Repricing the cross smile: an analytic joint density

Derivatives contracts on multiple foreign exchange rates must be priced to avoid arbitrage by contracts on the cross-rates. Given the triangle of smiles for two underlyings and their cross, **Peter Austing** *provides an analytic formula for a joint probability density such that all three vanilla markets are repriced. The method extends to N dimensions and leads to simple necessary conditions for a triangle of smiles to be arbitrage-free in the model*

When valuing a derivatives contract whose payout depends on two assets, the correlation between the random processes followed by those two assets must be taken into account. In most asset classes, there is no liquid instrument to determine that correlation. This makes the exposure to correlation hard to hedge, but straightforward from a modelling point of view since a single number, perhaps calculated from a historic time series of spot returns, can be used.

Foreign exchange is different. Let us consider the concrete case of a contract involving the euro/dollar and sterling/dollar exchange rates at a given expiry, T. We denote the spot rates S_1 and S_2 respectively. Then, since the euro/sterling exchange rate $S_3 = S_1/S_2$ is also liquidly traded, any model we use would need to correctly reprice euro/sterling vanilla options in order to avoid arbitrage. This places a heavy constraint on the choice of correlation between the two driving assets. We will call the currency pairs that, for the purpose of our modelling, we use as a basis for exchange rates the driving pairs and the other exchange rates that can be determined from them the crosses. In this case, euro/dollar and sterling/dollar are the drivers, and euro/sterling is the cross.

In the Black-Scholes model (1973), the condition relating the correlation to the volatilities is well known to all foreign exchange analysts. We assume that the drivers have a common domestic currency, dollars in our example. Then, choosing the dollar bond as numeraire, the Black-Scholes processes are:

$$S_1 = F_1 e^{-\frac{1}{2}\sigma_1^2 T + \sigma_1\sqrt{T}X_1} \tag{1}$$

$$S_2 = F_2 e^{-\frac{1}{2}\sigma_2^2 T + \sigma_2\sqrt{T}X_2} \tag{2}$$

$$E\left[X_1 X_2\right] = \rho \tag{3}$$

where F_i are the forwards, σ_i the volatilities and X_i normal variables with correlation ρ.

The risk-neutral process for the cross is given by the quotient $S_3 = S_1/S_2$, which is lognormal with volatility σ_3 as long as:

$$\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2 = \sigma_3^2 \tag{4}$$

This formula is known as the triangle rule. As long as it holds, the two-asset Black-Scholes model (1) to (3) will give correct Black-Scholes prices for European-style contracts that depend only on the cross rate S_3.

It is standard practice among market participants to back correlations out from at-the-money volatilities using the triangle rule. As long as options on the cross are traded liquidly, this allows us to replace exposure to correlation risk with a vega that can be hedged easily.

Once we have a correlation, all of the standard multi-asset methods are available to us. For example, we could simulate S_1 and S_2 with two correlated local volatility (Dupire, 1994) processes. Or, for a European-style payout, we could construct a Gaussian copula. We can then do the experiment of using our chosen method to value the contract:

$$\left(S_1 / S_2 - K\right)_+ S_2 \tag{5}$$

which is actually just a vanilla option on the cross, with the payout converted into dollars at expiry. We can do this for a number of strikes, and so plot the smile implied for the cross by our model. It will not match the true market smile for the cross, being typically too flat. Worse, there is no reason it would reprice even the at-the-money option correctly.

Thus we have an arbitrage that is likely to lead to a bleed of money through the life of the trade as we vega-hedge the cross. If some part of the trade, perhaps hidden to us, amounts to a European-style payout on the cross alone, we will be in danger of selling a trade that can be arbitraged at inception.

To address this issue, we would like to construct a joint probability density with the property that vanilla options on drivers and cross are correctly repriced. This problem has been tackled by Bennett & Kennedy (2004) by constructing a copula that can be calibrated numerically to the cross smile (and see Salmon & Schleicher, 2006, for a review of copula methods in foreign exchange). The purpose of this article is to construct such a density analytically. Our construction provides natural no-arbitrage conditions on the triangle of smiles, and it extends to N dimensions providing a joint probability distribution that matches all foreign exchange asset and cross smiles. Certain contracts (best-ofs, worst-ofs and multi-asset digitals) can be valued analytically in the model. Others can be priced with numerical integration, including, for example, quanto options whose valuation can be rather elusive (Jäckel, 2010).