rule the approximation errors follow on the basis of the options included.

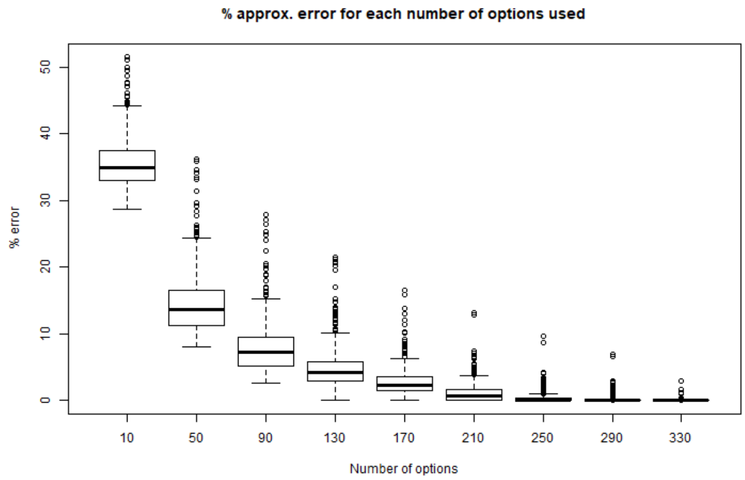

Figure 19: Boxplot of percentage approximation errors between the replicated VIX, using the CBOE methodology and the simulated indices using fewer options in the calculation, for each number of options used.

In Figure 19 the percentage approximation errors are plotted against the number of options used, in form of box plots. It can be seen that when only five put and five call options are used, between 50 and 70 percent of the VIX can already be described. Additionally, the approximation error converges exponentially towards zero with increasing number of options. The spread of the error is also reduced. However, the quality of the approximation should be enjoyed with caution, as the box plots show many outliers upwards. In sum, Figure 19 shows us that the options which are closest to K_0 at a certain time-point, or most likely ATM, make up the largest part of the VIX at a certain point of time. The reason for this behaviour is due to the distribution of prices across the strikes, as the mid-prices impact directly the value of the VIX (Figure 21 in the Appendix).
In summary, it is possible to direct the VIX in a desired direction by posting aggressive orders on favourable options in order to increase certain quotes. These quotes allow to change the strike tails of OTM puts and calls and influence the VIX, as well as higher quotes. If you look at a pure change of the tails, our analyses show that the deeper OTM the options in the tails are, the smaller the influence of these quotes on the VIX is. However, since the influence of the tails is strongly related to the prices of the options

and the weighting of the option prices in the VIX formula, a direct influence through the tails is difficult to achieve. From this and the previous analyses it becomes clear that the influence of the tails basically depends on the prices. Therefore it is also difficult to estimate which effort would achieve which yield.

8 The VIX and its sub-optimal market micro structure and market mechanism

Three deficiencies of the market micro structure and the market mechanism of the VIX and its derivatives can be observed: First, the VIX calculation itself is based on a wide range of OTM call and put options which are relatively illiquid with potentially large transaction costs. In strong contrast to that, VIX derivatives are very liquid instruments with low transaction costs. To put it more directly: The prices of VIX futures and options, which have a notional amount outstanding of several billion dollars are derived from the quotes of illiquid S&P 500 options. As a result, a relatively small number of trades in illiquid, deep OTM options can move the price of those options significantly, with corresponding and disproportionately large effect on the VIX. In addition, VIX derivatives can only be cash-settled, because the underlying commodity is the VIX itself, which is just a mathematical number, such that there is nothing to take delivery of. The cash settlement feature of VIX derivatives has the immediate consequence that any mispricings that might occur during the settlement auction are immediately converted into profit and losses for the investors. Second, the reference price for VIX futures at expiration is not the spot price of the VIX that day, but is rather a price determined by a special algorithmic auction held by Exchange (2018d). The settlement value itself is determined mostly based on open prices, which differs from the calculation methodology of the normal VIX value, which is calculated using mid-quotes. Third, for the settlement auction to determine the reference price for VIX derivatives only options with exactly 30 days to expiry are used. On the previous day the VIX calculation includes S&P 500 options with 31 days and 24 days to expiry. The series with 24 days to expiry will all disappear on the next day in the settlement calculation. Furthermore, on the settlement day, the VIX calculation (after the auction) will include S&P 500 options with 30 days and 37 days to expiry. However, for the VIX, we always have a near-term (23 to 30 days to expiry) and next-term (31 to 37 days to expiry) series of options that are used for the calculation. It is clear from this discrepancy in methodology that there will be deviations in the VIX value itself from its settlement auction value. This sheds some light on the observations that were made by other researchers that see a large jump in the VIX between the settlement value and the previous-day value. Clearly, two different methodologies are used. They differ both in the actual formula (opening prices of options vs mid-quotes) as well as the number of instruments that are used. We do not say that the CBOE should change that, we rather point out that this leads to major discrepancies between the VIX settlement value and the VIX value on the previous day as well as on the settlement day. We consider those deviations to naturally occur, whereas many academics and practitioners have raised this observation as one indication for manipulation in the VIX Griffin and Shams (2017).

9 Conclusion and summary

We have theoretically derived the VIX and identified a problem with the approximation methodology of the VIX formula. In our opinion, an unnecessary approximation is made, which leads to the fact that the true vola is underestimated. In addition, we investigated the impact of cutting the tails after two consecutive zero bids and found out that this measure also underestimates the VIX. In history, this has accounted for up to 7.5% of the value of VIX. We advise CBOE to reconsider this methodology, as it contributes to the VIX, to artificially underestimating the market vola and gives market participants further opportunities to influence the VIX to their advantage. For example, it is easier to post an order for two consecutive zero-bids to expand the tails of the options, than post orders on several zero-bid options at once. We looked at eight ways of influencing VIX using its theoretical formula. There it turned out that by aggressively posting orders, on the one hand the quote of the options can be changed and on the other hand by skilful placing of these orders, the width of the tails can be influenced. We have seen that by constructing the weights in the VIX formula deep OTM puts have a much higher weight, than options near K_0. This means that someone who wants to specifically influence the VIX will achieve the most effect in deep OTM options which are very illiquid and therefore receive little attention. We have simulated which range of options has how much influence on the VIX and found out that in the history the first ten options close to K_0 make up up to 60% of the VIX value, if you extend the option range, the additional options have less and less influence. We advise CBOE to revise the theoretical formula to obtain a more accurate representation of market volatility. Potential manipulators also have easier play based on the weight of the strikes.

We have identified three problems in the market structure of VIX derivative pricing. On the one hand, illiquid options are used to price very liquid instruments. And on the other hand, the settlement value of the VIX is determined with a different calculation method using open prices and not option quotes as usual. Exactly for the minute 8:30 a.m. only options which have exactly 30 days until expiry are used. Since an interpolation between 30 and 37 days until expiry is used at the remaining times of the day, the VIX to settlement partly deviates strongly from the actual VIX value. We would like to draw the CBOE's attention to the fact that this settlement method creates a discrepancy between the derivatives on the VIX and the VIX itself. This assures that deviations resulting from this method are directly converted into profit or loss for investors.

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List of Figures

1 Replication error of the VIX for daily close values. 12
2 Deviation in seconds, of EOD intraday timestamp with option data timestamp (03:15 p.m.). 12
3 Simulated correction-term time-series of the near-term σ^2. 17
4 Simulated sum-term time-series of the near-term σ^2. 17
5 Percentage distribution of the simulated correction- and sum-term for near-term σ^2. 18
6 Number of put and call options which are included in our VIX replication. . 18
7 Number of put and call options for the VIX replication, separated by puts and calls. 19
8 Simulated intraday number of options used in the VIX replication over four different days. Number of options separated by put and call. Associated replicated VIX on a separate y-axis. 20
9 Absolute and percentage difference of two simulated VIX indices, one using the CBOE methodology and one using a modified version without a Tailor-approximation. 22
10 Absolute and percentage difference of two simulated VIX indices, one using the CBOE methodology and one using all options after two consecutive zero-bids. 22
11 Difference between the number of put and call options used while simulating two VIX indices, one using the CBOE methodology and one including options after two consecutive zero-bids, separated by put and call options. . 23
12 How much weight has the price of an option, given a certain strike, focusing only on the distances between each strike (for a specific day) 27
13 How much weight has the price of an option, given a certain strike, focusing only on the strike value itself (for a specific day). 27
14 How much weight has the price of an option, given a certain strike, considering the strike value and the distance between strikes (for a specific day). This weight is used for calculating σ^2. 28
15 Influence of options included in the calculation of the VIX 29
16 Differences between the replicated VIX, using the CBOE methodology and the simulated indices using fewer options in the calculation. 29
17 Percentage differences between the replicated VIX, using the CBOE methodology and the simulated indices using fewer options in the calculation. . . . 30
18 Thirty days rolling correlation between the replicated VIX, using the CBOE methodology and the simulated indices using fewer options in the calculation. 30

19 Boxplot of percentage approximation errors between the replicated VIX, using the CBOE methodology and the simulated indices using fewer options in the calculation, for each number of options used. 31

List of Tables

1 Zerobid rule puts 44
2 Zerobid rule calls 44

A Appendix

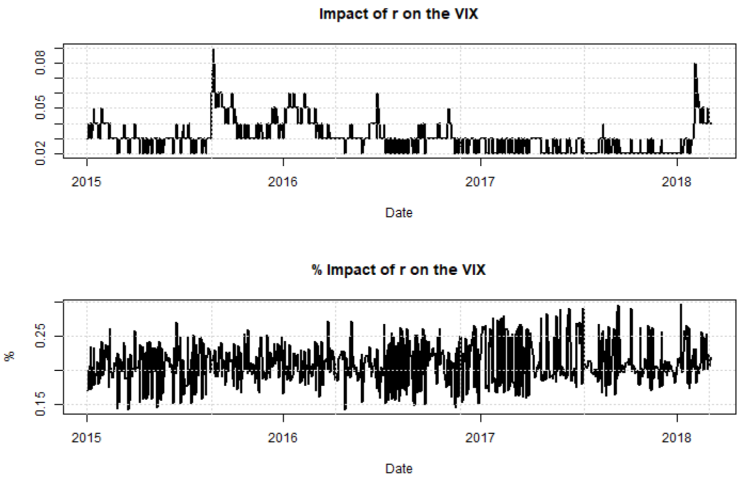

Figure 20: We here use two different interest rates (0% and 5%) and compare the impact on VIX. As we can see a difference of 5% has barely no impact (approximately 0.3%)

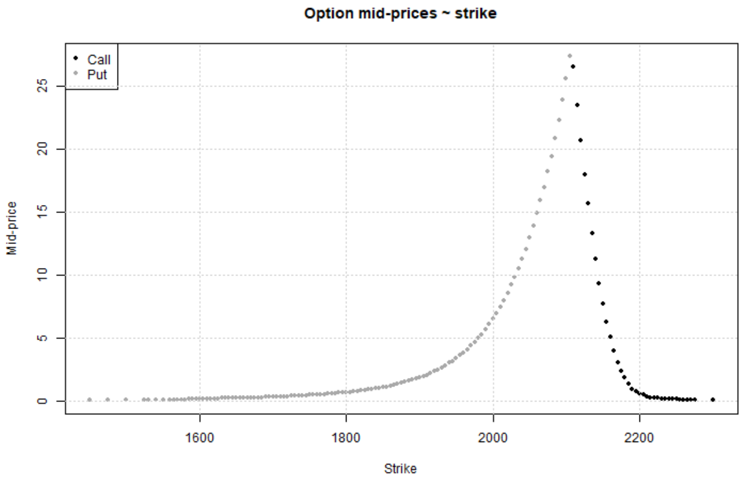

Figure 21: This Figure shows a snapshot of the OTM put and call mid prices, for a certain time point over their strike prices. The shape of the prices is the same in every time point. This means that the put prices always have wider tails than the call prices.

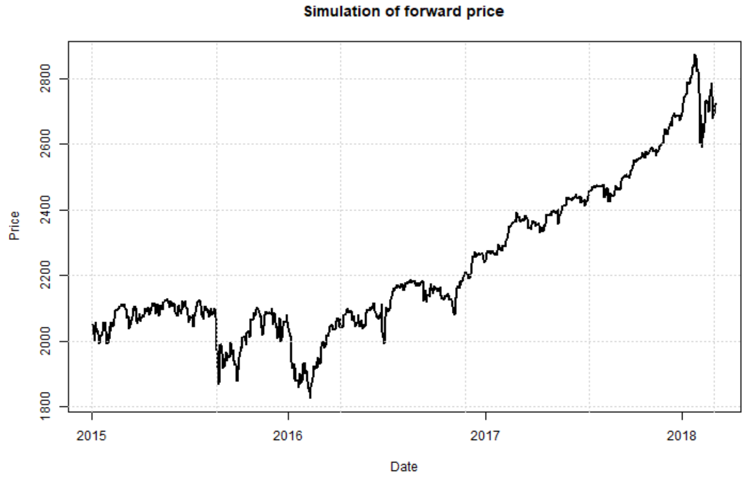

Figure 22: Here we compute the forward price of the VIX index, given historical option data.

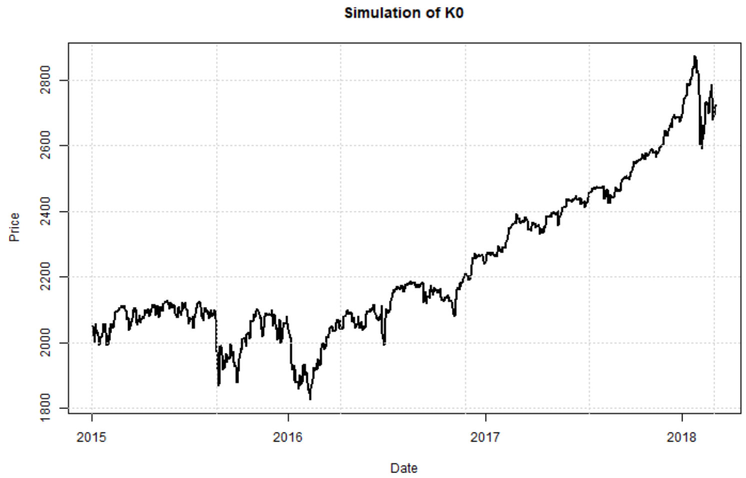

Figure 23: Here we compute the K_0 strike price of the VIX formula, given historical option data.

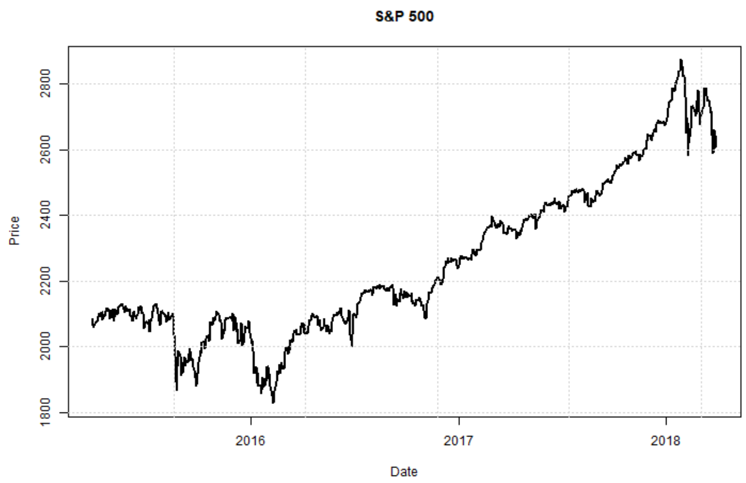

Figure 24: This Figure shows the S&P 500 over the last three years.

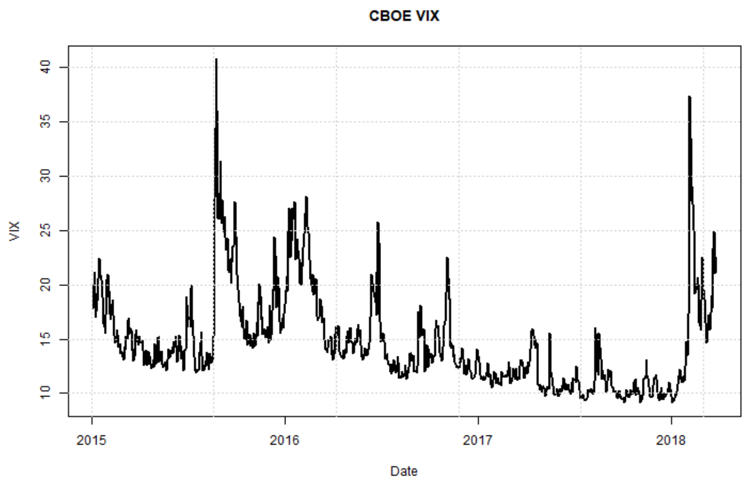

Figure 25: This Figure shows the daily close value of the VIX index over the last three years.

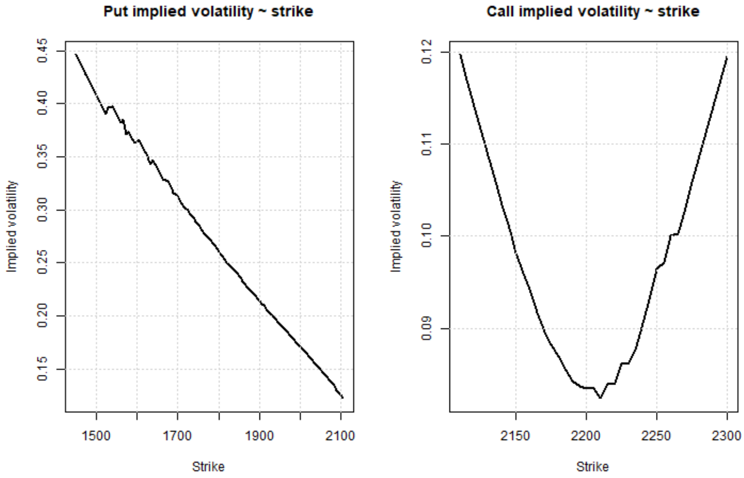

Figure 26: In this Figure the implied volatility skew and smile of puts and calls is shown, over the strikes.

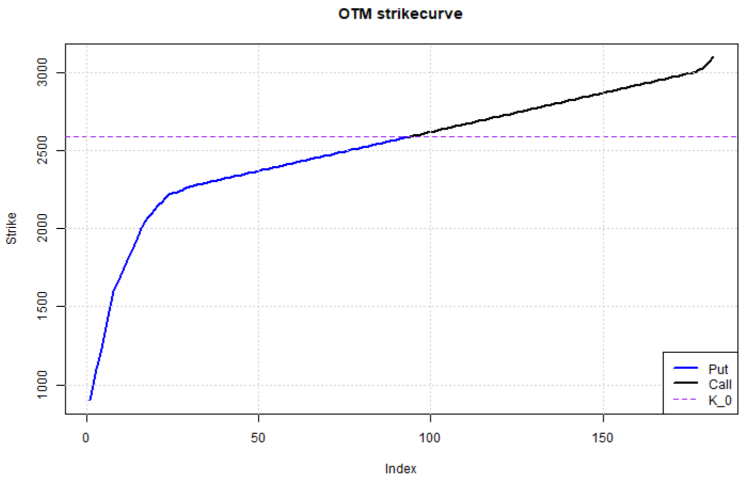

Figure 27: In this Figure a strike-curve, generated by OTM option strikes over the last years, is shown. Remarkable are the constant strike-intervals in the middle of the strike range, compared to the increasing distances in the tails of the range.

Put Strike	Bid	Ask	Include?
1345	0	0.15	Not considered following two zero bids
1350	**0.05**	**0.15**	
1355	**0.05**	**0.35**	
1360	0	0.35	No
1365	0	0.35	No
1370	0.05	0.35	Yes
1375	0.1	0.15	Yes
1380	0.1	0.2	Yes
...	...	...	...

Table 1: Zerobid rule puts

Call Strike	Bid	Ask	Include?
...	...	...	...
2095	0.05	0.35	Yes
2100	0.05	0.15	Yes
2120	0	0.15	No
2125	0.05	0.15	Yes
2150	0	0.1	No
2175	0	0.05	No
2200	0	0.05	Not considered following two zero bids
2225	**0.05**	**0.1**	
2250	0	0.05	
...	...	...	

Table 2: Zerobid rule calls