Foundations of a pathwise volatility framework with explicit fast reversion limits

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Abstract

Our first main results are the existence and uniqueness of solutions to a 'generalised CIR initial-value problem', which originates from the classical Cox-Ingersoll-Ross (CIR) stochastic differential equation after a 'time-change'. These results hold on a pathwise basis for *any* continuous driver, and solutions are shown to accommodate *any* practicable continuous volatility path. We then prove a certain 'fast reversion' pathwise convergence of our solutions to a hitting-time limit on the (D, M) topologies of Skorokhod (1956), thereby also accommodating discontinuous paths. In the classical Wiener setting, this limit is an inverse Gaussian Lévy process. By continuous mapping, analogous results follow for any functional, such as a derivative payoff or price, of our volatility paths. As an example, the classical Heston log-price limit is shown to be *very* closely related, but not equivalent, to the normal-inverse Gaussian Lévy process. This exposes the origin of the marginal connection between these two processes revealed in Mechkov (2015).

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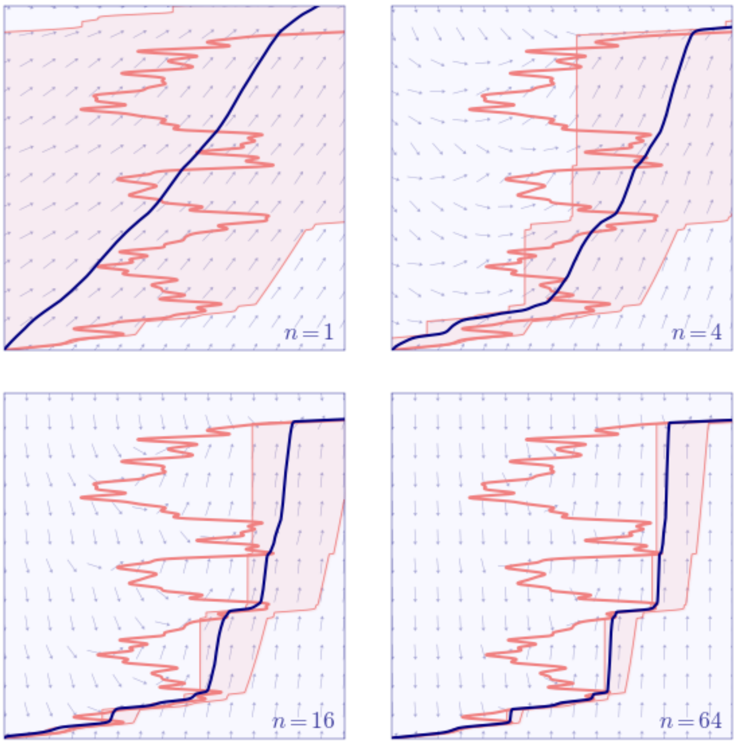
Here we exhibit the pathwise convergence of a cumulative volatility trajectory (blue) to a hitting-time limit, as the reversionary timescale $:= n^{-1}$ is varied from a year to a week.



The blue line 'gets sucked' to the red one...

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1 Introduction

Motivation. This work was motivated in part by a desire to find a satisfactory 'origin' of the following surprising result. Let $\mathrm{IG}(\delta,\gamma)$ denote an inverse Gaussian (IG) random variable, with characteristic function $\hat{\phi}:\mathbb{R}\to\mathbb{C}$ that satisfies

$$\log\hat{\phi}(u)=(\gamma-\sqrt{\gamma^2-2iu})\delta.$$

Proposition 1.1 (Marginal integrated CIR fast reversion convergence)**.** *Let the family $\{X_n\}_{n>0}$ of processes solve the CIR Itô SDEs*

$$\mathrm{d}X_t^n=n\left(\sqrt{X_t^n}\mathrm{d}W_t+(1-X_t^n)\mathrm{d}t\right),\quad X_0^n=X_0>0.$$

Then for any $t>0$, we have $\int_0^t X_u^n\mathrm{d}u=:\bar{X}_t^n\xrightarrow{d}\bar{X}_t\overset{d}{=}\mathrm{IG}(t,1)$ as $n\to\infty$.

Proof. This follows convergence of characteristic functions; sketched in the appendix along with an alternative proof of the related result due to Mechkov (2015). □

We will elaborate on *why* we consider this pursuit justified shortly, but, needless to say, the CIR process is *very* popular in mathematical finance, originally for the modelling of interest rates; Cox, Ingersoll and Ross (1985), and now as much volatility; Heston (1993). Indeed, what partly makes the above result surprising is that it is not widely known, if at all, yet undergraduates could certainly prove it.

Readers of 'standard' mathematical finance, please be warned, because the path we must take to this origin passes through ODE theory and several function topologies, prior to the reintroduction of a probability measure. Along this path, we believe our most useful references have been Coddington and Levinson (1955) and Whitt (2002).

Main results. We first state what we consider to be our main results, followed by a corollary that generalises Proposition 1.1. Let $C=C([0,\infty),\mathbb{R})$ denote the space of real continuous paths originating from 0.

Theorem 1.2 (Functional generalised CIR properties)**.** *For any $\omega\in C$, $n,c>0$, the generalised CIR initial-value problem, defined by*

$$\varphi_n'(t)=f_n\big(t,\varphi_n(t)\big),\quad f_n(t,x):=n\Big(\omega(x)+t-x\Big)+c,\tag{1.1}$$

admits a unique bijective solution $\varphi_n:[0,T_n)\to[0,\infty)$, where $T_n\in(0,\infty]$ satisfies

$$T_n=\sup\Big\{t>0:\varphi_n(t)<\infty\Big\}>0\vee\sup\Big\{x-\omega(x):x\geq 0\Big\}-c/n.$$

Moreover, all differentiable and bijective paths $\varphi_n:[0,T_n)\to[0,\infty)$ solve this problem for some $\omega\in C$.

Now let $D = D([0,T),\mathbb{R})$ denote the space of real càdlàg paths up to $T \in (0,\infty]$, and (D,M) denote the (D,M_1') topology introduced in Puhalskii and Whitt (1997); a minor extension of (D,M_1) due to Skorokhod (1956).

Theorem 1.3 (Functional generalised CIR convergence)**.** *The family $\{\hat{\varphi}_n\}_{n>0}$ of inverse solutions $\hat{\varphi}_n := \varphi_n^{-1}$ converges uniformly to a limit $\hat{\varphi} \in C$ as $n \to \infty$, and the family $\{\varphi_n\}_{n>0}$ converges on the topology (D,M) to a limit $\varphi \in D$, where*

$$\hat{\varphi}(t) := \max\Big\{s - \omega(s) : s \leq t\Big\}, \quad \varphi(t) := \inf\Big\{s > 0 : s - \omega(s) > t\Big\}.$$

In addition, the families $\{\omega \circ \varphi_n\}_{n>0}$ and $\{n^{-1}\varphi_n'\}_{n>0}$ implicitly present in (1.1) have well-defined limits, but typically not in (D,M).

The following application of these results exposes the precise origin of Proposition 1.1. We would like to additionally summarise a functional generalisation of the Heston and normal-inverse Gaussian (NIG) connection due to Mechkov (2015), but we prove in Section 4.2 that one actually does not exist, requiring the introduction of 'Whitt's excursion topology' from Whitt (2002). Now let $\mathscr{C} := \sigma(C)$ be the σ-field induced by the uniform norm $\|\cdot\|$ and Z the canonical process on $(C,\mathscr{C},\mathbb{P})$ for any probability measure $\mathbb{P}$.

Corollary 1.4 (Pathwise generalised CIR convergence)**.** *For $n, X_0 > 0$, the generalised CIR time-change equation*

$$X_t^n = n\bigg(\big(Z \circ \bar{X}_n\big)_t + \int_0^t (1 - X_u^n)\mathrm{d}u\bigg) + X_0,$$

where $\bar{X}_t^n := \int_0^t X_u^n \mathrm{d}u$ and $(Z \circ \bar{X}_n)_t = Z_{\bar{X}_t^n}$, admits a solution for any $Z(\omega) \in C$, with all of the above results for φ_n applying to the process $\bar{X}_n$. This means we have $\bar{X}_n \to \bar{X}$ as $n \to \infty$ for any $\mathbb{P}$, where $\bar{X}$ is the fast reversion hitting-time limit defined by

$$\bar{X}_t := \inf\Big\{s > 0 : s - Z_s > t\Big\}.$$

In the classical setting where $\mathbb{P}$ is the Wiener measure making Z Brownian motion, then $\bar{X}$ is the IG Lévy process of Applebaum (2009).

Of course, there is an analagous corollary of this result (and all others) applicable to the classical CIR process, driven by an Itô integral. However, this is clearly not amenable to a pathwise result on the entire space C (equivalently any measure $\mathbb{P}$), given that the Itô integral has no meaning on a pathwise basis. We therefore leave such a specific application for future work and are content for now with the immediately consequential weak results in these regards.

A few comments on the generality of these results are in order, as this work is not just about a time-changed CIR process. Firstly, their applicability to *any* positive, differentiable and bijective path φ_n essentially means that they apply to *any* continuous volatility path, thus model, specified by the choice of $\mathbb{P}$. The following collection of known results, which may be found in Skorokhod (1956) and Billingsley (1999), then summarises the further scope for applications; pathwise or probabilistic.

Proposition 1.5 (Continuous mapping)**.** *Let $h : (D, M) \to (S, m)$ and $\Phi : (S, m) \to (\mathbb{C}, \|\cdot\|)$ be continuous for some metric space (S, m). Then, $h(\varphi_n) \to h(\varphi)$ on (S, m) and $\Phi(h(\varphi_n)) \to \Phi(h(\varphi))$ on $(\mathbb{C}, \|\cdot\|)$. If $\Phi \circ h$ is dominated by a $\mathbb{P}$-integrable function, then $\mathbb{E}[\Phi(h(\bar{X}_n))] \to \mathbb{E}[\Phi(h(\bar{X}))]$. The same conclusion follows if $\Phi \circ h$ is only continuous almost everywhere, with respect to the limit distribution $\mathbb{P}(\Phi \circ h)^{-1}$ on $(\mathbb{C}, \|\cdot\|)$.*

As just one example, we may think of h here as defining a 'model' like Heston's and Φ a 'derivative payoff' under some $\mathbb{Q}$ equivalent to $\mathbb{P}$.

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Background. We should point out that we thought a satisfactory origin of our problem *would* be found within Itô calculus. However, a change of perspective on the CIR SDE proves necessary to obtain anything more than weak results, and this turns out to enable the complete generalisation to our pathwise framework.

The central part of this work thus relates to deterministic solutions $\{\varphi_n\}_{n>0}$ of the additively-separable initial-value problems (IVP)

$$\varphi_n'(t) = f_n\big(t, \varphi_n(t)\big), \quad f_n(t, x) = a_n(x) + b_n(t) + c_n, \quad t \in [0, T_n) \tag{1.2}$$

for some time horizon $T_n \in (0, \infty]$. These may be interpreted as pathwise expressions of Itô SDEs, after performing the Brownian motion-preserving time change, due to Dambis (1965) and Dubins and Schwarz (1965) (DDS). This DDS result is really the only fragment of Itô calculus that truly remains in this work.

We consider (1.2) with $a_n = n(\omega - \mathrm{e})$, $b_n = ne$ and $c_n = c > 0$, where $\omega \in C$ and $\mathrm{e} \neq e$ is the identity in C, to define a generalised CIR (GCIR) IVP, due to its relationship with the CIR SDE, which we clarify in the next section. So we henceforth define f_n by

$$f_n(t, x) = n\bigg(\omega(x) + t - x\bigg) + c.$$

where ω characterises a trajectory of 'cumulative noise'. We are especially interested in a 'fast reversion' limit of this problem, as is introduced in Mechkov (2015), and we define here by $n \to \infty$, .

Theoretically, our main reason for being interested in this limit comes from certain representations of infinite-dimensional processes, such as fractional Brownian motion, with low Hölder regularities, which are proving successful in 'rough' volatility modelling. These representations reveal a correspondence with unbounded reversion speeds. See Muravlev (2011) and Abi Jaber and El Euch (2018) for examples, and Bayer, Friz and Gatheral (2015) for an introduction to rough volatility. We refer to the calibration results for 'classical' models of volatility shared in De Col, Gnoatto and Grasselli (2013) and Bergomi (2016) for evidence of high reversion in practice. Informal comparisons between rough and fast-reverting processes are also available at bit.ly/2CXwzJT, particularly slides 7–9.

Outline of work. In establishing the main results above, we rely on the intuitive fact that any solution $\varphi_n \in C_1$, when considered a subset of $[0, \infty) \times \mathbb{R}$, cannot violate being tangent to the field $F_n := (1, f_n)$, which for us is continuous on its entire domain. So, given any closed Γ_n with $0 \in \Gamma_n \subset [0, \infty) \times \mathbb{R}$, a solution φ_n originating from 0 and existing over any $[0, \epsilon)$ may be 'continued' to a point of the boundary $\partial\Gamma_n$ at which (perturbations of) F_n points 'out' of Γ_n. For example, notice that any right-most point of $\partial\Gamma_n$ always provides such a point.

This intuitive fact essentially provides uniqueness, as well as existence, of solutions, because one finds that any existing solution φ_n, may itself be considered such a region Γ_n, with $\partial\Gamma_n = \Gamma_n$! This is equivalent to saying that our solutions are stable, in the sense that perturbations from them violate being tangent to F_n, which instead points back to whence φ_n came. Notice that this is not the case for the common counterexample to uniqueness; $\varphi' = \varphi^\alpha$, $\alpha \in (0, 1)$ on $[0, \infty)^2$, for which the associated field $(1, x^\alpha)$ allows perturbations from the solution $\varphi = 0$ to grow.

Additionally, this line of reasoning provides both upper and lower uniform bounds on, unusually, the inverse functions $\hat{\varphi}_n = \varphi_n^{-1}$. These bounds converge as $n \to \infty$, so lead to the result $\hat{\varphi}_n \to \hat{\varphi}$ on $(C, \|\cdot\|)$. This result maps to our main fast reversion result $\varphi_n \to \varphi$, which takes place on the (D, M) topology, which is summarised in Whitt (2002). This certainly appears to us a rare type of functional convergence, as most practically useful choices of $\omega \in C$ lead to a discontinuous limit $\varphi \in D \setminus C$, despite the differentiable family $\{\varphi_n\}_{n>0}$. Graphical examples of this result are shared in Section 3.4, and python code is available in a standalone jupyter notebook at bit.ly/2Q96GtF, to produce your own.

Once these existence, uniqueness and convergence results are obtained, a limitless set of corollaries follow, both deterministic and probabilistic. Not wanting to be sidetracked and overwhelmed by these, we focus properly on just one in Section 4, where we work towards surprising results relating to a generalised Heston model, before concluding.

2 Preparing for pathwise results

In this section we prepare our problem to enable its pathwise generalisation, which demands the DDS reparameterisation of the classical CIR process. We also summarise the applicable functional topologies on which our main convergence results take place, the best reference for which is Whitt (2002).

As in the introduction, let $Z = \{Z_t\}_{t\geq 0}$ denote the canonical stochastic process on the space $(C, \mathscr{C}, \mathbb{P})$, specified for $\omega \in C$ by $Z_t(\omega) = \omega(t)$. When we use $Z = W$, then the Wiener measure $\mathbb{P} = \mathbb{W}$ is selected on $\mathscr{C}$, so we are working on $(C, \mathscr{C}, \mathbb{W})$, where W is Brownian motion.

2.1 Time-changing the CIR process

For $n > 0$, let CIR stochastic processes $X_n = \{X_t^n\}_{t\geq 0}$ be defined as the solution of the Itô SDEs

$$\mathrm{d}X_t^n = n\Big(\sqrt{X_t^n}\mathrm{d}W_t + (1 - X_t^n)\mathrm{d}t\Big), \quad X_0^n = X_0 > 0. \tag{2.1}$$

These are known to almost surely exist and be unique in L_1 by the results of Skorokhod (1965) and Yamada and Watanabe (1971). We consider here a single-parameter SDE, as generalisations may be accommodated by scalar transformations of W and X_n.

Define local martingales M_n by the components $M_t^n = \int_0^t \sqrt{X_u^n}\mathrm{d}W_u$ that are implicitly present in (2.1). Then the DDS theorem provides that processes B_n, each defined by

$$B_n = M_n \circ \tau_n, \quad \tau_t^n = \min\Big\{s \geq 0 : \bar{X}_s^n = t\Big\}, \quad \bar{X}_t^n := \int_0^t X_u^n \mathrm{d}u$$

constitute (non-canonical) Brownian motions on $(C, \mathscr{C}, \mathbb{W})$. Furthermore, these almost surely verify $M_n = B_n \circ \bar{X}_n$. See Karatzas and Shreve (1988) for details.

Solutions of (2.1) are therefore almost surely equivalent to those of a corresponding integral equation with the components $\int_0^t \sqrt{X_u^n}\mathrm{d}W_u$ replaced by $(B_n \circ \bar{X}_n)_t$. We call the resulting integral equation a time-change equation (TCE), since the DDS theorem essentially redefines the time index. Solutions of this TCE are weakly equivalent to those of the same equation driven instead by the canonical Brownian motion W.

Instead of using (2.1), in this work we define weakly-equivalent CIR stochastic processes by solutions of the CIR TCE

$$X_t^n = n\Big((W \circ \bar{X}_n)_t + \int_0^t (1 - X_u^n)\mathrm{d}u\Big) + X_0 \tag{2.2}$$

originating from $(C, \mathscr{C}, \mathbb{W})$, and generalised CIR (GCIR) stochastic processes equivalently, but originating from any other space $(C, \mathscr{C}, \mathbb{P})$ with $\mathbb{P} \neq \mathbb{W}$.

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Notice that in the case of $\mathbb{P} = \mathbb{W}$, any conclusion drawn from (2.2) may be mapped back to an almost sure one for the classical SDE (2.1), driven instead by B_n, provided $(W \circ \bar{X}_n)_t = \int_0^t \sqrt{X_u^n} \mathrm{d}B_u^n$ holds almost surely. Doing this is not our present focus, however, as we mean to place equal emphasis on cases where $\mathbb{P} \neq \mathbb{W}$, when the stochastic integral $\int_0^t \sqrt{X_u^n} \mathrm{d}B_u^n$ generally has no meaning.

To elaborate on this point a little, suppose we naïvely 'solve' the integral equation $(W \circ \bar{X}_n)_t = \int_0^t \sqrt{X_u^n} \mathrm{d}B_u^n$ for

$$B_t^n = \int_0^t \frac{\mathrm{d}(W \circ \bar{X}_n)_u}{\sqrt{X_u^n}} = n^{-1} \int_0^t \frac{1}{\sqrt{X_u^n}} \mathrm{d}X_u^n - \int_0^t \frac{1 - X_u^n}{\sqrt{X_u^n}} \mathrm{d}u$$

and so might claim to understand the space $(C, \mathscr{C}, \mathbb{W}_n)$ for some Wiener measure $\mathbb{W}_n := \mathbb{W} B_n^{-1}$ on which B_n is Brownian motion. Then, although this would lead to some almost sure properties for classical CIR SDE solutions, unfortunately, we are still required to make meaning of B_n obtained here as $n \to \infty$. Although interesting, we do not attempt to do this yet, as this is somewhat tangential to the pathwise, probability-free, approach we otherwise take. So, in the classical setting out results and corollories remain weak ones.

Before we dispose of the measure $\mathbb{P}$, it is worth noting that if the canonical process Z driving (2.2) is 'rough', then we of course have ourselves a new model of rough volatility. However, it is not yet clear how helpful this type of model would be in practice *in comparison* to the use of a more convenient driver and the fast reversion regime. Recall our introductory discussion relating the two. We would suggest that for most applications, the benefits of defining $\mathbb{P}$ to be a distribution of fractional Brownian motion, for example, would fall short of the analytical and computational complications that comes with this, but this remains to be verified.

2.2 Working *sans probabilités*

In a manner related mildy to Hans Föllmer's *Calcul d'Itô sans probabilités*, now consider (2.2) without a probability measure. We are left on the topological space $(C, \mathscr{C})$, identified more precisely as the Polish normed vector space $(C, \|\cdot\|)$. Fixing any $\omega \in C$, under the correspondences

$$X_n(\omega) = \varphi_n', \quad \bar{X}_n(\omega) = \varphi_n, \quad X_0 = c, \quad ' = \partial.$$

one arrives at the IVP in Theorem 1.2

$$\varphi_n'(t) = f_n\big(t, \varphi_n(t)\big), \quad f_n(t,x) = n\bigg(\omega(x) + t - x\bigg) + c, \quad t \in [0, T_n) \tag{2.3}$$

for $\varphi_n \in C_1$ and some $T_n \in (0, \infty]$. We consider it helpful to keep the connection with the GCIR TCE in mind when working with this IVP, and to a lesser extent the connection of the TCE with the SDE.

The analysis of this IVP is our primary focus *without* additional assumptions on ω, such as its Lipschitz or more generally Hölder continuity. As a starting point, notice that existence and uniqueness of solutions must be addressed, as for (2.2) with any $\mathbb{P} \neq \mathbb{W}$. We spend much time deriving other functional properties of the GCIR family $\{\varphi_n\}_{n>0} \subset C_1$ of paths, the derivatives φ_n' of which identify with 'pathwise volatility' trajectories. Not being tied to a specific probability measure, the majority of our results apply to any, and so any corresponding family $\{\bar{X}_n\}_{n>0}$ (and functionals thereof) of GCIR processes.

As we have suggested, no limit φ of $\{\varphi_n\}_{n>0}$ exists in C as $n \to \infty$, let alone C_1, for any practicable choice of $\omega \in C$. Rather, one finds a limit most naturally in D, the space of càdlàg paths. Next we therefore cover some topological details, required to appreciate the notion of distance implicit in a statement like $\varphi_n \to \varphi$ as $n \to \infty$, for some $\varphi \in D$.

2.3 The relevant topologies

Skorokhod's popular J_1 metric, introduced in Skorokhod (1956), is generally considered to embody a view that measuring distances in time is similar to those in space, rather than being materially more or less difficult. Although this view seems a helpful one, that the J_1 metric best embodies it does appear to expose a lack of appreciation for the particulars of it. It is easy to write this from our position, because a result $\varphi_n \to \varphi$ for our problem cannot be found on this 'standard' topology on D, yet our consideration of space and time as equal proves necessary.

Whitt (2002) provides an excellent introduction to non-standard topologies on D. There it is also proved that the (D, M_1) topology, may be induced by a Polish modification of the original metric defined in Skorokhod (1956). This is analogous to how Billingsley (1999) establishes the same for (D, J_1), and would allow a straightforward application of Prokhorov's theorem, of Prokhorov (1956), popular for concluding weak convergence on (D, J_1).

The relevant topology for us is a natural extension of (D, M_1) labelled (D, M_1'), first introduced in Puhalskii and Whitt (1997) to alleviate problems at the origin, otherwise avoided by working with paths over some $(0, T)$. We will simply refer to this as (D, M). Due to the path we take, we will not need to get dirty with the innards of this topology; we just require the continuity of a certain functional $R : (C, \|\cdot\|) \to (D, M)$, which is not the case onto (D, M_1), let alone (D, J_1).

The functional R is a right inverse, 'dual' to a maximal functional M using the terminology of Whitt (1971), each defined by

$$M(\omega)(t) = \max\left\{\omega(s) : s \in [0, t]\right\}, \quad R(\omega)(t) = \inf\left\{s > 0 : \omega(s) > t\right\}$$

for $\omega \in C$, $t \in [0, T)$ and assuming $\inf \varnothing = \infty$. That R is continuous as claimed is proved in Puhalskii and Whitt (1997).

So as to not jump too far ahead, let us briefly remark that the IG process $\bar{X} = \{\bar{X}_t\}_{t\geq 0}$ which appears in Proposition 1.1 admits a hitting-time representation $\bar{X} = R(\mathrm{e} - W)$ in terms of Brownian motion W. This is more commonly written out as something like

$$\bar{X}_t = \inf\left\{s > 0 : s - W_s > t\right\} \stackrel{d}{=} \mathrm{IG}(t, 1),$$

as in Applebaum (2009).

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Now we summarise the (D, M) topology for intuitive purposes by defining the standard metric from which it may be induced (not its more complicated Polish modification). We simply denote this metric by d, because we will not introduce any others explicitly. Put very tersely, this metric may be understood as a uniform metric 'on the space of parametric representations of the extended graphs of the elements of D'. What follows is hopefully clearer.

Definition 2.1 (M metric on D)**.** For $x \in D$, let its graph $\Gamma_t(x) \subset [0, t] \times \mathbb{R}$ up to $t \in [0, T)$ be defined by

$$\Gamma_t(x) = \left\{(u, v) \in [0, t] \times \mathbb{R} : v = (1 - \alpha)x(u_-) + \alpha x(u),\ \alpha \in [0, 1]\right\},$$

with the convention $x(0_-) = 0$. The set $\Gamma_t(x)$ is ordered 'from end to end' by letting $(u_1, v_1) \leq (u_2, v_2)$ if $u_1 < u_2$ or $u_1 = u_2$ and $|v_1 - x(u_{1-})| \leq |v_2 - x(u_{2-})|$. Now let $\Pi_t(x)$ denote the set of parametric representations of $\Gamma_t(x)$; these are continuous functions (p, q), with p mapping $[0, 1]$ onto the first component of $\Gamma_t(x)$ and q the second, which are non-decreasing under this ordering. A restricted metric between $x_1, x_2 \in D$ up to $t \in (0, T)$ is defined by

$$d_t(x_1, x_2) = \inf\left\{\|p_2 - p_1\| \vee \|q_2 - q_1\| : (p_i, q_i) \in \Pi_t(x_i),\ i = 1, 2\right\}$$

and the M metric may then be specified by

$$d(x_1, x_2) = \int_0^T e^{-t}\big(d_t(x_1, x_2) \wedge 1\big)\mathrm{d}t.$$

We define the M metric using an integral of this form in order to naturally accommodate any $T \in (0, \infty]$. That d should define a metric or even exist at all is not trivial, but details are provided in Whitt (2002).

Our main convergence result takes place on this (D, M) topology, but using Proposition 1.5 may be mapped to any other, (S, m). In the Heston example that we cover, we are required to understand what we call 'Whitt's excursion topology', introduced and denoted by (E, M_2) in Whitt (2002). We will label this (D, H), to remind us that we are required to relax the metric to the Hausdorff one, but we cover this when it is required in Section 4, since it is rather unconventional.

3 Properties of generalised CIR solutions

As remarked in the previous section, if one is only interested in the classical case of the TCE (2.2), with $\mathbb{P} = \mathbb{W}$ making W Brownian motion, then the classical existence result of Skorokhod (1965) applies, alongside the DDS time-change. Uniqueness of solutions turns out to have no influence on our limits, although is provided in L_1 under $\mathbb{W}$ by Yamada and Watanabe (1971).

In this section we establish that the GCIR IVP (2.3) actually admits a unique solution for any driving $\omega \in C$, so too the GCIR TCE (2.2) for any $\mathbb{P} \neq \mathbb{W}$. For example, let $\mathbb{P}$ be a measure which makes the canonical process on $(C, \mathscr{C}, \mathbb{P})$ driving (2.2) fractional Brownian motion; this leads to one of an infinitude of volatility models which our results cover.

Until the convergence result of Section 3.4, we consider $n > 0$ as fixed, so drop its use in subscripts. This aligns notation with Coddington and Levinson (1955). A solution φ therefore refers to φ_n otherwise, and should not be confused with the limit $\varphi_n \to \varphi$ there. We also cover forward Euler scheme convergence for fixed $n > 0$ in this section; not to be confused with 'fast reversion' $n \to \infty$ convergence elsewhere!

For $\omega \in C$ and $c > 0$, we are therefore faced with the problem of finding some $\varphi \in C_1$ and $T \in (0, \infty]$ such that

$$\varphi'(t) = f\big(t, \varphi(t)\big), \quad f(t, x) = n\bigg(\omega(x) + t - x\bigg) + c, \quad t \in [0, T). \tag{3.1}$$

Before we address the existence of such φ, it helps to formalise the intuitive notion, from the introduction, that valid trajectories of φ cannot 'cross' (that is, cannot violate being tangent to) the vector field $F := (1, f)$, defined on $[0, \infty) \times \mathbb{R}$. This will help us be considerably more efficient in what follows.

Proposition 3.1 (Conditions for a solution bound)**.** *Let $\varphi \in C_1$ solve* (3.1)*. Suppose $\gamma \in D$ is non-decreasing, that $\gamma(0) > 0$ and $f(t, \gamma(t)) < 0$ for $t \in [0, T)$. Then, $\varphi < \gamma$ over $[0, T)$.*

Proof. This essentially follows a couple of applications of the intermediate value theorem. $\gamma(0) > 0$ ensures γ starts above φ. As $\gamma \in D$ is non-decreasing, φ cannot exceed γ without a point of equality. Let $t' > 0$ be the first such point, so $\varphi(t') = \gamma(t')$. Then $\varphi'(t') = f(t', \gamma(t')) < 0$, so there exists some $t'' < t'$ such that $\varphi(t'') > \varphi(t')$. Since γ is non-decreasing, this would imply $\varphi(t'') > \gamma(t'')$, and so the existence of another point $t''' < t''$ such that $\varphi(t''') = \gamma(t''')$. This contradicts the defined properties of t', therefore its existence. □

Remark 3.2. This result may be adapted in two ways. Firstly, if instead $f(t, \gamma(t)) = 0$ for $t \in [0, T)$, but there exists some x such that $f(t, \gamma(t) + x') < 0$ for all $x' \in (0, x)$ and $t \in [0, T)$, then the weaker $\varphi \leq \gamma$ conclusion holds. Secondly, reversing the assumptions, so that γ is non-increasing and $f(t, \gamma(t)) \geq 0$, leads to the opposite result $\varphi \geq \gamma$. In this case, $\gamma(0) = 0$ becomes acceptable, since $\varphi'(0) = c > 0$.

The basic assumptions used in these results allows us to establish the first necessary properties of solutions $\varphi \in C_1$.

Proposition 3.3 (Injectivity and first bounds)**.** *Let $\varphi \in C_1$ solve* (3.1)*. Then φ is injective and satisfies $0 \leq \varphi \leq \gamma$ on $[0, T)$, where $\gamma := R(\mathrm{e} - \omega - c/n) \in D$.*

Proof. The curve 0 satisfies the reverse properties of Proposition 3.1, mentioned in the remark above, since $f(t, 0) = nt + c > 0$ for any $t \geq 0$. This provides $\varphi \geq 0$. Now consider the continuous parametric curve on which $f = 0$, given by

$$\Big\{ \big(x - \omega(x) - c/n, x\big) : x \in [0, \infty) \Big\}.$$

As far as this curve extends into the positive quadrant, any subset of it which also defines a path in D will satisfy the properties of Proposition 3.1. Taking $\gamma \in D$ to be the path which for all $t \in [0, T)$ coincides with the minimum of such elements in D yields $\gamma = R(\mathrm{e} - \omega - c/n)$.

For injectivity, notice that for any $t \in [0, T)$, $f(t, x) > 0$ for all $x \in [0, \gamma(t))$. So in this region φ is *strictly* increasing. The violation of injectivity therefore requires $\varphi = \gamma$ over some $[t, t']$ with $t' > t$. By the definition of γ, we would have $\varphi' = 0$ here, so γ constant. This contradicts its strictly increasing nature of γ. □

Remark 3.4. Notice that the bound γ introduced here resides in $D \setminus C$ provided that $\omega \in C$ satisfies $\omega(x') - \omega(x) > x' - x$ for any $x' > x$. This leads to the limit of any $\{\varphi_n\}_{n>0}$ being in $D \setminus C$ too, clarifying our related point in the introduction. It likely is not yet obvious, but the injectivity of any solution $\varphi \in C_1$ just established, which is equivalent to saying that $\varphi' = 0$ at countably many points and $\varphi' > 0$ otherwise, is actually a functional counterpart to the reflection property of the classical CIR process at zero.

3.1 Existence and continuation

If a solution $\varphi \in C_1$ to (3.1) exists, then it adheres to Proposition 3.3. So we may save time by not looking for such a solution outside of the region defined by the bounds 0 and $\gamma := R(\mathrm{e} - \omega - c/n)$.

The next result essentially applies the Cauchy-Peano existence theorem in its basic form, with the knowledge that f is continuous on any closed subset of $[0, \infty) \times \mathbb{R}$, in order to establish existence of some $\varphi \in C_1$ over some interval. See Chapter 2, Theorem 1.2 of Coddington and Levinson (1955) for background details.

Proposition 3.5 (Generalised CIR existence)**.** *A solution $\varphi \in C_1$ of (3.1) exists over $[0, \alpha)$, where*

$$\alpha = \sup_{t,x>0} \Big\{ t \wedge x/m(t,x) \Big\}, \quad m(t,x) = n\Big(t + M(\omega - \mathrm{e})(x)\Big) + c.$$

Proof. Notice that $m(t,x)$ provides the maximum of f in the closed subset (rectangle) of $[0, \infty) \times \mathbb{R}$ defined by $t, x > 0$. In fact m provides the maximum *absolute* value of f, if we look only between the bounds 0 and γ provided by Proposition 3.1.

Since f is continuous on any closed subset of $[0, \infty)^2$, we may fix any $t, x \in (0, \infty)$ and apply the Cauchy-Peano theorem to obtain a solution $\varphi \in C_1$ over $[0, \alpha')$ where $\alpha' = t \wedge x/m(t,x)$. By 'optimising' this result over any such t and x, this interval takes the form of $[0, \alpha)$. □

Remark 3.6 (Restarted Cauchy-Peano procedure). There might appear to be some danger in neglecting the region above γ, in which one could find f arbitrarily negative, depending on ω. This is because the proof of Cauchy-Peano existence relies on the convergence of polygonals which may still access such regions. We may avoid this by simply restarting the Cauchy-Peano procedure on the curve γ, should a polygonal path touch it. This amendment to the procedure can only improve the rate of convergence. In fact, since $\partial_t f(t,x) = n$ is independent of ω, it always holds that such a restarted polygonal will remain an 'ϵ-approximating' solution (defined momentarily) to (3.1) over an interval of exactly ϵ/n in length.

As we will use ϵ-approximating solutions a little more formally later, we provide the following.

Definition 3.7 (ϵ-approximating solution)**.** Any $\varphi \in C$ over $[0,T)$ is an ϵ-approximating solution to the IVP (3.1) if, on some $[0,T)\backslash S$, we have $\varphi \in C_1$ and $|\varphi'(t) - f(t,\varphi(t))| \leq \epsilon$, where $S \subset [0,T)$ is a countable set of points.

The existence established in Proposition 3.5 is useful, as it provides a non-empty interval of existence for any $\omega \in C$. If ω is such that γ provides a *finite* upper bound to φ over some interval, however, we may use the 'continuation' of ODE solutions to essentially defer the application of the previous result. See Theorem 4.1, Chapter 1 and Theorem 1.3, Chapter 2 of Coddington and Levinson (1955) for details and helpful continuation remarks.

Proposition 3.8 (Generalised CIR continuation)**.** *A solution $\varphi \in C_1$ of (3.1) exists over $[0,T)$, where $T = \beta + \alpha(\beta)$, with*

$$\beta = 0 \vee \sup\Big\{x - \omega(x) : x > 0\Big\} - c/n$$

and

$$\alpha(t) = \sup_{t',x'>0}\Big\{t' \wedge x'/m\big(t+t', \gamma(t_-) + x'\big)\Big\},$$

using the convention $\gamma(0_-) = 0$.

Proof. First notice that $[0,\beta)$ provides the time interval over which γ extends into $[0,\infty)^2$ and remains finite, so $\gamma(t_-) < \infty$ for $t \leq \beta$. Assume $\beta > 0$, as the case $\beta = 0$ coincides exactly with the previous result. A solution φ established through Proposition 3.5 may therefore be 'continued' up to the boundary provided by the curves 0 and γ over $[0,\beta)$. By using the polygonal procedure of Remark 3.6, this solution may not 'stop existing' at the boundaries provided by 0 or γ, but will eventually reach the right hand edge, so exist over $[0,\beta)$.

Now if $\beta = \infty$, the result follows trivially, so assume $\beta < \infty$. Then the solution φ just established may be continued over $[0,\beta]$. We are then guaranteed that $\varphi(\beta) \leq \gamma(\beta_-)$, so by shifting the application of Proposition 3.5 to apply from the point $(\beta, \gamma(\beta_-))$, rather than the origin, we get existence of a solution over $[0,T)$ for $T = \beta + \alpha(\beta)$. ☐

The previous result feels like an 'optimal' application of Cauchy-Peano existence to our problem. In practice, if ω is to represent a sensible trajectory of noise, it is unnecessarily strong and we will probably have $\beta = \infty$, rendering $\alpha(\beta)$ completely redundant. In fact one finds $\alpha(\beta) \to 0$ as $n \to \infty$ for any $\beta < \infty$, so the component $\alpha(\beta)$ has no use in finding a common interval $[0,T)$ over which all elements of a family $\{\varphi_n\}_{n>0}$ exist. This is precisely $[0,\beta)$. The following corollary helps our intuition for existence.

Corollary 3.9 (Condition for indefinite existence)**.** *Suppose $\omega \in C$ is recurrent, meaning that for any $x > 0$ there exists some $x' > x$ such that $\omega(x') = 0$. Then a solution φ of (3.1) exists on $[0, \infty)$.*

Proof. Since ω is recurrent, there exists a positive and unbounded set of points $\{x_k\}_{k=1}^{\infty}$ such that $\omega(x_k) = 0$ for each. Then assuming φ exists over some $[0, T)$, and using β as defined in Proposition 3.8, we have

$$T \geq \beta := 0 \vee \sup\Big\{x - \omega(x) : x > 0\Big\} - c/n \geq \sup\{x_k\}_{k=1}^{\infty} - c/n = \infty,$$

where we rely on the unbounded nature of $\{x_k\}_{k=1}^{\infty}$. □

By Proposition 3.3, recurrence therefore provides the existence of an *injective* solution $\varphi \in C_1$ over $[0, \infty)$. Once a better lower bound to φ than 0 is established (an unbounded one), we will be able to upgrade this to the bijectivity of φ from and to $[0, \infty)$. For now, we may say that any restricted solution $\varphi : [0, T) \to [0, \varphi(T))$ is bijective, so has a well defined inverse $\varphi^{-1} : [0, \varphi(T)) \to [0, T)$ given by $R(\varphi)$, which we label $\hat{\varphi}$. Having a well defined inverse leads us onto the following.

3.2 Bounds on solution families

Consider the bounds of 0 and $\gamma = R(\mathrm{e} - \omega - c/n)$ on a solution φ, derived in Proposition 3.3. Applying the functional R, we obtain

$$M_+(\mathrm{e} - \omega - c/n) \leq \hat{\varphi} \leq \infty$$

where $M_+ = 0 \vee M$. Towards established a better lower bound than 0 for φ, it proves slicker to work with $\hat{\varphi}$, and to refine the upper bound of ∞ here. We proceed with a looser lower bound of $\hat{\varphi}$, given by $\hat{\gamma}_- := M(\mathrm{e} - \omega) - c/n$.

Proposition 3.10 (Bounds on inverses)**.** *Let φ solve (3.1), and $\hat{\varphi} := \varphi^{-1}$ over $[0, \varphi(T))$. Then,*

$$\hat{\gamma}_- := M(\mathrm{e} - \omega) - c/n \leq \hat{\varphi} \leq M(\mathrm{e} - \omega) + \sqrt{c^2/n^2 + 2\mathrm{e}/n} - c/n =: \hat{\gamma}_+.$$

Proof. The bound $\hat{\gamma}_- \leq \hat{\varphi}$ is a corollary of 3.1, as just explained. To get $\hat{\gamma}_+$, we first write (3.1) in the equivalent form

$$\varphi(t) = \int_0^t f(u, \varphi(u))\mathrm{d}u = \int_0^t \mathrm{d}u\Big\{nu + c - n(\mathrm{e} - \omega)\big(\varphi(u)\big)\Big\},$$

then using the non-decreasing natures of $M(\mathrm{e} - \omega)$ and φ, get

$$\varphi(t) \geq \int_0^t \mathrm{d}u\Big\{nu + c - nM(\mathrm{e} - \omega)\big(\varphi(t)\big)\Big\}. \tag{3.2}$$

Now treating $t \in [0,T)$ as fixed, the right hand side of (3.2) curiously coincides with that obtained from the case $\omega = \mathrm{e}$, under the transformation $c \mapsto c' = c - nM(\mathrm{e}-\omega)(\varphi(t))$. This suggests that any solution could be bounded by a simple transformation of the case $\omega = \mathrm{e}$, which yields a parabolic solution; $\varphi(t) = ct + nt^2/2$.

Indeed, resolving (3.2) provides $\varphi(t) \geq c't + nt^2/2$, which may be inverted for

$$t \leq \sqrt{c'^2/n^2 + 2\varphi(t)/n} - c'/n$$

for any $t \in [0,T)$. Now simply making the transformation $t \mapsto \hat{\varphi}(t)$ and using $\varphi(\hat{\varphi}(t)) = t$ provides the neat upper bound

$$\hat{\varphi}(t) \leq M(\mathrm{e}-\omega)(t) + \sqrt{c'^2/n^2 + 2t/n} - c/n.$$

Since $c' < c$, this may be relaxed to $\hat{\gamma}_+$, providing $\hat{\gamma}_- \leq \hat{\varphi} \leq \hat{\gamma}_+$. $\square$

Corollary 3.11 (Improved bounds on solutions)**.** *Let $\varphi \in C_1$ solve (3.1) over $[0,T)$. Then we have $\gamma_- \leq \varphi \leq \gamma_+$, where the curves $\gamma_\pm \in D$ are given by*

$$\gamma_\pm(t) := R(\hat{\gamma}_\mp)(t) = \inf\Big\{x > 0 : \hat{\gamma}_\mp(x) > t\Big\}.$$

Proof. As the curves $\hat{\gamma}_\pm, \hat{\varphi}$ are all non-decreasing, applying the right inverse functional R to the inequality $\hat{\gamma}_- \leq \hat{\varphi} \leq \hat{\gamma}_+$ yields $R(\hat{\gamma}_+) \leq \varphi \leq R(\hat{\gamma}_-)$, which is the result. $\square$

Given that the lower bound γ_- here is itself unbounded, we have confirmation of the existence of some T such that $\varphi : [0,T) \to [0,\infty)$ is bijective, and which verifies

$$T = \sup\Big\{t > 0 : \varphi(t) < \infty\Big\}.$$

From now on, we therefore assume T to satisfy this if not otherwise stated. Since the recurrence of ω is sufficient for $T = \infty$, this assumption also makes φ a bijection from and to $[0,\infty)$.

Now knowing that the GCIR IVP maps ω onto a bijective path φ, raises the question of whether any bijective $\varphi : [0,T) \to [0,\infty)$ solves (3.1) for some ω. That is, does (3.1) *itself* set up a bijection between continuous drivers ω and bijective paths φ. The answer is yes, although to complete the argument we require uniqueness, covered in the next section.

Proposition 3.12 (Surjectivity onto bijective paths)**.** *Let $\varphi : [0,T) \to [0,\infty)$ be differentiable and bijective for some $T \in (0,\infty]$. Then φ solves (3.1) over $[0,T)$, with driver $\omega \in C$ given by*

$$\omega(x) = x - \hat{\varphi}(x) + \Big(\varphi'(\hat{\varphi}(x)) - c\Big)/n$$

where $\hat{\varphi} = \varphi^{-1}$.

Proof. Since φ is bijective, the inverse $\hat{\varphi} : [0, \infty) \to [0, T)$ is well defined. So evaluating (3.1) at any $t = \hat{\varphi}(x) \in [0, T)$ provides this result, noting that $\varphi(\hat{\varphi}(x)) = x$. $\square$

This result proves that our framework is actually general enough to accommodate *any* practicable volatility path $\varphi' \in C$, driven by some $\omega \in C$. The constraints we have indirectly placed on the path $\varphi \in C_1$ is just its positivity and bijectivity, so the precise constraints on the volatility path φ' is just that we have $\varphi'(t) = 0$ only at countably-many $t \in [0, T)$, and $\varphi'(t) > 0$ otherwise. Uniqueness, in the next section, will give us *injectivity* onto bijective paths, and so the problem (3.1) provides a bijection itself, from drivers in C onto the space of bijective paths.

3.3 Uniqueness and simulation

Although uniqueness of solutions is not central to our convergence results, establishing this property is practically helpful, especially because of its implication on the forward Euler approximation scheme. The following proof utilises a method that appears to generalise provided $\partial_t f(t, x)$ is *strictly* increasing.

Proposition 3.13 (Generalised CIR uniqueness)**.** *Any solution φ of (3.1) is unique.*

Proof. By this we really mean φ is unique over any $[0, t)$ where $t < T$, so assume $T < \infty$. Let $\varphi_i \in C_1$ be solutions of (3.1) for $i = 1, 2$. As these solutions are both injective over $[0, T)$, we have $R(\varphi_i) = \hat{\varphi}_i$, where we use $\hat{\varphi}_i = \varphi_i^{-1}$ to help notationally. Now consider the function $\rho \in C_1$ defined by $\rho = \hat{\varphi}_2 - \hat{\varphi}_1$, so $\rho(x)$ measures a difference in time for φ_1 and φ_2 to reach the level x, and is well defined over $[0, x')$ for $x' = \varphi_1(T) \wedge \varphi_2(T)$. We may assume without loss of generality that $\rho \geq 0$ over some interval $[0, x''] \subset [0, x')$.

Since $\partial_t f(t, x) = n$ for all t, x, we then have

$$\varphi_2'\big(\hat{\varphi}_2(x)\big) - \varphi_1'\big(\hat{\varphi}_1(x)\big) = \int_{\hat{\varphi}_1(x)}^{\hat{\varphi}_2(x)} \partial_u f(u, x) \mathrm{d}u = n\rho(x) \geq 0 \tag{3.3}$$

assuming $x \in [0, x'']$. This tells us that φ_1' and φ_2' cannot 'point away from one another' at any level x, and that φ_1 and φ_2 must approach one another as their level is increased, if they are ever apart. The same applies to $\hat{\varphi}_1$ and $\hat{\varphi}_2$, by the elementary property $\hat{\varphi}_i'(x) = 1/\varphi_i'(\hat{\varphi}_i(x))$ of inverse functions. Formally, (3.3) provides

$$1/\hat{\varphi}_2'(x) - 1/\hat{\varphi}_1'(x) \geq 0,$$

and so $\hat{\varphi}_2'(x) - \hat{\varphi}_1'(x) \leq 0$. But since $\rho' = \hat{\varphi}_2' - \hat{\varphi}_1'$ follows clearly from its definition, we see that ρ defines a non-increasing function. The proof is therefore completed by noting $\rho(0) = 0$, thus $\rho = 0$, $\hat{\varphi}_2 = \hat{\varphi}_1$ and finally $\varphi_2 = \varphi_1$ over the domain $[0, T)$ of existence. $\square$

Now we have established all of the components making up Theorem 1.2, so clarify where to find them here.

Proof of Theorem 1.2. That φ_n exists over some domain $[0, T)$ which maps onto $[0, \infty)$ requires the lower bound γ_- of Corollary 3.11, which is itself unbounded. Thus φ_n may be continued, as in Proposition 3.12, to the boundary of the domain on which it is finite, and this defines T. Uniqueness of this solution φ_n is provided by Proposition 3.13, and that any differentiable and bijective $\varphi_n : [0, T) \to [0, \infty)$ solves (1.1) is covered by Proposition 3.12. $\square$

◦ ◦ ◦

A practically useful consequence of uniqueness is the next result. Consider again the ϵ-approximating polygonals, which are relied upon in the proof of Cauchy-Peano existence, thus Proposition 3.5. The Ascoli lemma plays an important role in this proof, because *not all* sequences of ϵ-approximating polygonals are guaranteed convergent. A corollary of uniqueness, however, is that *all* sequences $\{\varphi_k\}_{k=1}^{\infty}$ of ϵ_k-approximating polygonal paths must converge, provided $\epsilon_k \to 0$ as $k \to \infty$. See Coddington and Levinson (1955) yet again for more details.

Corollary 3.14 (Convergence of forward Euler scheme)**.** *The forward Euler scheme defined by the 'restarted' Cauchy-Peano procedure in Remark 3.6 converges to the unique solution of* (3.1) *over* $[0, t]$*, for any* $t < T$*.*

Proof. Suppose $\{\varphi_k\}_{k=1}^{\infty}$ is a sequence of ϵ_k-approximating solutions to (3.1), with $\epsilon_k \to 0$. Then $\varphi_k \to \varphi$ on $(C, \|\cdot\|)$, since φ is unique. Now we just need to verify that our forward Euler scheme produces such a sequence of ϵ_k-approximating solutions

Let $\{\varphi_k\}_{k=1}^{\infty}$ be a sequence of forward Euler approximations over some $[0, t]$ each with maximum step sizes $\delta_k \to 0$. Then each φ_k is an ϵ_k-approximating solution, where $\epsilon_k = \|\varphi_k' - f(\cdot, \varphi_k(\cdot))\|$, considered over $[0, t] \setminus S$, where S is as in Definition 3.7.

If $\epsilon_k \not\to 0$ is assumed, then for some $\epsilon > 0$, there are infinitely many $\epsilon_k > \epsilon$ despite having $\delta_k \to 0$. This would suggest that on a compact region, f varies by at least ϵ over a line of length at most $\sqrt{1 + m^2}\delta_k$ for *any* such δ_k, where m is the maximum value of f over the domain of conseridation. This would however violate the continuity of f over any compact subset of $[0, \infty) \times \mathbb{R}$, so we must have $\epsilon_k \to 0$, therefore $\varphi_k \to \varphi$. $\square$

We now start to index $n > 0$ again, where relevant, as this ends our exposition of convergence-unrelated properties of the family $\{\varphi_n\}_{n>0}$ of GCIR solutions.

3.4 Fast reversion convergence

It is clear from Proposition 3.10 that our bounds satisfy $\hat{\gamma}_n^{\pm} \to M(\mathrm{e}-\omega)$ on $(C, \|\cdot\|)$. So it is straightforward to obtain the following.

Proposition 3.15 (Generalised CIR convergence)**.** *Let φ_n solve (2.3) and $\hat{\varphi}_n = \varphi_n^{-1}$. Then $\hat{\varphi}_n \to \hat{\varphi} := M(\mathrm{e}-\omega)$ on $(C, \|\cdot\|)$ and $\varphi_n \to \varphi := R(\mathrm{e}-\omega)$ on (D, M), as $n \to \infty$.*

Proof. From the convergence of the bounds $\gamma_n^{\pm}$ above, we have

$$\|\hat{\varphi} - \hat{\varphi}_n\|_t \leq \|\gamma_n^+ - \gamma_n^-\|_t = \sqrt{c^2/n^2 + 2t/n} \to 0$$

for any $t > 0$. This establishes the neat result $\hat{\varphi}_n \to \hat{\varphi}$ on $(C, \|\cdot\|)$.

Now, $\hat{\varphi}_n \to \hat{\varphi}$ on $(C, \|\cdot\|)$ gives the same on the weaker topology (D, M), and in Puhalskii and Whitt (1997) it is shown that R constitutes a continuous map from and to (D, M). So, this provides $R(\hat{\varphi}_n) \to R(M(\mathrm{e}-\omega))$ on (D, M). Writing $\hat{\varphi}_n = R(\varphi_n)$ and using the basic properties $R \circ R = M$ and $R \circ M = R$ of these functionals summarised in Whitt (1971), as well as $M(\varphi_n) = \varphi_n$, leads straight to

$$\varphi_n = M(\varphi_n) = R(R(\varphi_n)) \to R(M(\mathrm{e}-\omega)) = R(\mathrm{e}-\omega) = \varphi.$$

And so we have finally established our main goal of $\varphi_n \to \varphi$ on (D, M), as $n \to \infty$. □

Remark 3.16. We have completely avoided the particulars of parametric representations in this proof, by exploiting the continuity of R from $(C, \|\cdot\|)$ onto (D, M). However, for any $\varphi(t) < \infty$, the uniform distance $\|M(\mathrm{e}-\omega) - \hat{\varphi}_n\|_{\varphi(t)}$ precisely coincides with the distance between φ_n and $R(\mathrm{e}-\omega)$ in the (D, M) topology, when *only* parametric representations with equal spatial component are considered. Since the metric d_t represents the infimum instead over *all* parametric representations, we see that

$$d_t(\varphi_n, R(\mathrm{e}-\omega)) \leq \|M(\mathrm{e}-\omega) - \hat{\varphi}_n\|_{\varphi(t)} \to 0.$$

We consider this result $\varphi_n \to R(\mathrm{e}-\omega)$ on (D, M) to provide the 'purest' (sans probabilités) origin of Proposition 1.1 and so eventually the result of Mechkov (2015). Proposition 3.15 goes a long way to establishing Theorem 1.3, with the exception of the last part which deals with the families $\{\omega \circ \varphi_n\}_{n>0}$ and $\{n^{-1}\varphi_n'\}_{n>0}$. The treatment of these comes in Section 4.1. For now we provide some more intuition on our results through examples.

3.5 Graphical examples

Now we demonstrate the functional convergence result $\varphi_n \to \varphi$ on (D, M), of Theorem 1.3, graphically. We fix $c = 1$ in (2.3), and let n vary from 0 to 2^{10}. We plot curves in the unit square $(t,x) \in [0,1]^2$, including the driving path ω implicitly (thick red) through the parametric curve

$$\Big\{\big(x-\omega(x),x\big)\Big\}_{x\in[0,1]}.$$

This curve is a key component of the parametric bounds, and the limit. The bounds $\gamma_n^{\pm}$ of φ_n define a region (shaded red) for each n, given by

$$\Big\{(t,x)\in[0,1]^2 : x\in\big[\gamma_n^-(t),\gamma_n^+(t)\big]\Big\}.$$

A forward Euler approximation (thick blue) of each unique solution $\varphi_n = \bar{X}_n(\omega)$ of (2.3) is also shown, which uses 2^{15} steps, since the convergence rate is not understood. Finally we show the (normalised) vector field $F_n = (1, f_n)$, as discussed in the introduction, which provides excellent intuition for the properties of φ_n.

In the following examples that we share on the next two pages, and for all tested cases of ω and c that we don't, the convergences $\varphi_n \to \varphi := R(\mathrm{e}-\omega)$ on (D, M) and $\hat{\varphi}_n \to M(\mathrm{e}-\omega) := \hat{\varphi}$ on $(C, \|\cdot\|)$ are clear. A standalone python jupyter notebook is available at bit.ly/2Q96GtF, which will produce similar convergence plots for a driving path $\omega \in C$ of the user's choice. High-resolution PDF images of our plots shared here are also available at bit.ly/2GHZwNC.

4 Continuous mapping

All the results of this section are believed to hold, but we expect to establish them with more rigour in the next version of this work. For now we keep this application of our results relatively short, focusing only on the composition map specifically, and leave others for future work. Now we prove Proposition 1.5, which does not constitute anything new but summarises the scope of applications for our results.

Proof of Proposition 1.5. That we should have $h(\varphi_n) \to h(\varphi)$ on (S, m) and $\Phi(h(\varphi_n)) \to \Phi(h(\varphi))$ on $(\mathbb{C}, \|\cdot\|)$ are consequences, or definitions, of h and Φ being continuous. If there exists a $\mathbb{P}$-integrable dominating function to $\Phi \circ h$, then the dominated convergence theorem provides $\mathbb{E}[\Phi(h(\bar{X}_n))] \to \mathbb{E}[\Phi(h(\bar{X}))]$. Theorem 3.2.3 of Skorokhod (1956) or the continuous mapping theorem of Billingsley (1999) proves that the same conclusion only requires continuity of $\Phi \circ h$ $\mathbb{P}$-almost everywhere. □

4.1 Composition

The humble composition map is the only we consider in detail, for two reasons. Firstly, it will allow us 'complete' our understanding of the components present in (2.3): $\omega \circ \varphi_n$

Figure 1: Forward Euler approximations to $\varphi_n(t) = \bar{X}_t^n(\omega)$ (thick blue) for n as shown, $\omega(x) = -\sin(6\pi x)/6$ (thick red) and $c = 1$, on the unit square.

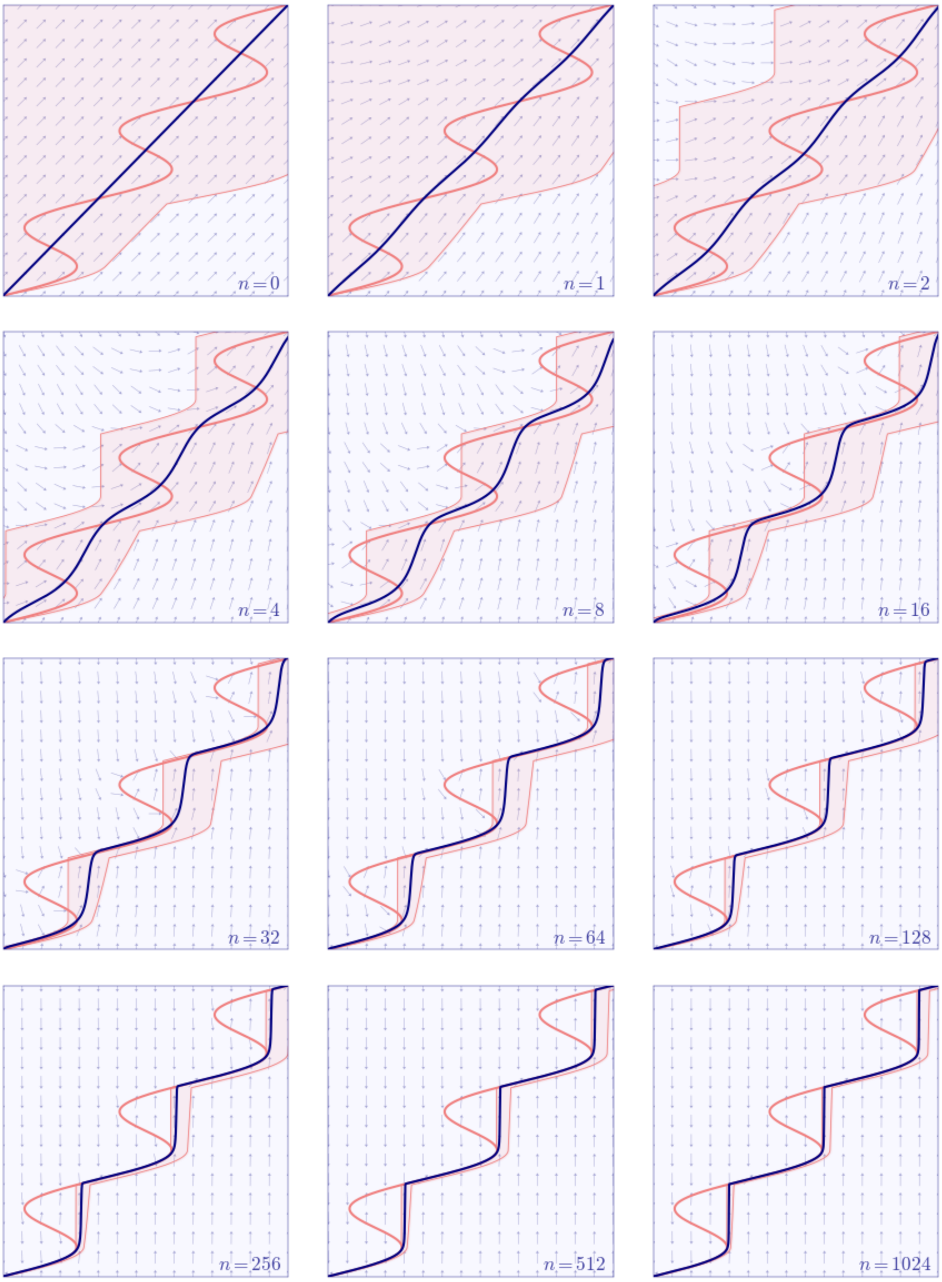
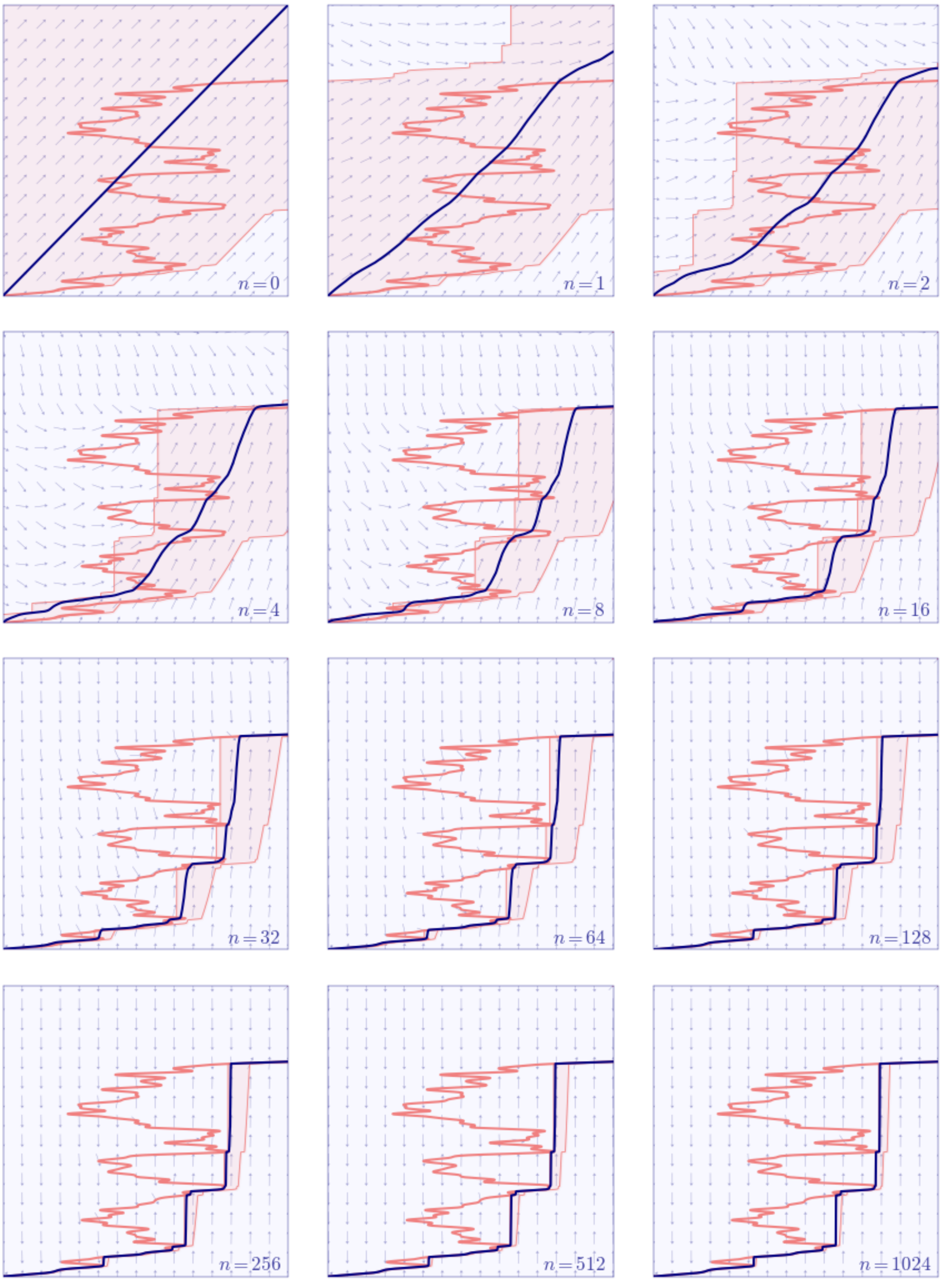


Figure 2: Forward Euler approximations to $\varphi_n(t) = \bar{X}_t^n(\omega)$ (thick blue) for n as shown, with ω given by a truncated Karhunen-Loève series (thick red) and $c = 1$, on the unit square. Specifically, we set $\omega(x) = \sqrt{2}\sum_{k=1}^{128} \xi_k \sin(k\pi x)/(k\pi)$ and for $\xi = (\xi_k)_{k=1}^{128}$ use python's numpy.random module with seed(2).



directly, and φ'_n indirectly. Both behave in surprising ways as $n \to \infty$; in particular having no limit in D in general. Secondly, the consideration of $\bar{\omega} \circ \varphi_n$ for $\omega \neq \bar{\omega} \in C$ will provide the simplest transition from our volatility paths φ_n to a price path. The obvious (log-Heston) example being something like $H_n := \bar{\omega} \circ \varphi_n + \omega \circ \varphi_n - \frac{1}{2}\varphi_n$. First we clarify the following surprising negative result then demonstrate it graphically.

Proposition 4.1 (Discontinuity of composition)**.** *Let $\omega, \bar{\omega} \in C$ and φ_n solve (2.3) driven by ω. Then, despite having $\varphi_n \to \varphi := R(\mathrm{e} - \omega)$ on (D, M), we in general find $\bar{\omega} \circ \varphi_n \not\to \bar{\omega} \circ \varphi$ as well as $\omega \circ \varphi_n \not\to \omega \circ \varphi = \varphi - \mathrm{e}$ on (D, M).*

Proof. We may establish this on the Hausdorff topology, without dealing specifically with parametric representations, since (D, M) is finer. Proving $\omega \circ \varphi_n \not\to \omega \circ \varphi$ would cover both cases, but it is clearer to start with $\bar{\omega} \circ \varphi_n \not\to \bar{\omega} \circ \varphi$.

At continuity points of φ we do have $(\bar{\omega} \circ \varphi_n)(t) \to (\bar{\omega} \circ \varphi)(t)$, so assume $\varphi(t_-) < \varphi(t)$, and so a jump occurs in $\bar{\omega} \circ \varphi$ from $\bar{\omega}(\varphi(t_-))$ to $\bar{\omega}(\varphi(t))$. How $\bar{\omega}$ behaves over the interval $[\varphi(t_-), \varphi(t)]$ of its domain therefore does not affect $\bar{\omega} \circ \varphi$. In contrast, although $\varphi_n \to \varphi \in D$, φ_n is nevertheless continuous for all $n > 0$, and so is *always* affected by what $\bar{\omega}$ does over $[\varphi(t_-), \varphi(t)]$. Since $\bar{\omega}$ has no relation with any φ_n (unlike ω), in general nothing may stop $\bar{\omega} \circ \varphi_n$ reaching a lower value than $\bar{\omega}(\varphi(t_-)) \wedge \bar{\omega}(\varphi(t))$ nor a higher value than $\bar{\omega}(\varphi(t_-)) \vee \bar{\omega}(\varphi(t))$.

Because $\varphi_n \to \varphi$ on (D, M), meaning that the time component of respective parametric representations converge uniformly, *and* because discontinuities are countable, so continuity point at which $(\bar{\omega} \circ \varphi_n)(t) \to (\bar{\omega} \circ \varphi)(t)$ are dense, the effect of a discontinuity point of φ as $n \to \infty$ is that $\bar{\omega} \circ \varphi_n$ has to 'fold' into a temporally instantaneous excursion, in contrast to $\bar{\omega} \circ \varphi$, which certainly jumps. We may go further and say that $\bar{\omega} \circ \varphi_n$ *will* find a limit in (D, M) if the following equalities are satisfied

$$\min \left\{ \bar{\omega}(s) : s \in [\varphi(t_-), \varphi(t)] \right\} = \bar{\omega}(\varphi(t_-)) \wedge \bar{\omega}(\varphi(t))$$
$$\max \left\{ \bar{\omega}(s) : s \in [\varphi(t_-), \varphi(t)] \right\} = \bar{\omega}(\varphi(t_-)) \vee \bar{\omega}(\varphi(t))$$

but in the practice of our intented applications, with $\bar{\omega}$ not even assumed Lipschitz, this proves unlikely.

Since ω is related intimately to φ, these equalities could be naturally satisfied in the case of $\bar{\omega} = \omega$. It turns out that one *is*, but in general the other is not, and that these are swapped if instead $\bar{\omega} = -\omega$. To see this, first express the above equalities using the knowledge that $\omega \circ \varphi = \varphi - \mathrm{e}$, for

$$\min \left\{ \omega(s) : s \in [\varphi(t_-), \varphi(t)] \right\} = \varphi(t_-) - t$$
$$\max \left\{ \omega(s) : s \in [\varphi(t_-), \varphi(t)] \right\} = \varphi(t) - t.$$

Now recall that discontinuity points of φ arise only when ω *increases* locally at a rate greater than or equal to e, which ensures the satisfaction of the first equality but violation of the second. $\square$

Using the second equality, we may therefore define a positive excursion height $\mathscr{E}$ above any point $\varphi(t)$ by

$$\mathscr{E}(\omega,t) := 0 \vee \mathscr{M}(\omega,t), \quad \mathscr{M}(\omega,t) := \max\Big\{\omega(s) : s \in [\varphi(t_-),\varphi(t)]\Big\} + t - \varphi(t).$$

Geometrially, $\mathscr{M}$ has a very neat representation. Writing this as

$$\mathscr{M}(\omega,t) = \max\Big\{s + (\omega - \mathrm{e})(s) - (-t) : s \in [\varphi(t_-),\varphi(t)]\Big\} - \varphi(t),$$

we see that a positive excursion exists at t beyond $\varphi(t)$ if, when passing between levels $\varphi(t_-)$ and $\varphi(t)$, a distance backwards in *time* to the curve $\omega - \mathrm{e}$ exceeds the distance in *space* up to $\varphi(t)$. In Figure 3 we repeat Figure 1 but replace curves φ_n with $\omega \circ \varphi_n + \mathrm{e}$. These two paths have related, but not equivalent, limits as $n \to \infty$. The following operator will help us resolve this issue.

Definition 4.2 (Filled composition)**.** Let $D_\uparrow \subset D$ be the space of non-decreasing càdlàg paths over some $[0,T)$ for $T \in (0,\infty]$. Then for $\bar{\omega} \in C$ and $\varphi \in D_\uparrow$, define the continuous set-valued function $\bar{\omega} \bullet \varphi$ by

$$(\bar{\omega} \bullet \varphi)(t) := \big[\mathscr{E}_-(t), \mathscr{E}_+(t)\big]$$

for $t \in [0,T)$, where 'excursion boundaries' $\mathscr{E}_\pm(t) = \mathscr{E}_\pm(\bar{\omega},\varphi,t)$ are defined by

$$\mathscr{E}_-(t) := \min\Big\{\bar{\omega}(s) : s \in [\varphi(t_-),\varphi(t)]\Big\}, \quad \mathscr{E}_+(t) := \max\Big\{\bar{\omega}(s) : s \in [\varphi(t_-),\varphi(t)]\Big\}.$$

Notice it is always the case that $(\bar{\omega} \circ \varphi)(t) \in (\bar{\omega} \bullet \varphi)(t)$, and at continuity points of φ in fact $(\bar{\omega} \bullet \varphi)(t) = \big\{(\bar{\omega} \circ \varphi)(t)\big\}$, although this will be written $\bar{\omega} \bullet \varphi = \bar{\omega} \circ \varphi$.

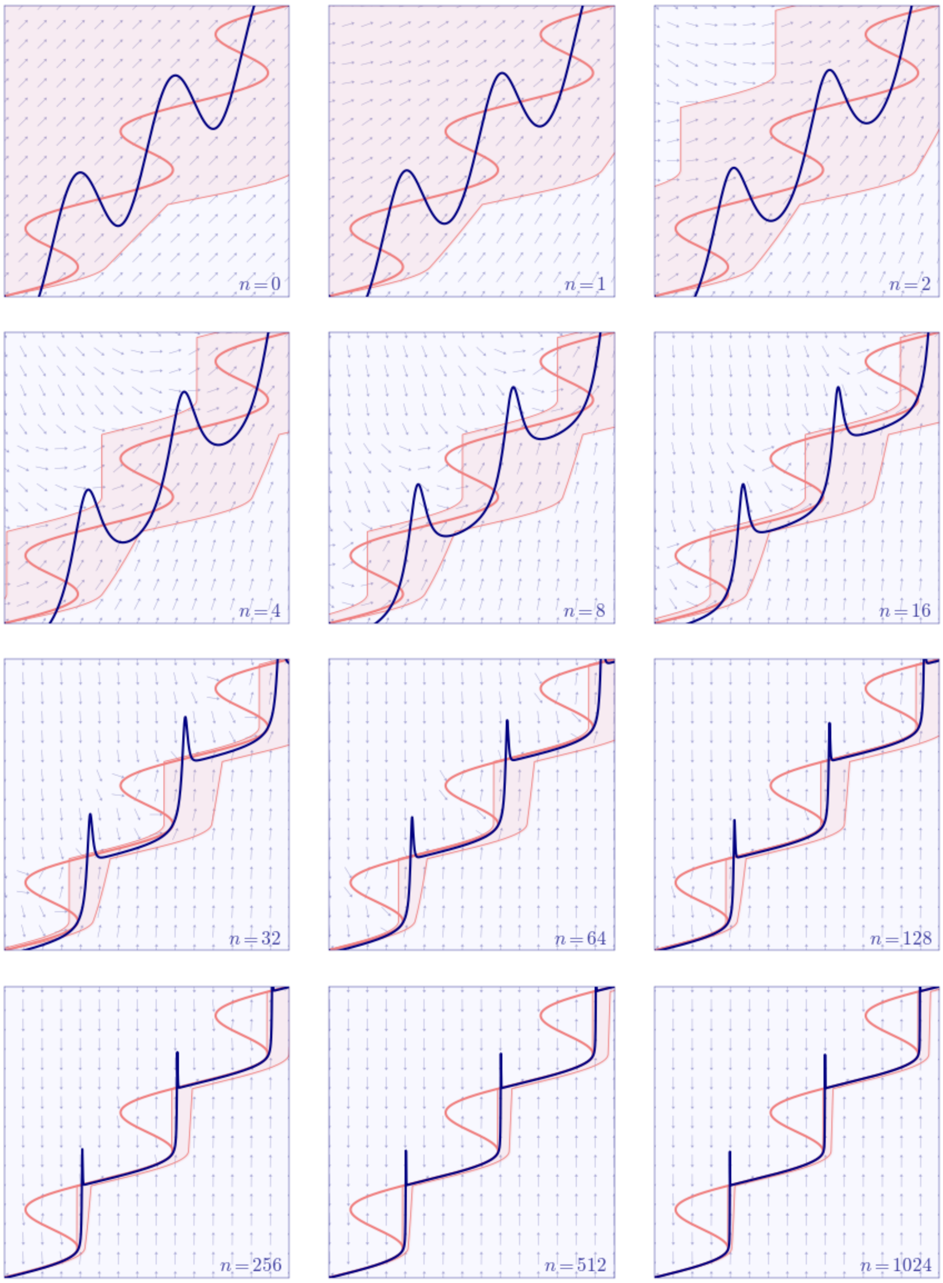
The following definition is not strictly as in Whitt (2002), which is more general than we require, but is related.

Definition 4.3 (Whitt's excursion topology)**.** Let E be the space of set-valued functions originating from the filled composition map,

$$E := \Big\{\bar{\omega} \bullet \varphi : (\bar{\omega},\varphi) \in C \times D_\uparrow\Big\}$$

and (E,H) the Hausdorff topology on the space $\Gamma(E)$ of graphs.

Figure 3: Here we repeat Figure 1, but show $\omega \circ \varphi_n + \mathrm{e}$ (thick blue) instead of φ_n, for n as shown. Convergence $\omega \circ \varphi_n + \mathrm{e} \not\to \varphi$ is visible, due to excursions.



Proposition 4.4 (Continuity of filled composition)**.** *For $\bar\omega \in C$, and φ_n, φ as above, although in general $\bar\omega \circ \varphi_n \not\to \bar\omega \circ \varphi$ on (D, M), it holds that $\bar\omega \bullet \varphi_n = \bar\omega \circ \varphi_n \to \bar\omega \bullet \varphi$ on (E, H). So, in particular, $\omega \bullet \varphi_n \to \omega \bullet \varphi$ on (E, H), and at continuity points of φ, $\omega \bullet \varphi = \omega \circ \varphi = \varphi - \mathrm{e}$.*

Proof. The hard work for this is covered by the equivalent negative result of Proposition 4.1. We showed there that $\bar\omega \bullet \varphi_n = \bar\omega \circ \varphi_n$ tends to $\bar\omega \circ \varphi = \bar\omega \bullet \varphi$ at φ-continuity points, but otherwise may lead to an excursion outside of the interval $[\varphi(t_-), \varphi(t)]$. These additional excursions at φ-discontinuity points are precisely accommodated by the definition of $\bar\omega \bullet \varphi$. So, the Hausdorff distance vanishes as $n \to \infty$, and convergence takes place on this (E, H) topology. □

It is important to appreciate that the filled composition map violates distributivity in the sense that $(\omega + \bar\omega) \bullet \varphi \neq \omega \bullet \varphi + \bar\omega \bullet \varphi$, because so do the max and min functions. Take for example $\bar\omega = -\omega$ to readily see this. We will raise this in the final section, but otherwise this completes the treatment of this map.

Understanding the limiting behaviour of φ_n and $\omega \circ \varphi_n$ leads naturally to the behaviour of the component $n^{-1}\varphi_n'$ in (2.3), therefore some understanding of φ_n'. This might be considered important, as this corresponds directly to a volatility path, as opposed to the integrated volatility path φ_n.

Proposition 4.5 (Generalised CIR fast-reversion limit)**.** *Let φ_n solve (3.1). Then $n^{-1}\varphi_n' \to (\omega - \mathrm{e}) \bullet \varphi + \mathrm{e}$ on (E, H). Geometrically, this limit is better understood by*

$$((\omega - \mathrm{e}) \bullet \varphi)(t) + t := [0, \mathscr{E}_+(t)], \quad \mathscr{E}_+(t) = \max\left\{(\omega - \mathrm{e})(s) - (-t) : s \in [\varphi(t_-), \varphi(t)]\right\}.$$

This confirms that at φ-discontinuity points, we have $\varphi_n' \to \infty$, and clearly if φ is differentiable at t, we have $\varphi_n'(t) \to \varphi'(t)$.

Notice that $\mathscr{E}_+$ is equal to 0 at φ-continuity points, but geometrically represents the maximum backwards distance in *time* from t to the curve $\omega - \mathrm{e}$, between the levels $\varphi(t_-)$ and $\varphi(t)$. Now clearly at φ-discontinuity points $\varphi_n' \to \infty$ and at φ-differentiable points $\varphi_n' \to \varphi'$. It seems also possible to classify what happens if, for example, ω is Lipschitz or has $\alpha \in (0, 1)$ Hölder regularity. (This is most easily done by the limiting behaviour of $\hat\varphi'$ and the relationship $\varphi'(t) = 1/\hat\varphi_n'(\varphi(t))$.) However, the conclusion is much the same: φ_n tends towards an object outside of E, which has excursions on the interval $[0, \infty)$ at the (countable) φ-discontinuty points, and is a finite single value otherwise.

Proof. The IVP (2.3) may be expressed as

$$n^{-1}(\varphi_n' - c) = (\omega - \mathrm{e}) \circ \varphi_n + \mathrm{e} = (\omega - \mathrm{e}) \bullet \varphi_n + \mathrm{e}$$

and since $(\omega - \mathrm{e}) \bullet \varphi_n \to (\omega - \mathrm{e}) \bullet \varphi$ on (E, H), simply letting $n \to \infty$ provides the result, given $n^{-1}c \to 0$. That we have $(\omega - \mathrm{e}) \bullet \varphi + \mathrm{e} = [0, \mathscr{E}_+]$ simply follows from their definitions. $\square$

We are *finally* in a position to consider the limiting behaviour of a generalised log-Heston path. As these will hold for *any* driving path, extension to derivative prices is trivial, without the need to check properties such as tightness of induced distributions.

4.2 Generalised Heston

We start with the following result which is a straightforward application of the previous section.

Proposition 4.6 (Generalised fast reversion Heston convergence)**.** *Let φ_n solve (2.3) and as before $\varphi := R(\mathrm{e} - \omega)$ for $\omega \in C$. Now define a log-Heston path by $H_n := H_n(\omega, \bar{\omega}) := (\bar{\omega} + \omega - \frac{1}{2}\mathrm{e}) \circ \varphi_n$ for $\bar{\omega} \in C$ and $H_\bullet := (\bar{\omega} + \omega - \frac{1}{2}\mathrm{e}) \bullet \varphi$ where the filled composition operator $\bullet$ is as in Definition 4.2. Then $H_n \to H_\bullet$ on (E, H).*

Proof. Since the path $\bar{\omega} + \omega - \frac{1}{2}\mathrm{e}$ is in C simply applying Proposition 4.4 provides $(\bar{\omega} + \omega - \frac{1}{2}\mathrm{e}) \circ \varphi_n \to (\bar{\omega} + \omega - \frac{1}{2}\mathrm{e}) \bullet \varphi$ on (E, H), which is the result. $\square$

Now we cover something which feels truly remarkable.

Proposition 4.7 (Near-convergence of generalised Heston to a NIG process)**.** *Place the Wiener measure on $\Omega := C([0, \infty), \mathbb{R}^2)$, from which $(\omega, \bar{\omega})$ is drawn, and define $H_\circ := (\bar{\omega} + \omega - \frac{1}{2}\mathrm{e}) \circ \varphi$, which differs from $H_\bullet$ (only) by excursions at φ-discontinuity points. Then, we have the following convergence in Lebesgue measure $\mathbb{L}$ for any $t \in [0, T)$:*

$$\mathbb{L}\Big\{\alpha \in [0, 1] : (\bar{X}, H_n)(\alpha t) \to (\bar{X}, H_\circ)(\alpha t)\Big\} = 1,$$

as well as convergence of all finite-dimensional distributions. $(\bar{X}, H_\circ)$ is the IG-NIG Lévy motion of Barndorff-Nielsen and Shephard (2001)! However, due to excursions, we have $H_n \not\to H_\circ$ on (E, H).

Proof. We know that $(\bar{X}, H_n)(t) \to (\bar{X}, H_\circ)(t)$ at $\bar{X}$-continuity points, and since discontinuity points are countable, we therefore get this type of convergence in Lebesgue measure over any compact interval. Convergence of *all* finite-dimensional distributions then follows by the known stochastic continuity of $\bar{X}$, which is the case for any Lévy process, and so in particular this IG one. That $(\bar{X}, H_\circ)$ is IG-NIG Lévy motion just follows from the representation of this process, as in our appendix, equivalent to

$H_\circ := (\bar{\omega} + \omega - \frac{1}{2}\mathrm{e}) \circ \varphi$. Finally, that we have $H_n \not\to H_\circ$ on (E, H) is straightforward, since we know that $H_n \to H_\bullet$ on (E, H), and these two limits differ by excursions. □

We provide this one other probabilistic application relating to derivative proces convergence, and leave others for future work.

Example 4.8 (Generalised Heston derivatives). For $\Omega := C([0,\infty), \mathbb{R}^3)$ and $\mathscr{F} := \sigma(\Omega)$, let Z be the canonical process on $(\Omega, \mathscr{F}, \mathbb{P})$ and X_n solve the generalised CIR time-change equation (2.2), repeated below. Further, define $\bar{X}_n$, H_n, $\bar{X}$, and $H_{\circ\bullet}$ by

$$X_t^n = n\Big(\big(Z_0 \circ \bar{X}_n\big)_t + \int_0^t (1 - X_u^n)\mathrm{d}u \Big) + X_0, \quad \bar{X}_t^n := \int_0^t X_u^n \mathrm{d}u,$$

$$H_n := \big(Z_1 \circ \bar{X}_n\big) + \big(Z_2 \circ \bar{X}_n\big) - \frac{1}{2}\bar{X}_n,$$

$$\bar{X}_t := \inf\Big\{ s > 0 : s - Z_s^0 > t \Big\}, \qquad H_{\circ\bullet} := \big(Z_1 \circ \bullet \bar{X}\big) + \big(Z_2 \circ \bullet \bar{X}\big) - \frac{1}{2}\bar{X}$$

so that H_n constitutes a generalised Heston process and $H_{\circ\bullet}$ its 'prospective' limits as $n \to \infty$. For $T < \infty$, define derivative payoffs $\Phi_i : \omega \to \Phi_i(\omega) \in \mathbb{C}$ for $i = 1, 2, 3$, by

$$\Phi_1(\omega) := \prod_{k=1}^m e^{iu_k \omega(t_k)}, \quad \Phi_2(\omega) := \max\Big\{ e^{\omega(u)} : u \in [0, T) \Big\}, \quad \Phi_3(\omega) := \int_0^T e^{\omega(u)}\mathrm{d}u$$

for $m \in \mathbb{Z}_+$, $u_k \in \mathbb{R}$ and $t_k \in [0, T)$. Assume $H_\circ$ is $\mathbb{P}$-stochastically continuous, Then, we have $\mathbb{E}[\Phi_1(H_n)] \to \mathbb{E}[\Phi_1(H_\bullet)] = \mathbb{E}[\Phi_1(H_\circ)]$. If further $\Phi_2(H_n)$ and $\Phi_3(H_n)$ are $\mathbb{P}$-integrable, then we have $\mathbb{E}[\Phi_2(H_n)] \to \mathbb{E}[\Phi_2(H_\bullet)] \neq \mathbb{E}[\Phi_2(H_\circ)]$, but $\mathbb{E}[\Phi_3(H_n)] \to \mathbb{E}[\Phi_3(H_\bullet)] = \mathbb{E}[\Phi_3(H_\circ)]$.

The obvious example of a measure $\mathbb{P}$ that satisfies all of these properties is one under which (Z_1, Z_2) is Brownian motion and $Z_0 = Z_1$ is verified pathwise. In this case, $(\bar{X}, H_\circ)$ is IG-NIG Lévy motion, with $H_\circ$ having the NIG normal variance-mean mixture representation

$$H_t^\circ = (Z_2 \circ \bar{X})_t + \frac{1}{2}\bar{X}_t - t$$

since $Z_1 \circ \bar{X} = \bar{X} - \mathrm{e}$. These results should be interpreted in practice as 1. finite-dimensional convergence of the time-changed *and the classical* Heston process to the NIG process (and so any finite-dimensional derivative price), 2. failure of convergence of (theoretical) barrier option prices to that predicted by the NIG process, 3. convergence of (theoretical) Asian option prices to that predicted by the NIG process. We use the term theoretical here as, in practice, these options only ever reference finitely many time points, and so in this case we will have convergence.

To cover some technicalities, 1. relies on the stochastic continuity of $H_\circ$ under this specific choice $\mathbb{P}$, which is equivalent to the stochastic continuity of the NIG process;

known to hold for any Lévy process. 2. violates convergence along paths where $e^{H_\bullet}$ exhibits *positive* excursions, but 3. assigns no (Lebesgue) measure to these excursions. $\mathbb{P}$-integrability for 1. is trivial as $\Phi_1(H_n)$ is dominated by 1. $\mathbb{P}$-integrability for 2. follows either Doob's maximal or the BDG inequalities, since e^{H_n} is an exponential martingale. $\mathbb{P}$-integrability for 3. follows since $\Phi_3(H_n)$ is dominated by $\Phi_2(H_n)$.

5 Conclusion

Without wanting to repeat ourselves, we have accumulated many functional properties of volatility paths within a pathwise framework that we call generalised CIR. As stated in the introduction, we consider our main results to be the existence and uniqueness of our paths, and the explicit fast reversion convergence result which takes place on the (D, M) topology; a minor extension of that in Skorokhod (1956) due to Puhalskii and Whitt (1997).

It has felt important to focus primarily on functional results here, due in part because it is very easy to be overwhelmed and sidetracked by the scope of probabilistic applications, summarised by Proposition 1.5. However, we have established that the generalised Heston process defined in 4.8, although not converging precisely to a NIG process, will lead to the convergence of any practicable (finite-dimensional) derivative price to that implied by the NIG process. This generalises the marginal result obtained in Mechkov (2015), which in part inspired this work.

To provide an example of a probabilistic application not present in this work (at least not yet), consider the effect of a Girsanov measure change in the GCIR process. This may be considered a continuous map, and so by letting $n \to \infty$ we are taught how the limiting IG process transforms; this turns out to correspond to an Escher transform. By applying the results of Sato (1999), this may be confirmed by inspection, but once we transition to the generalised Heston model and more complicated functionals, our convergence result provides a general method for establishing Lévy measure changes consistent with simple Girsanov transformations. NIG transformation formulas and FX applications are available for now at bit.ly/2U2K2pm.

For now, our own interests lie primarily in the theoretical consequences of our results, as there are many to still be addressed. Firstly, it is tempting to consider generalisations of the IVP in our introduction

$$\varphi_n'(t) = f_n\big(t, \varphi_n(t)\big), \quad f_n(t,x) = a_n(x) + b_n(t) + c_n, \quad t \in [0,T),$$

for example, by simply considering other bijective $ne \neq b_n \in C_1$, or raising the dimension to correspond with higher-dimensional Itô SDEs. However, since we have proved that the GCIR framework is accommodate any volatility path, and so model through the measure choice $\mathbb{P}$, we consider these generalisation to be already accommodated for the most part in our framework.

For those interested in jumps, it would appear that our framework may be taken in this direction by simply letting $\omega \in D$. This would rest upon the application of the Carathéodory existence theorem in place of Cauchy-Peano used here. We don't presently envisage problems with this avenue, although one should be comfortable with the resulting solutions and properties holding in the 'extended' sense of Carathéodory.

We have chosen to focus here on a GCIR framework, but one might prefer to adopt another, such as GeOU; generalised exponential Ornstein-Uhlenbeck. The resulting IVP, taking $a_n = -n\mathrm{e}$, $b_n = n(\omega + \mathrm{e})$ and $c_n = c \in \mathbb{R}$ above, may be solved directly, so existence and uniqueness is trivial. However, with these benefits come complications. What helped us a great deal in the GCIR case was being able to obtain the characteristic function of the integrated process, and so having a strong indication of the type of limiting process. In the OU case, one would most naturally work with an integrated volatility process like $\bar{X}^n_t := \int_0^t e^{X^n_u} \mathrm{d}u$, which not being affine has no known closed from characteristic function.

Perhaps of most interest to us is to refine the topology on which our type of convergence results take place, or even the object one uses to define a volatility path, which should provide a better foundation for applying Proposition 1.5 and so a better understanding of how other models, derivative payoffs and prices will behave. We expand upon this here and provide a possible solution.

Parametric volatility. We finish with this unusual discussion, by questioning the approach taken here to convergence results. This is inspired in part by the space F introduced at the end of Whitt (2002) (a space of parametric representation equivalence classes), as well as the approach taken by physicists to the modeling of particle trajectories as 'world lines'.

Before one assumes this to be an attempt to promote another concept of physics to finance, consider the logging of trade data; is time *ever* used to index such data? Rather, a practically arbitrary identification parameter is used, alongside which a timestamp and price (temporal and spatial locations), among other details, are recorded. We do not stand to lose anything if we model paths in this parametric way.

We instead started in a completely 'standard' framework, with volatility paths φ_n constituting particular elements of C. Establishing consequential results from this standard framework is important, but there are shortcomings of this approach.

One should appreciate that, although we proved a limit $\varphi := R(\mathrm{e} - \omega)$ exists in (D, M), it is clearly not the case that our continuous paths φ_n actually 'become discontinuous' as $n \to \infty$. It was a convenient, but not entirely helpful, *choice* to assume the limit as such, because in doing so, some information is lost. We could say that (D, M) can be 'leaky' in this respect.

By accepting losses of information, we accept complications, which are very easy to attribute elsewhere. As an example, our treatment of the composition map required three things; the introduction of the space E, the new operator $\bullet$ and the relaxation to a Hausdorff metric. From the standard perspective, these introduction appear necessary, but are they just consequences of an unnecessary choice? Can we model volatility paths and functionals thereof in a more helpful way, to avoid transitions into more complicated spaces and the introduction of courser metrics?

$\circ \circ \circ$

Suppose that from the outset we consider a (cumulative) volatility path X as a world line; a parametric representation of a path in both space and time, jointly indexed by an essentially arbitrary parameter $\tau \in [0, \infty)$. For example, for non-decreasing $t, x \in C$, representing abstract paths of time and space respectively, define

$$X := \Big\{ \big(t(\tau), x(\tau)\big) \in [0, \infty)^2 : \tau \in [0, \infty) \Big\}$$

and notice the invariance of X to continuous mappings of the index τ. For short let us write $X(\tau) = \big(t(\tau), x(\tau)\big)$, and $X(\tau) \sim \tilde{X}(\tau)$ if two representations are connected by an explicit mapping of their index.

The 'standard' framework used here and elsewhere may be considers as a special case of this parametric approach, with $t = \mathrm{e}$. We would write our case of $\varphi_n \in C_1$ in parametric form as

$$X_n(\tau) = \big(\tau, \varphi_n(\tau)\big).$$

Crucially, wanting to avoid the loss of information in an assumed limit $\varphi := R(\mathrm{e} - \omega) \in D$, we are free to make the transformation $\tau \mapsto \hat{\varphi}_n(\tau) := \varphi_n^{-1}(\tau)$, for

$$X_n(\tau) \sim \tilde{X}_n(\tau) := \big(\hat{\varphi}_n(\tau), \tau\big) \to \big(\hat{\varphi}(\tau), \tau\big) =: X(\tau).$$

Recall the definition $\hat{\varphi} := M(\mathrm{e} - \omega) \in C$, and notice that this parametric result therefore holds in the uniform product topology.

The power of this choice is evidenced by the consideration of a case like the composition $\omega \circ \varphi_n$, for which we established the limit $\omega \bullet \varphi$ in (E, H). In a parametric framework, defining $H_n(\tau) := \big(\tau, (\omega \circ \varphi_n)(\tau)\big)$ one simply obtains

$$H_n(\tau) \sim \tilde{H}_n(\tau) := \big(\hat{\varphi}_n(\tau), \omega(\tau)\big) \to \big(\hat{\varphi}(\tau), \omega(\tau)\big) =: H(\tau)$$

which takes place on the same uniform topology, and precisely retains the information telling us what H does while $\varphi \in D$ would otherwise jump. This information is represented in this parametric form by what ω does over τ intervals where $\hat{\varphi}$ is static.

We do not claim yet that there are great practical advantages to modeling or simulating a path in this way, by employing an arbitrary indexing parameter, but by remaining in this uniform topology we have clearly avoided technicalities of others.

In Whitt (2002) it is suggested that *if only one topology on the function space D is to be considered, then it should be the M_1 topology.* This is of course taken out of its context, but in ours, we would suggest that one should further try to avoid the introduction of 'leaky' topologies like (D, M) from the outset, rather than face growing topological complexities.

As the modeling of volatility paths in this parametric way would clearly be very unconventional, we are content to leave this for future work, and believe the present approach and results to be more informative and helpful for the mathematical finance community.

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References

E. Abi Jaber and O. El Euch (2018): Multi-factor approximation of rough volatility models. Available at arXiv: arxiv.org/abs/1801.10359.

D. Applebaum (2009): Lévy Processes and Stochastic Calculus Second Edition. *Cambridge University Press.*

O. E. Barndorff-Nielsen and N. Shephard (2001): Modelling by Lévy Processess for Financial Econometrics. In: Lévy Processes. *Birkhäuser*, Boston, MA.

C. Bayer, P. Friz and J. Gatheral (2015): Pricing under rough volatility. *Quantitative Finance*, 16(6), 887–904.

L. Bergomi (2016): Stochastic Volatility Modelling. *Chapman & Hall/CRC Press.*

P. Billingsley (1999): Convergence of Probability Measures. *Wiley.*

E. A. Coddington and N. Levinson (1955): Theory of ordinary differential equations. *Tata McGraw-Hill*, Twelfth reprint, 1991.

J. C. Cox, J. E. Ingersoll, and S. A. Ross (1985): A theory of the term structure of interest rates. *Econometrica*, 53(2) 385–407.

K. È. Dambis (1965): On decomposition of continuous submartingales. *Teor. Veroyatnost. i Primenen.*, 10(3), 438–448.

A. De Col, A. Gnoatto and M. Grasselli (2013): Smiles all around: FX joint calibration in a multi-Heston model. *Journal of Banking & Finance*, 37, 3799–3818.

L. Dubins and G. Schwarz (1965): On continuous martingales. *Proceedings of the National Academy of Sciences of the United States of America*, 53(5), 913–916.

D. Dufresne (2001): The integrated square-root process. *Research paper number 90, Univ. Melbourne*.

J. Gatheral, T. Jaisson and M. Rosenbaum (2018): Volatility is rough. *Quantitative Finance*, 18(6), 933–949.

S. L. Heston (1993): A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options. *The Review of Financial Studies*, 6(2), 327–343.

I. Karatzas and S. E. Shreve (1988): Brownian Motion and Stochastic Calculus. *Springer-Verlag*.

S. Mechkov (2015): Fast-reversion limit of the Heston model. Available at SSRN: ssrn.com/abstract=2418631.

A. A. Muravlev (2011): Representation of a fractional Brownian motion in terms of an infinite-dimensional Ornstein– Uhlenbeck process. *Russian Mathematical Surveys*, 66(2), 439–441

G. Peano (1890): Demonstration de l'intégrabilité des équations différentielles ordinaires. *Mathematische Annalen*, 37(2), 182–228.

Y. V. Prokhorov (1956): Convergence of random processes and limit theorems in probability theory. *Theory of Probability and Applications I*, 2(2), 157–214.

A. Puhalskii and W. Whitt (1997): Functional large deviations principles for first-passage-time processes. *Annals of Applied Probability*, 7(2), 362–381

K. Sato (1999): Lévy Processes and Infinitely Divisible Distributions. *Cambridge University Press.*

A. V. Skorokhod (1956): Limit theorems for stochastic processes. *Theory of Probability and its Applications*, 1(3), 261–290

A. V. Skorokhod (1965): Studies in the theory of random processes. *Addison-Wesley.*

T. Yamada and S. Watanabe (1971): On the uniqueness of solutions of stochastic differential equations. *J. Math. Kyoto Univ. (JMKYAZ)*, 11(1), 155–167

W. Whitt (1971): Weak convergence of first passage time processes. *Journal of applied probability*, 8, 417–422

W. Whitt (2002): Stochastic-Process Limits. *Springer.*

A Marginal convergence

We only provide a sketch proof here, since this result is implied by our main work. A alternative proof of the conclusion $H_t^n \xrightarrow{d} \mathrm{NIG}(\alpha, \beta, \delta t, \mu t)$ is in Mechkov (2015).

Theorem A.1 (Marginal Heston 'fast-reversion' convergence)**.** *For $\varsigma, n > 0$, let X_n solve the CIR Itô SDE*

$$\mathrm{d}X_t^n = n\left(\varsigma\sqrt{X_t^n}\mathrm{d}W_t^0 + (1 - X_t^n)\mathrm{d}t\right), \quad X_0^n = X_0 > 0,$$

and H_n be defined for $\sigma > 0$, $\rho \in (-1, 1)$ by

$$H_t^n := \sigma \int_0^t \sqrt{X_u^n}\mathrm{d}\big(\bar{\rho}W_u^1 + \rho W_u^0\big) - \frac{1}{2}\sigma^2 \int_0^t X_u^n \mathrm{d}u$$

where $\bar{\rho} := \sqrt{1 - \rho^2}$. Then, for any $t > 0$, $\bar{X}_t^n := (\sigma\bar{\rho})^2 \int_0^t X_u^n \mathrm{d}u$ and H_t^n satisfy

$$(\bar{X}_t^n, H_t^n) \xrightarrow{d} (\bar{X}_t, H_t), \quad \bar{X}_t \stackrel{d}{=} \mathrm{IG}(\delta t, \gamma), \quad H_t \stackrel{d}{=} \mathrm{NIG}(\alpha, \beta, \delta t, \mu t), \quad \text{as } n \to \infty,$$

where

$$\beta = \frac{2\rho - \sigma\varsigma}{2\sigma\bar{\rho}^2\varsigma}, \quad \delta = \frac{\sigma\bar{\rho}}{\varsigma}, \quad \gamma = \frac{1}{\sigma\bar{\rho}\varsigma}, \quad \mu = -\frac{\sigma\rho}{\varsigma}$$

and $\alpha = \sqrt{\beta^2 + \gamma^2}$.

Proof. (Sketch.) The characteristic function of the random number $\bar{X}_t^n$ may be obtained by solving its backward Kolmogorov equation. See Dufresne (2001), for example. This converges to the characteristic function of the IG distribution as $n \to \infty$, and so by Lévy's continuity theorem one obtains $\bar{X}_t^n \xrightarrow{d} \bar{X}_t \stackrel{d}{=} \mathrm{IG}(\delta t, \gamma)$.

By using the DDS theorem, the Heston random number H_t^n satisfies

$$H_t^n \overset{d}{=} \left(W_1 \circ \bar{X}_n\right)_t + \frac{\rho}{\bar{\rho}}\left(W_0 \circ \bar{X}_n\right)_t - \frac{1}{2\bar{\rho}^2}\bar{X}_t^n$$

and so we may conclude using Billingsley (1999)

$$H_t^n \xrightarrow{d} H_t \overset{d}{=} \left(W_1 \circ \bar{X}\right)_t + \frac{\rho}{\bar{\rho}}\left(W_0 \circ \bar{X}\right)_t - \frac{1}{2\bar{\rho}^2}\bar{X}_t$$

as $n \to \infty$. The IG random number $\bar{X}_t$ satisfies $\sigma\bar{\rho}\varsigma\left(W_0 \circ \bar{X}\right)_t = \bar{X}_t - (\sigma\bar{\rho})^2 t$, almost by definition (otherwise characteristic functions can be checked), and so one eventually obtains the neat normal variance-mean mixture representation

$$H_t \overset{d}{=} \left(W_1 \circ \tilde{X}\right)_t + \beta\tilde{X}_t + \mu t$$

with β and μ as given. Since the mixing density, that of $\bar{X}_t$, is $\mathrm{IG}(\delta t, \gamma)$, this shows that $H_t \overset{d}{=} \mathrm{NIG}(\alpha, \beta, \delta t, \mu t)$, and actually that $(\bar{X}_t^n, H_t^n) \xrightarrow{d} (\bar{X}_t, H_t)$ as $n \to \infty$, where $(\bar{X}_t, H_t)$ is distributed as the (IG, NIG) Lévy motion of Barndorff-Nielsen and Shephard (2001). $\square$

CIR characteristic function. Here we provide what we consider the tidiest CIR characteristic function available, guaranteed well-defined in an open ball around the origin. This extends in a straightforward way to the Heston characteristic function.

Let $X = (X_1, X_2, X_3)$, where X_1 solves the CIR SDE

$$\mathrm{d}X_t^1 = n\left(\varsigma\sqrt{X_t^1}\mathrm{d}W_t + (1 - X_t^1)\mathrm{d}t\right), \quad X_0^1 \geq 0$$

and X_2, X_3 are defined by the implicit components

$$X_t^2 := \int_0^t X_u^1 \mathrm{d}u, \quad X_t^3 := \varsigma\int_0^t \sqrt{X_u^1}\mathrm{d}W_u.$$

Then for $u = (u_1, u_2, u_3) \in \mathbb{C}^3$, we have $\mathbb{E}[e^{u \cdot X_t} | X_s] = e^{\varphi_0(t-s) + \varphi_1(t-s)X_s^1}$, where

$$\varphi_0(\tau) = \frac{1}{\varsigma^2}(1 - \varsigma^2 u_3)\tau - \frac{2}{n\varsigma^2}\log\left(\cosh\left(\frac{1}{2}n\vartheta_1\tau\right) + \frac{\vartheta_2}{\vartheta_1}\sinh\left(\frac{1}{2}n\vartheta_1\tau\right)\right)$$

$$\varphi_1(\tau) = \frac{1}{n\varsigma^2}\left(1 - \varsigma^2 u_3 - \vartheta_1\frac{\vartheta_2 + \vartheta_1\tanh(\frac{1}{2}n\vartheta_1\tau)}{\vartheta_1 + \vartheta_2\tanh(\frac{1}{2}n\vartheta_1\tau)}\right)$$

and

$$\vartheta_1 := \sqrt{1 - 2\varsigma^2(u_2 + u_3)}, \quad \vartheta_2 := 1 - \varsigma^2 u_3 - n\varsigma^2 u_1.$$