Vector autoregressive Moving Average Process

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Road Map

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1: Introduction

- Extension of Standard VAR process

- VAR(p)

$$y_t = v + A_1 y_{t-1} + \cdots + A_p y_{t-p} + \varepsilon_t$$

Where ε_t is auto correlated not a white noise

- MA(q)

$$\varepsilon_t = u_t + M_1 u_{t-1} + \cdots + M_q u_{t-p}$$

- Where u_t is zero mean white noise with nonsingular covariance matrix $\sum u$

- Combination of VAR(p) and MA(q) is VARMA (p, q)

$$y_t = v + A_1 y_{t-1} + \cdots + A_p y_{t-p} + u_t + M_1 u_{t-1} + \cdots + M_q u_{t-q}$$

2:Properties of MA Finite Process

- MA(1)

$$y_t = \mu + u_t + M_1 u_{t-1}$$

Where $y_t\ is\ = (y_{1t}, \dots, y_{Kt})'$ and u_t is zero mean white noise wit non-singular covariance matrix $\sum_u$ and $E(y_t) = \mu$, further assume that $\mu = 0$

$$u_t = y_t - M_1 u_{t-1}$$

Successive substitution we get (MA(1) to VAR(∞))

$$u_t = y_t - M_1 u_{t-1}$$
$$u_t = y_t - M_1(y_{t-1} - M_1 u_{t-2}) = y_t - M_1 y_{t-1} + {M_1}^2 u_{t-2}$$
$$= y_t - M_1 y_{t-1} + \cdots + (-M_1)^n y_{t-n} + (-M_1)^{n+1} u_{t-n-1}$$
$$= y_t + \sum_{i=1}^{\infty} (-M_1)^i y_{t-i} \,,$$

$$y_t = -\sum_{i=1}^{\infty} (-M_1)^i \, y_{t-i} + u_t$$

is VAR(∞) ,(Wold Type representation as in ch:2)

if $M_1^i \to 0 \; as \; i \to \infty$.

This requires that the eigenvalues of M1 are all less than 1 in modulus, i.e.,

$$det(I_K + M_1 z) \neq 0 \qquad for \; z \in \; \mathbb{C}, |z| \leq 1$$

condition is analogous to the stability condition for a VAR(1)

- Similarly MA(q) process can be represented as VAR(∞)

$$y_t = u_t + M_1 u_{t-1} + \cdots + M_q u_{t-q}, \qquad t = 0, \pm 1, \pm 2, \ldots ,$$

$$y_t = \sum_{i=1}^{\infty} \Pi_i \, y_{t-i} + u_t \qquad \text{VAR}(\infty)$$

If

$$\det\left(I_K + M_1 z + \cdots + M_q z^q\right) \neq 0 \qquad for\ z \in \mathbb{C}, |z| \leq 1$$

then, the MA(q) process is called invertible .

- MA(q) to VAR by using lag operator

$$y_t = (I_K + M_1 L + \cdots + M_q L^q) u_t$$

$$y_t = M(L) u_t$$

- $M(L) := I_K + M_1 L + \cdots + M_q L^q$ MA operator is invertible if it satisfies the above condition. Then we can write,

$$M(L)^{-1} y_t = u_t$$

$$M(L)^{-1} = \Pi(L) = I_K - \sum_{i=0}^{\infty} \Pi_i L^i$$

Where $\Pi_1 = M_1$

and $\Pi_i = M_i - \sum_{j=0}^{i-1} \Pi_{i-j} M_j, \quad i = 2, 3, \ldots,$

$M_j := 0 \quad for\ j > q$

These recursions used to compute the MA coefficients of a pure VAR process (ref ch2)

3: Stationarity of MA process

$E(y_t) = \mu = 0$

$\Gamma_y(h) = E(y_t y'_{t-h})$

$$= \begin{cases} \sum_{i=0}^{q-h} M_{i+h} \Sigma_u M'_i, & h = o, 1, \dots, q, \\ 0 & h = q+1, q+2, \dots, \end{cases}$$

With $M_0 := I_K$

- If h > q then the vector y_t and y_{t-h} are uncorrelated $\Gamma_y(h)$ =0
- The autocovariance $\Gamma_y(h)'$ is independent of time (stationary)

- Non-invertible MA(q) process

That violate the invertible condition and has no roots on the complex unit circle, i.e., if

$$\det\left(I_K + M_1 z + \cdots + M_q z^q\right) \neq 0 \qquad for\ |z| = 1$$

- This process can also be stationary (proof by Hannan and Deistler, 1988)

- For instance two different univariate MA(1) process below yielding same autocovariance structure.

$A)\quad y_t = u_t + m u_{t-1}$

its autocovariance is

$$E(y_t y_{t-h}) = \begin{cases} (1-m^2)\sigma_u^2 & for\ h = 0 \\ m\sigma_u^2 & for\ h = \pm 1, \\ 0 & otherwise \end{cases}$$

Where $\sigma_u^2 := Var(u_t)$

B) $y_t = v_t + \frac{1}{m} v_{t-1}$ where v_t is white noise process with same autocovariance $\sigma_v^2 := Var(v_t) = m^2 \sigma_u^2$

When $|m|$>1 , then we may choose invertible presentation

$$v_t = \left(1 + \frac{1}{m} L\right)^{-1} y_t = \sum_{i=0}^{\infty} \left(\frac{-1}{m}\right)^i y_{t-i}$$

$$= \left(1 + \frac{1}{m} L\right)^{-1} (1 + mL) u_t$$

- If $|m| = 1$ and , hence 1+mz=0 for some z on the unit circle (z=1 or -1), an invertible representation does not exist.

4: VARMA (p,q) process as Pure MA and Pure VAR

$$y_t = v + A_1 y_{t-1} + \cdots + A_p y_{t-p} + u_t + M_1 u_{t-1} + \cdots + M_q u_{t-q}, \qquad t = 0, \pm 1, \pm 2, \ldots,$$

Where u_t is zero mean white noise with non-singular covariance matrix $\sum_u$

- Let suppose MA part is denoted by ε_t

and $\varepsilon_t = u_t + M_1 u_{t-1} + \cdots + M_q u_{t-q}$

- Then VARMA (p, q) look like VAR process as,

$$y_t = v + A_1 y_{t-1} + \cdots + A_p y_{t-p} + \varepsilon_t$$

- If this process is stable, that is, if

$$\det\left(I_K - A_1 z - \cdots - A_p z^p\right) \neq 0 \qquad for\ |z| \leq 1$$

then it can be represented as MA(∞)

$$y_t = \mu + \sum_{i=0}^{\infty} D_i \varepsilon_{t-i}$$

$$= \mu + \sum_{i=0}^{\infty} D_i (u_{t-i} + M_1 u_{t-i-1} + \cdots + M_q u_{t-i-q}$$

$= \mu + \sum_{i=0}^{\infty} \Phi_i u_{t-i}$ ……….Pure MA process

The μ is $= (I_K - A_1 - \cdots - A_p)^{-1} v$

$$\sum_{i=0}^{\infty} D_i z^i = (I_K - A_1 z - \cdots - A_p z^p)^{-1}$$

The Φ_i are (K x K) matrices

$$\sum_{i=0}^{\infty} \Phi_i z^i = \left(\sum_{i=0}^{\infty} D_i z^i\right)(I_K - M_1 z - \cdots - M_q z^q)$$

VARMA (p,q) in lag operator notation as MA process

$$y_t = v + A_1 y_{t-1} + \cdots + A_p y_{t-p} + u_t + M_1 u_{t-1} + \cdots + M_q u_{t-q}$$

After using lag operator

$$A(L) y_t = v + M(L)\, u_t$$

Where $A(L) := I_K - A_1 L - \cdots - A_p L^p$ and

$M(L) := I_K + M_1 L + \cdots + M_q L^q$

Premultiplying with $A(L)^{-1}$ we get

$$y_t = A(1)^{-1} v + A(L)^{-1} M(L) u_t$$

$$y_t = \mu + \sum_{i=0}^{\infty} \Phi_i u_{t-i}$$

Hence, multiplying from the left by A(L) gives

$$(I_K - A_1 L - \cdots - A_p L^p)\left(\sum_{i=0}^{\infty} \Phi_i L^i\right)$$

$$= (I_K + \sum_{i=0}^{\infty}\left(\Phi_i - \sum_{j=1}^{i} A_j \Phi_{i-j}\right) L^i$$

$$= I_K + M_1 L + \cdots + M_q L^q$$

After comparing coefficients

$$M_i = \Phi_i - \sum_{j=1}^{i} A_j \Phi_{i-j}, \qquad i = 1,2,\ldots,$$

with $\Phi_0 := I_K$, $A_j := 0$ for $j > p$ and $M_i := 0$ for $i > q$, Rearranging gives

$$\Phi_i = M_i + \sum_{j=1}^{i} A_j \Phi_{i-j}, \qquad\qquad i = 1,2, \ldots ,$$

If MA operator M(L) satisfies the invertibility condition

$$\det\left(I_K + M_1 z + \cdots + M_q z^q\right) \neq 0 \quad for\ z \in \ \mathbb{C}, |z| \leq 1$$

Then VARMA process is called invertible then we have

- Pure VAR representation

$$\mathrm{y}_t - \sum_{i=1}^{\infty} \Pi_i \mathrm{y}_{t-i} = M(L)^{-1} A(L) \mathrm{y}_t = M(1)^{-1} v + u_t$$

Π_i matrices are obtained by comparing coefficients,

$$I_K + \sum_{i=1}^{\infty} \Pi_i L^i = M(L)^{-1} A(L)$$

Multiplying by M(L) from the left

$$(I_K + M_1 L + \cdots + M_q L^q \left(I_K - \sum_{i=1}^{\infty} \Pi_i L^i \right)$$

$$= I_K + \sum_{i=1}^{\infty} \left(M_i - \sum_{j=1}^{i} \mathrm{M}_{i-j} \Pi_j \right) L^i$$

$$= I_K - A_1 L - \cdots - A_p L^p$$

with $\mathrm{M}_0 := I_K,\ and\ M_i := 0$ for $i > q. Setting\ A_i := 0$ for $\mathrm{i} > p$ and comparing coefficients gives,

$$-A_i = M_i - \sum_{j=1}^{i-1} \mathrm{M}_{i-j}\Pi_j - \Pi_i$$

or

$$\Pi_i = A_i + M_i - \sum_{j=1}^{i-1} \mathrm{M}_{i-j}\Pi_j \qquad for\ i = 1,2,\ldots$$

if y_t is a stable and invertible VARMA process, then the pure MA representation is called the *Canonical* or *prediction error MA representation.*

$$y_t = \mu + \sum_{i=0}^{\infty} \Phi_i u_{t-i}$$

5: VAR(1) representation of a VARMA process

- Suppose y_t has the $VARMA(p,q)$ and for simplicity $\nu = 0$

$$\mathrm{Y}_t := \begin{bmatrix} y_t \\ \vdots \\ y_{t-p+1} \\ u_t \\ \vdots \\ u_{t-q+1} \end{bmatrix}, \qquad \mathrm{U}_t := \begin{bmatrix} u_t \\ 0 \\ \vdots \\ 0 \\ u_t \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} u_t \\ 0 \\ \vdots \\ 0 \end{matrix}}\right\} Kp\times 1) \\ \left.\vphantom{\begin{matrix} u_t \\ 0 \\ \vdots \\ 0 \end{matrix}}\right\} (Kq \quad \times 1) \end{matrix}$$

$$(K(p+q) \times 1)$$

$$\mathbf{A} := \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{\mathbf{12}} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \qquad [K(p+q) \times K(p+q)]$$

- With this notation we get VAR (1) process of Y_t.

$$Y_t = \boldsymbol{A} Y_{t-1} + U_t$$

$$A_{11} := \begin{bmatrix} A_1 & \cdots & A_{p-1} & A_p \\ I_K & \cdots & 0 & 0 \\ & \ddots & & \vdots \\ 0 & \cdots & I_K & 0 \end{bmatrix}_{(K_p \times K_p)} \qquad \mathbf{A}_{12} := \begin{bmatrix} M_1 & \cdots & M_{q-1} & M_q \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}_{(K_p \times K_q)}$$

$$\mathbf{A}_{21} := 0 \quad (K_q \times K_p) \qquad \mathbf{A}_{22} := \begin{bmatrix} 0 & \ldots & 0 & 0 \\ I_K & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & \ldots & I_K & 0 \end{bmatrix}_{(K_q \times K_q)}$$

- If VAR order is one, we will choose p=2 and set $A_2 = 0$
- VAR (1) process is stable if and only if y_t is stable.

$$\det\left(I_{K(p+q)} - \boldsymbol{A}z\right) = \det\left(I_{Kp} - \boldsymbol{A}_{11}z\right)\det\left(I_{Kq} - \boldsymbol{A}_{22}z\right)$$
$$= \det\left(I_K - A_1 z - \cdots - A_p z^p\right)$$

- Determinant can be found by partition Matrix (See Apendix A.10)

- # Wold type representation

If y_t hence Y_t is stable, VARMA (p,q) can be represented as MA (∞)

$$Y_t = \sum_{i=0}^{\infty} \mathbf{A}^i \, U_{t-i}$$

Premultiplying by the J matrix,

$$JY_t = \sum_{i=0}^{\infty} \boldsymbol{J}\mathbf{A}^i \, U_{t-i}$$

As $J := [I_K : 0 : \cdots 0]$

$$y_t = \sum_{i=0}^{\infty} \mathrm{J}\mathbf{A}^i \, HJU_{t-i} = \sum_{i=0}^{\infty} \mathrm{J}\mathbf{A}^i \, Hu_{t-i}$$

Where $H = \begin{bmatrix} I_K \\ 0 \\ \vdots \\ 0 \\ I_K \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $\Big\}Kp{\times}K)$ $\Big\}(Kq_{\times}K)$ and Thus $\Phi_i = J\boldsymbol{A}^i H$

$$y_t = \sum_{i=0}^{\infty} \Phi_i \, u_{t-i}$$

- Example: VARMA (1,1)

$$y_t = A_1 y_{t-1} + u_t + M_1 u_{t-1}$$

For this process

$$Y_t = \begin{bmatrix} y_t \\ u_t \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} A_1 & M_1 \\ 0 & 0 \end{bmatrix}, \quad U_t = \begin{bmatrix} u_t \\ u_t \end{bmatrix}$$

$$\mathrm{J} = [I_K : 0] \quad (K \times 2K) \qquad\qquad H = \begin{bmatrix} I_K \\ I_K \end{bmatrix} \ (2K \times K)$$

$$\Phi_0 = JH = I_K,$$
$$\Phi_1 = J\mathbf{A}H = [A_1 : M_1]H = A_1 + M_1,$$
$$\Phi_2 = J\mathbf{A}^2 H = J\begin{bmatrix} A_1^2 & A_1 M_1 \\ 0 & 0 \end{bmatrix} H = A_1^2 + A_1 M_1,$$

$$\vdots$$

$$\Phi_i = J\mathbf{A}^i H = J\begin{bmatrix} A_1^i & A_1^{i-1} M_1 \\ 0 & 0 \end{bmatrix} H = A_1^i + A_1^{i-1} M_1, \qquad i = 0,1,2,\ldots.$$

6: Autocovariance and Autocorrelation of VARMA (p,q)

The *K*-dimensional, zero mean, stable VARMA(*p, q*) process is

$$y_t = A_1\, y_{t-1} + \cdots + A_p y_{t-p} + u_t + M_1 u_{t-1} + \cdots + M_q\, u_{t-q}$$

- the autocavariance can be obtained by Pure MA representation
- If y_t has the canonical MA representation as

$$y_t = \sum_{i=0}^{\infty} \Phi_i\, u_{t-i}$$

The autocovariance are

$$\Gamma_y(h) := E(y_t y'_{t-h}) = \sum_{i=0}^{q-h} \Phi_{h+i} \Sigma_u \Phi'_i,$$

Alternative Method: The more convenient process is multiplying VARMA process by y'_{t-h} and taking expectations gives

$$E(y_t y'_{t-h}) = A_1 E(y_{t-1} y'_{t-h}) + \cdots + A_p E\left(y_{t-p} y'_{t-h}\right)$$

$$+E(u_t y'_{t-h}) + \cdots + M_q E(u_{t-q} y'_{t-h})$$

From Pure MA representation, It can be seen that E($u_t y'_s$) = 0 for $s < t$.

Hence we get $h > q$

$$\Gamma_y(h) = A_1 \Gamma_y(h-1) + \ldots + A_p \Gamma_y(h-p)$$

If $h = 0$ *(initial matrix)*, covariance matrix of VAR(1) process is

$$\Gamma_y(0) = \mathbf{A}\Gamma_y(0)\mathbf{A}' + \Sigma_U$$

Here$\Sigma_U = E(U_t U_t')$ is the covariance matrix of white noise.

Applying the *vec* operator

$$\text{vec}\,\Gamma_y(0) = (I_{K^2(p+q)^2} - \mathbf{A}\otimes\mathbf{A})^{-1}\,\text{vec}(\Sigma_U)$$

- Inverse only exist when $(\mathrm{I} - \mathbf{A}\otimes\mathbf{A})$ is stable
- First evaluate $\Gamma_y(0)$ from given $\mathbf{A}$ and Σ_U
- Recursion only valid for $h > q$
- Computation of autocovariance requires that $p > q$.
- If not we will add one more lag of y_t by taking zero coefficient matrix until VAR(p) is greater then MA(q)

Example:

Considering VARMA(1,1) process as $p = q$ so we are adding second lag of y_t

$$y_t = A_1 y_{t-1} + A_2 y_{t-2} + u_t + M_1 u_{t-1}$$

With $A_2 := 0$, now

$$Y_t = \begin{bmatrix} y_t \\ y_{t-1} \\ u_t \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} A_1 & 0 & M_1 \\ I_K & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$U_t = \begin{bmatrix} u_t \\ 0 \\ u_t \end{bmatrix}, \qquad \Sigma_U = \begin{bmatrix} \Sigma_u & 0 & \Sigma_u \\ 0 & 0 & 0 \\ \Sigma_u & 0 & \Sigma_u \end{bmatrix}$$

$$\text{vec}(\Gamma_y(0)) = (I_{9K^2} - \mathbf{A} \otimes \mathbf{A})^{-1} \, \text{vec}(\Sigma_U)$$

By having the starting up matrices, the recursion may be applied,

$$\Gamma_y(h) = A_1 \Gamma_y(h-1) \qquad \text{for } h = 2,3, \ldots$$

Autocorrelations of VARMA(p,q) process is

$$R_y(h) = D^{-1} \Gamma_y(h) D^{-1}$$

- Where D is the diagonal Matrix with the square roots from $\Gamma_y(0)$ on the main Diagonal.

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Thank You