# Hedging under rough volatility

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#### Abstract

In this chapter we first briefly review the existing approaches to hedging in rough volatility models. Next, we present a simple but general result which shows that in a one-factor rough stochastic volatility model, any option may be perfectly hedged with a dynamic portfolio containing the underlying and one other asset such as a variance swap. In the final section we report the results of a back-test experiment using real data, where VIX options are hedged with a forward variance swap. In this experiment, using a rough volatility model allows to almost completely remove the bias and reduce the overall hedging error by a factor of 27% compared to traditional diffusion-based models.

Keywords: Rough fractional stochastic volatility, forward variance, martingale representation, hedging, back testing, volatility options, VIX options

### 1 Introduction

The many advantages of rough volatility models have been outlined in previous chapters. One of the only potential challenges that remain to be addressed in practice is an (apparent) difficulty to hedge derivatives in rough models. Hedging in rough volatility models can seem intricate since the dynamics of rough volatility models involve a fractional Brownian motion. In this chapter we demonstrate how this apparent challenge can be overcome in different modelling scenarios exhibiting different levels of generality, which allow us to derive (often explicit) hedging strategies. For building a hedging portfolio, one essentially needs to compute conditional expectations of the form

$$C_t = \mathbb{E}[f(S_T)|\mathcal{F}_t],\tag{1}$$

where *f* is a deterministic payoff function, and determine the associated martingale representations. Classical theory tells us that the option payoff can be replicated at time t for a price  $\mathbf{P}_{T-t}f(S_t)$ , where,  $(\mathbf{P}_t)_{t\geq 0}$  is the semigroup on  $C^b(\mathbb{R}^2)$  generated by the infinitesimal generator A associated with the instantaneous covariance of S and with the (local) martingale problem characterising the law of the process S. Under the assumption that an equivalent local martingale measure exists (and even beyond that, see [19], classical theory ([1, 8, 16]) gives conditions when a contingent claim can be hedged (optimally). In a classical Markovian setting, an optimal trading strategy h can be derived directly from the solution  $C_t$  to the Cauchy problem  $\partial_t C_t(s) - AC_t(s) = 0$  associated with the generator A cf. [19]. The non-Markovian nature (at least in the finite-dimensional sense) of rough models coming from the fractional Brownian driver leaves most of these results out of scope for rough volatility. In addition, this very (non-Markovian) nature of rough volatility models also prohibits the direct use of (PDE-based) efficient numerical methods for a tractable evaluation of prices (1) and associated hedging portfolios, which can make hedging more challenging than in classical models. In the case of affine rough models it is possible to exploit the affine structure to derive efficient pricing and hedging. For the general case Monte Carlo methods have been derived [5, 10, 18] for pricing under rough volatility<sup>1</sup> which can in some cases be computationally slow, but by using deep neural networks it is possible to speed up these pricing methods by several orders of magnitude as demonstrated in [4, 14]. Also, deep neural networks can aid the direct computation of hedging strategies as in [7, 15]. Indeed, the deep hedging framework is applicable in great generality, including rough volatility models as recently demonstrated in [15]. These hedging portfolios are obtained, based on the idea that every investment strategy gives rise to a profit and loss, whose distribution can be optimised with respect to specific risk measures. More specifically, Horvath Teichmann and Zuric [15] compute hedging strategies for the rough Bergomi model, numerically building on results of Gassiat [11] and Viens and Zhang [20] and demonstrate the applicability of deep hedging for the calculation of hedging strategies in rough models.

The special role of the forward variance curve in rough models As mentioned, in some cases within the rough volatility framework, it is possible to derive hedging strategies more explicitly. In the rough Heston model for example, El Euch and Rosenbaum [9] obtain explicit hedging strategies that lead to *perfect hedging*. For obtaining these results, it is central to identify the relevant state variables: These are in rough Heston models namely (i) the underlying  $(S_t)_{0 \le t \le T}$  and (ii) the so-called

<sup>&</sup>lt;sup>1</sup>For small values of the Hurst parameter the computational cost of calculation of prices increases.

forward variance curve

$$(\mathbb{E}[V_{\theta+t}|\mathcal{F}_t])_{\theta\geq 0},\tag{2}$$

where  $V_t$  is the instantaneous variance of the underlying price at time t. In the setting of [9], the conditional expectation (1) above can then be written explicitly as

$$C_t = C(T - t, S_t, (\mathbb{E}[V_{\theta + t}|\mathcal{F}_t])_{\theta \ge 0})$$
(3)

where *C* is some deterministic function. Indeed it can be observed more broadly, that replicating portfolios in rough volatility models typically contain the *underlying asset* and the *forward variance curve*. In fact, not only in the case of the rough Heston model but in all affine rough models, a close relation to the forward variance curve can be drawn from the affine structure, as highlighted in Chapter 8. This gives rise to one of the perspectives presented in Chapter 8, viewing rough affine models as forward variance models. Viens and Zhang in [20] confirm this idea for general Volterra-type stochastic differential equations. In fact, while for some practitioners the idea of using the forward variance curve for hedging (even vanilla options) may come as a surprise, the observation of the importance of the the forward variance curve was already emphasized in [3] and is also very close in spirit to the approach developed by Bergomi in [6].

**Martingale problems and Markovianity** Viens and Zhang present in [20] a martingale approach, for general<sup>2</sup> Volterra stochastic differential equations. While for affine rough models it has been noted that martingale problems (connected to hedging problems) are more convenient thanks to the affine structure, the introduction of a martingale component is also key in the general case in [20] for recovering the flow property, which makes it possible to derive a certain "Markov" property for rough models.

Pricing and hedging of *volatility options* under rough volatility models has been considered in [13], where the special role of the forward variance is re-confirmed as well as martingality considerations revisited. In particular, it is shown that by focusing on the forward variance instead of the instantaneous volatility, one recovers the martingale framework and in particular the classical martingale representation property of option prices. This makes it possible to compute the hedge ratios, and to show that options can be hedged with a finite number of liquid assets, as in the classical setting. To calibrate VIX option smiles via rough volatility we consider extended lognormal models by adding volatility modulation through an independent stochastic factor in the Volterra integral which preserves part of the analytical tractability of the lognormal setting by extending it through an affine structure,

<sup>&</sup>lt;sup>2</sup>This in particular includes the rough Heston model; the rough Bergomi model; fractional Ornstein-Uhlenbeck process as well as affine Volterra processes.

which makes it possible to develop approximate option pricing and calibration algorithms based on Fourier transform techniques.

In this chapter, we showcase these ideas in relatively transparent and illustrative settings. We discuss the role of the forward variance curve to establish (perfect) hedging in rough models, and present a hands-on empirical study illustrating the role of the Hurst parameter (driving the roughness of the paths) on the hedging performance for hedging in VIX options.

## 2 A theoretical framework

The purpose of this section is to illustrate an infinite-dimensional Markov nature of rough volatility models, which enables us to hedge options without any "memory" of the past. While fractional Brownian motions have (long or short) memory properties, we see that the memory is stored in an option market.

### 2.1 The model

Here we consider a 2 factor model; there is a (two-sided) 2-dimensional standard Brownian motion  $(\overline{W}^1, \overline{W}^2)$  on a probability space  $(\Omega, \mathcal{F}, P)$  with a filtration  $\{\mathcal{F}_t\}$ being the augmentation of the one generated by the Brownian motion. We consider a hypothetical option market where call and put options are traded for all strike prices  $K \ge 0$  and maturities  $T \ge 0$ . Their prices at time  $t \ge 0$  are denoted by  $C_t(K, T)$  and  $P_t(K, T)$  respectively. The underlying asset price process of the options is denoted by *S* and we suppose

$$C_t(K,T) = E_Q[(S_T - K)_+ | \mathcal{F}_t], \ P_t(K,T) = E_Q[(K - S_T)_+ | \mathcal{F}_t], \ S_t = C_t(0,T)$$

for all  $K \ge 0$ ,  $T \ge 0$  and  $t \ge 0$ , where Q is an equivalent measure to P of which the existence is assumed. Here and hereafter we assume risk-free rates are zero for brevity.

Here we introduce a SABR/Bergomi-type stochastic volatility model

$$dS_t = f(S_t) \sqrt{V_t^t} \left[ \rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2 \right],$$
  

$$dV_t^u = V_t^u g(u - t) dW_t^1, \quad t < u$$
(4)

where  $(W^1, W^2)$  is a 2-dimensional  $\{\mathcal{F}_t\}_{t\geq 0}$ -Brownian motion under Q, f and g are deterministic Borel functions on  $[0, \infty)$ , and  $\rho \in (-1, 1)$ . We assume g is locally square integrable, so that we have explicit expressions

$$V_t^u = V_s^u \exp\left\{\int_s^t g(u-v) dW_v^1 - \frac{1}{2} \int_s^t g(u-v)^2 dv\right\}$$
(5)

for  $0 \le s \le t \le u$ . Note that  $E_Q[V_t^t|\mathcal{F}_s] = V_s^t$  for  $t \ge s$ . The case f(s) = s,  $g(u) = \eta u^{H-1/2}$  corresponds to the rough Bergomi model introduced by [3]. A volatility process driven by a fractional Brownian motion can be treated in this framework. For example, if the log volatility is a stationary fractional Ornstein-Uhlenbeck process (see [2])

$$\log v_t = \frac{1}{2} \int_{-\infty}^t g(t-s) \mathrm{d}\bar{W}_s^1, \ g(u) = \eta u^{H-1/2} - \eta \lambda e^{-\lambda u} \int_0^u v^{H-1/2} e^{\lambda v} \mathrm{d}v$$

under *P* and the volatility risk premium is deterministic, then we have (4) and (5) with f(s) = s,  $V_u^t = E_Q[v_t^2|\mathcal{F}_u]$  for  $u \le t$  and a suitable family  $\{V_0^t\}_{t\ge 0}$  of  $\mathcal{F}_0$  measurable random variables (recall that  $\overline{W}^1$  is a two-sided Brownian motion). We call the curve

$$\hat{V}_t: \theta \mapsto V_t^{t+\theta}$$

the forward variance curve at time *t*.

**Proposition 1** The forward variance curve  $\{\hat{V}_t\}_{t\geq 0}$  is a Markov process with state space  $C[0,\infty)$ .

*Proof:* By (5), we have for  $t \ge s$ ,

$$\hat{V}_t(\theta) = \hat{V}_s(\theta + t - s) \exp\left\{\int_s^t g(\theta + t - u) \mathrm{d}W_u^1 - \frac{1}{2}\int_s^t g(\theta + t - u)^2 \mathrm{d}u\right\}.$$

Since the exponential term is independent of  $\mathcal{F}_s$ , the result follows.

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**Corollary 1** (*S*,  $\hat{V}$ ) *is a Markov process with state space*  $[0, \infty) \times C[0, \infty)$ *.* 

Now we discuss that  $\hat{V}$  is an observable state. By Itô's formula, we have

$$\int_{t}^{t+\theta} V_{u}^{u} \mathrm{d}u = \int_{t}^{t+\theta} \frac{S_{u}^{2}}{f(S_{u})^{2}} \mathrm{d}\langle \log S \rangle_{u},$$

which is the payoff of a weighted variance swap. The fair strike of this swap is

$$E_{Q}\left[\int_{t}^{t+\theta} \frac{S_{u}^{2}}{f(S_{u})^{2}} \mathrm{d}\langle \log S \rangle_{u} \middle| \mathcal{F}_{t}\right] = \int_{t}^{t+\theta} E_{Q}[V_{u}^{u}|\mathcal{F}_{t}] \mathrm{d}u = \int_{t}^{t+\theta} V_{t}^{u} \mathrm{d}u.$$

Therefore the forward variance curve  $\hat{V}$  is the derivative in  $\theta$  of this derivative price. It is uniquely determined by call and put option prices in a model-free manner as follows; assume 1/f is locally square integrable on  $(0, \infty)$  and let

$$h(x) = \int_{1}^{x} \int_{1}^{y} \frac{2}{f(z)^2} dz dy.$$

Then, again by Itô's formula,

$$h(S_{t+\theta}) = h(S_t) + \int_t^{t+\theta} h'(S_u) dS_u + \int_t^{t+\theta} \frac{S_u^2}{f(S_u)^2} d\langle \log S \rangle_u$$

and by an integration-by-parts formula,

$$h(S_{t+\theta}) = h(S_t) + h'(S_t)(S_{t+\theta} - S_t) + \int_0^{S_t} (K - S_{t+\theta})_+ h''(K) dK + \int_{S_t}^{\infty} (S_{t+\theta} - K)_+ h''(K) dK$$

This means a model-free replication of the weighted variance swap payoff is given as a static portfolio of call and put options with weight  $h''(K) = 2/f(K)^2$ . The replication price is given by

$$U_t(\theta) := 2 \int_0^{S_t} P_t(K, t+\theta) \frac{\mathrm{d}K}{f(K)^2} + 2 \int_{S_t}^\infty C_t(K, t+\theta) \frac{\mathrm{d}K}{f(K)^2}.$$

Finally we get  $\hat{V}_t(\theta) = \frac{\partial}{\partial \theta} U_t(\theta)$ .

Consequently, for a possibly path-dependent functional  $F = F(\{S_u\}_{u \in [t,T]})$ , its conditional expectation  $E_Q[F|\mathcal{F}_t]$  is a function of  $S_t$  and  $\{\hat{V}_t(\theta)\}_{\theta \ge 0}$ , which are observable from the option market at time t.

#### 2.2 Perfect hedging

We are considering an infinite dimensional Markov model. But we have only two factors and so, in light of the martingale representation theorem, every square integrable payoff is perfectly replicated with a dynamic portfolio of two traded assets. A natural choice of the two would be the underlying asset and the weighted variance swap (with a fixed maturity).

As a hedging instrument, the replication portfolio for the weighted variance swap is more convenient than the weighted variance swap itself because it is a local martingale. Let

$$U_t^T = \int_0^t (h'(S_0) - h'(S_u)) dS_u + 2 \int_0^{S_0} P_t(K, T) \frac{dK}{f(K)^2} + 2 \int_{S_0}^\infty C_t(K, T) \frac{dK}{f(K)^2}$$

be the time t value of the replication portfolio with maturity T initiated at time 0. We have then

$$U_t^T = E_Q \left[ \int_0^T V_u^u du \middle| \mathcal{F}_t \right]$$
  
=  $E_Q \left[ \int_0^T \left\{ V_0^u + \int_0^u V_s^u g(u-s) dW_s^1 \right\} du \middle| \mathcal{F}_t \right]$   
=  $\int_0^T V_0^u du + \int_0^t dW_s^1 \int_s^T V_s^u g(u-s) du.$ 

Therefore,

$$\mathbf{d}U_t^T = \mathcal{D}_g U_t^T \mathbf{d}W_t^1, \tag{6}$$

where

$$\mathcal{D}_g U_t^T = \int_t^T V_t^u g(u-t) \mathrm{d}u = \int_t^T \frac{\partial U_t^u}{\partial u} g(u-t) \mathrm{d}u.$$

**Proposition 2** For any  $F \in L^2(\mathcal{F}_{\tau}, Q), \tau \in (t, T)$ , there exists an adapted process  $(H^S, H^U)$  such that

$$F = E_{Q}[F|\mathcal{F}_{t}] + \int_{t}^{\tau} H_{v}^{S} \mathrm{d}S_{v} + \int_{t}^{\tau} H_{v}^{U} \mathrm{d}U_{v}^{T}.$$

*Proof:* By the martingale representation theorem, there exists  $(H^1, H^2)$  such that

$$F = E_Q[F|\mathcal{F}_t] + \int_t^\tau H_v^1 dW_v^1 + \int_t^\tau H_v^2 dW_v^2.$$

Since

$$dW_t^1 = \frac{1}{\mathcal{D}_g U_t^T} dU_t^T,$$
  
$$dW_t^2 = \frac{1}{\sqrt{1 - \rho^2}} \left[ \frac{1}{f(S_t) \sqrt{V_t^t}} dS_t - \rho dW_t^1 \right]$$

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by (6), we have the result.

## 3 Hedging VIX options: empirical analysis

In this section, we illustrate the advantages of rough volatility modeling for managing a simple VIX option. We consider three models for the VIX index: the Black-Scholes model (geometric Brownian motion), the CIR model, and the rough stochastic volatility model (where the volatility is the exponential of a fractional Brownian motion). Since we are interested in hedging short-term options, we use simplified version of the models without drift and neglect the effect of the interest rate. As a result, all models have only one parameter to be estimated (see below).

In each model, we perform a series of back-tests of dynamic hedging of a VIX option with a forward variance swap with the same maturity as the option and with the duration corresponding to that of the VIX (1 month). In all tests, the hedging portfolio is readjusted daily using the closing prices of the hedging instruments. The test is performed 1000 times, starting on each working day *t* between Jan 10, 2012 and Apr 29, 2016. The back-test is organized as follows:

• The parameter is estimated on the 88-day period preceding day *t*.

	<b>Black-Scholes</b>		CIR		RFSV	
	No hedge	Hedge	No hedge	Hedge	No hedge	Hedge
Mean	0.01445	0.005336	0.01363	0.003399	0.009919	0.0006345
Std. dev.	0.01896	0.003555	0.02069	0.006506	0.01880	0.004141
RMSE	0.02384	0.006412	0.02478	0.007341	0.02125	0.004190
Red. factor	3.7176		3.7176		5.0724	

Table 1: Empirical Performance of hedging strategies based on different models

- The initial value of the hedging portfolio is initialized with the ATM VIX option price with maturity 1.5 months computed within the model, and the quantity of the hedging asset in the portfolio is initialized with the corresponding model-based hedge ratio.
- For 29 working days following day *t*, each day the portfolio value is readjusted following the change in the value of the hedging asset, and the hedge ratio is recomputed.
- At the end of the 29 working day period, the P&L of the hedging portfolio is recorded, and the no-hedge P&L is recorded as the difference between the option price at the beginning and at the end of the period.

The back test uses synthetic forward variance curve data, computed from the historical prices of S&P index options, downloaded from the WRDS database. The detailed description of models and hedging procedures is given below. Table 3 presents the main results of the back test. We see that while the Black-Scholes and CIR benchmarks appear to have similar performance, the RFSV model exhibits a much lower bias, and a RMSE which is 27% lower than the other two models. Figure 1 plots the back-test PnL evolution as function of the starting date of the back test. The consistently low bias of the strategy based on the RFSV model is clearly visible here.

**Black-Scholes model** Let VIX<sub>t</sub> be the VIX index at time t, and let  $F_t^T$  be the T-forward variance swap at time t, which refers to the same period as the VIX index, that is,  $F_t^T = E_Q[VIX_T^2|\mathcal{F}_t]$ . Assume that the VIX index follows the log-normal dynamics VIX<sub>t</sub> =  $e^{X_t}$ , where X is an OU process  $dX_t = \kappa(\theta - X_t)dt + \gamma dW_t$  under the risk-neutral probability. Then, forward variance swap has dynamics

$$\mathrm{d}F_t^T = 2F_t^T e^{-\kappa(T-t)} \gamma \mathrm{d}W_t.$$

When close to term, the exponential factor can be neglected and we obtain the simple Black-Scholes dynamics. On the other hand,  $\gamma$  may be estimated from the volatility of VIX:

$$d\langle VIX \rangle_t = VIX_t^2 \gamma^2 dt.$$



Figure 1: Back-testing PnL as function of time for the three models we study

We are hedging a VIX option with pay-off

$$(VIX_T - K)_+$$
.

Introducing the VIX future

$$\mathrm{VIX}_t^T = E_Q[\mathrm{VIX}_T | \mathcal{F}_t],$$

neglecting the interest rate, the option price is given by

$$p(t, \text{VIX}_t^T) = \text{VIX}_t^T N(d_t^1) - KN(d_t^2), \quad d_t^{1,2} = \frac{\log \frac{\text{VIX}_t^T}{K} \pm \frac{\gamma^2(T-t)}{2}}{\gamma \sqrt{T-t}},$$

or in other words,

$$p(t, F_t^T) = \sqrt{F_t^T} e^{-\frac{\gamma^2}{2}(T-t)} N(d_t^1) - KN(d_t^2), \quad d_t^{1,2} = \frac{\log \frac{\sqrt{F_t^T} e^{-\frac{\gamma}{2}(T-t)}}{K} \pm \frac{\gamma^2(T-t)}{2}}{\gamma \sqrt{T-t}},$$

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and the hedge ratio is  $\frac{N(d_t^1)}{2\sqrt{F_t^7}}e^{-\frac{\gamma^2}{2}(T-t)}$ .

**CIR model** Assume that the VIX index follows the square root dynamics:

$$d\text{VIX}_t^2 = \kappa(\theta - \text{VIX}_t^2)dt + \gamma \text{VIX}_t dW_t.$$

Since we are hedging short maturity options and cannot estimate  $\kappa$  and  $\theta$  under the risk-neutral measure anyway, we assume that  $\kappa = 0$  so that

$$dVIX_t^2 = \gamma VIX_t dW_t.$$

The forward variance swap is then given by

$$F_t^T = E_Q[\text{VIX}_T^2 | \mathcal{F}_t] = \text{VIX}_t^2,$$

and follows the dynamics

$$\mathrm{d}F_t^T = \gamma \sqrt{F_t^T} \mathrm{d}W_t$$

We are hedging a VIX option with pay-off

$$(VIX_T - K)_+$$

with a forward variance swap. The price of the VIX option is given by

$$P(t, F_t^T) = \int_{K^2}^{\infty} (\sqrt{x} - K) p_{T-t}(F_t^T, x) \mathrm{d}x,$$

where  $p_T(v_0, x)$  is the density of the CIR process at time *T* with the starting value  $v_0$ . The parameter  $\gamma$  may be estimated by observing that  $\langle \text{VIX} \rangle_t = \frac{\gamma^2}{4}t$ .

**Rough fractional stochastic volatility** Assume now that the VIX index is given by

$$VIX_t = Ce^{X_t}$$
,

where C > 0 is a constant and X is a centered Gaussian process under the riskneutral probability. For all  $s \ge 0$ , let  $\mathcal{F}_s^0 := \sigma(X_r, r \le s)$ , and  $\mathcal{F}_s := \bigcap_{s < t} \mathcal{F}_t^0$ . The interest rate is taken to be zero. Fix a time horizon T, let  $Z_t(T) := E_Q[X_T|\mathcal{F}_t]$ , so that  $(Z_t(T))_{t\ge 0}$  is a Gaussian martingale and thus a process with independent increments, completely characterised by the function

$$c^{T}(t) := E_{Q}[Z_{t}(T)^{2}] = E_{Q}[E_{Q}[X_{T}|\mathcal{F}_{t}]^{2}].$$

If we assume in addition that  $c^{T}(\cdot)$  is continuous then  $(Z_{t}(T))_{t\geq 0}$  is almost surely continuous. Using the total variance formula, the forward variance swap can be characterised as

$$F_t^T := E_Q[\text{VIX}_T^2 | \mathcal{F}_t] = C^2 E_Q[e^{2X_T} | \mathcal{F}_t] = C^2 e^{2E_Q[X_T | \mathcal{F}_t] + 2\text{Var}[X_T | \mathcal{F}_t]} = C^2 e^{2(Z_t(T) + c^T(T) - c^T(t))}.$$

The time-*t* price of a Call on the VIX is given by  $P_t := E_Q[(VIX_T - K)^+ | \mathcal{F}_t]$ . Note that the VIX future is a continuous lognormal martingale with  $E_Q[VIX_T | \mathcal{F}_t] = VIX_t^T$  and, by the total variance formula,

$$\operatorname{Var}[\log \operatorname{VIX}_t | \mathcal{F}_t] = \operatorname{Var}[X_T | \mathcal{F}_t] = c^T(T) - c^T(t).$$

In other words,

$$P_t = \text{VIX}_t^T N(d_t^1) - KN(d_t^2), \quad d_t^{1,2} = \frac{\log \frac{\text{VIX}_t^T}{K} \pm \frac{1}{2} (c^T(T) - c^T(t))}{\sqrt{c^T(T) - c^T(t)}}.$$

Applying Itô's formula and keeping in mind the martingale property of the option price, we obtain

$$dP_t = N(d_t^1) dVIX_t^T$$
.

In terms of forward variance swap, we then have:

$$P_{t} = \sqrt{F_{t}^{T}} e^{-\frac{1}{2}(c^{T}(T) - c^{T}(t))} N(d_{t}^{1}) - KN(d_{t}^{2}), \quad d_{t}^{1,2} = \frac{\log \frac{\sqrt{F_{t}^{T}} e^{-\frac{1}{2}(c^{T}(T) - c^{T}(t))}}{K} \pm \frac{1}{2}(c^{T}(T) - c^{T}(t))}{\sqrt{c^{T}(T) - c^{T}(t)}},$$

and the option price dynamics takes the following form:

$$dP_{t} = \frac{N(d_{t}^{1})e^{-\frac{1}{2}(c^{T}(T)-c^{T}(t))}}{2\sqrt{F_{t}^{T}}}dF_{t}^{T}$$

Assuming that

$$X_t = \sigma W_t^H$$

where *W* is the fractional Brownian motion with the Hurst parameter *H*, we get, after some computations using the Mandelbrot-Van Ness representation:

$$c^{T}(t) = \frac{\sigma^{2}}{\Gamma^{2}(H+1/2)} \int_{0}^{\infty} \left[ (T+s)^{H-1/2} - s^{H-1/2} \right]^{2} ds + \frac{\sigma^{2}}{\Gamma^{2}(H+1/2)} \int_{0}^{t} (T-s)^{2H-1} ds$$
$$= f(T) - \frac{\sigma^{2}(T-t)^{2H}}{2H\Gamma^{2}(H+1/2)}$$

for some function f(T), which cancels out in the difference, so that

$$c^{T}(T) - c^{T}(t) = \frac{\sigma^{2}(T-t)^{2H}}{2H\Gamma^{2}(H+1/2)}$$

Contrary to the previous two models, this one formally has two parameters to be estimated:  $\sigma$  and H. To estimate the Hurst parameter, following [12], we define

$$m(q,\Delta) = \frac{1}{\lfloor T/\Delta \rfloor} \sum_{k=1}^{\lfloor T/\Delta \rfloor} |\log(\text{VIX}_{k\Delta}) - \log(\text{VIX}_{(k-1)\Delta})|^q,$$

and estimate *H* from the half slope of  $m(2, \Delta)$  as function of  $\Delta$  in the log-log coordinates (see Figure 2, where  $\Delta$  varies from 1 to 30 days). Since this procedure requires a relatively long dataset to be precise, we perform it only once, on the VIX index time series from April 17, 2001 to April 16, 2021. This gives an estimated Hurst parameter value of 0.377, and the procedure is quite stable: when estimating on the first 10 years of the dataset, one obtains 0.380 and on the last 10 years one obtains 0.379.

These estimated values of the Hurst index are much higher than the values found by [12] and many other authors using the daily time series of realized volatility (typically between 0.1 and 0.15). However, the VIX index is constructed from prices of one-month options on the S&P index, and using the implied volatility of one-month options as proxy for volatility, [17] find a value of H = 0.32, which is much closer to our result. The relatively high value of the Hurst index we find can thus be explained by the averaging effects associated with computing option prices.

In view of the stability of the Hurst index estimation, we fix the value H = 0.377 for all tests, rather than estimating it before each back-test. Note that the hedging performance of the model remains very similar for  $H \in [0.37, 0.39]$ . This leaves us with a single parameter,  $\sigma$ , to estimate before each back test, which is estimated by

$$\hat{\sigma} = \sqrt{\frac{m(2,\Delta)}{\Delta^{2H}}}.$$

To further illustrate the dependence of the hedging performance on the value of the Hurst index and the importance of using a 'rough volatility' specification,



Figure 2: Estimation of the Hurst parameter

we performed the same test for *H* values ranging between 0.2 and 0.5, with step of 0.01, where the value H = 0.5 corresponds to the Black-Scholes benchmark. Figure 3 shows the dependence of the hedging RMSE on the value of the Hurst parameter with the minimum attained around H = 0.38.

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Figure 3: Dependence of the hedging RMSE on the Hurst parameter value.

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