Documentation of Local Volatility Surface
— Based on Lognormal-Mixture Model

This draft: June 27, 2017

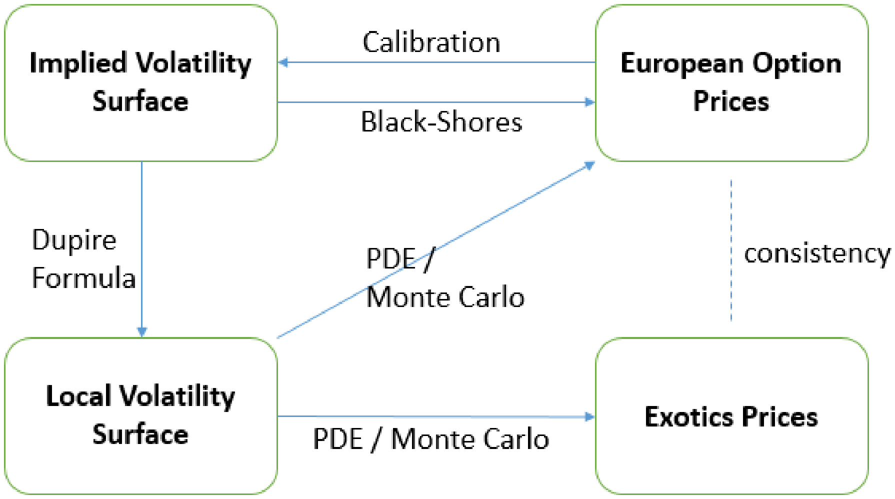
1 Summary

1.1 Local Volatility Surface

In our local volatility surface project, there are mainly two ways to build local volatility surface.

- Transform from implied volatility surface to local volatility surface based on Dupires work. In practice, there are three kind of methods to construct a smooth implied volatility surface.
  - Model calibration: such as Heston model, Lognormal-mixture model
  - Parametric method: such as SVI method
  - Nonparametric method: such as maximize relative entropy
- Set the problem of finding volatility as PDEs inverse problem and find a well-posed algorithm for recovering the implied local volatility

The local volatility surface is important in pricing exotic options. In this documentation, we focus on the first way: first build a good implied volatility surface based on lognormal-mixture model, and then transfer it to local volatility surface using Dupire's formula. Finally we re-price out-of-sample European options in China market with the calibrated volatility surface using PDE method, in order to test the performance of the surface. The summary graph is shown below.



2 Lognormal-Mixture Model

Brigo and Mercurio (2002) assume the marginal density of stock price is the mixture of lognormal densities and derive closed form formulas for option prices.

The dynamics of stick price S consists of N diffusion processes with dynamics given by

$$dS_t^i = (r-q)S_t^i dt + \sigma_i(t) S_t^i dW_t, \quad i = 1, \ldots, N \tag{2.1}$$

with initial value S_0^i. Furthermore, assuming $S_0^i = \xi_i S_0$, where S_0 is the spot price and ξ_i is the shifted factor aimed to achieve more flexibility.

For each t, the density function of S_t^i is denoted by $p_t^i(S)$. In the lognormal mixture model, the risk-neutral density of the spot price at a fixed maturity is modeled as a weighted sum of lognormal densities with different means and variances. Specifically, the risk-neutral probability density function of the stock price at any future time $T > 0$ is assumed to be in the following form

$$p_t(s) = \sum_{i=1}^{N} \omega_i p_t^i(s)$$

where ω_i is strictly positive constant and $\sum_{i=1}^{N} \omega_i = 1$. The density of p_t^i is given as

$$p_t^i(s) = \frac{1}{s\Sigma_i(t)\sqrt{2\pi}} exp\left\{-\frac{1}{2\Sigma_i^2(t)}\left[ln\frac{s}{S_0^i} - (r-q)t + \frac{1}{2}\Sigma_i^2(t)\right]^2\right\}$$

$$\Sigma_i(t) := \sqrt{\int_0^t \sigma_i^2(u)du} \tag{2.2}$$

with mean $\xi_i S_0 e^{(r-q)t}$ and variance Σ_i^2.

Thus, the expectation value of stock price at time t is

$$\int_0^{+\infty} sp_t(s)ds = \sum_{i=1}^{N} \omega_i \int_0^{+\infty} sp_t^i(s)ds = \sum_{i=1}^{N} \omega_i \xi_i S_0 e^{(r-q)t} = S_0 e^{(r-q)t} \tag{2.3}$$

with constrain $\sum_{i=1}^{N} \omega_i \xi_i = 1$ due to that the formula need to reprice the forward price.

Applying the Fokker-Plank equation

$$\frac{\partial}{\partial t} p_t(s) = -\frac{\partial}{\partial s}((r-q)sp_t(s)) + \frac{1}{2}\frac{\partial^2}{\partial s^2}(\sigma^2(t,s)s^2 p_t(s))$$

with P_t given by (2.2), to back out the diffusion coefficient σ, the following SDE for stock price can be derived as

$$dS_t = (r-q)S_t dt + \sqrt{\frac{\sum_{i=1}^{N} \omega_i \sigma_i^2(t) p_t^i(S_t)}{\sum_{i=1}^{N} \omega_i p_t^i(S_t)}} S_t dW_t. \tag{2.4}$$

Then the option price can be derived in closed form as

$$\begin{aligned} O &= e^{-rt} E^Q[(S_T - K)^+] \\ &= e^{-rt} \int_0^{+\infty} (s-K)^+ \sum_{i=1}^{N} w_i p_T^i(s) ds \\ &= \sum_{i=1}^{N} w_i e^{-rT} \sum_0^{+\infty} (s-K)^+ p_T^i(s) ds \\ &= \sum_{i=1}^{N} w_i O_i \end{aligned} \tag{2.5}$$

where O_i can be calculated with Black-Shores formula.

The following proposition has been proven by Brigo and Mercurio (2002).

Proposition 2.1 *Let us assume that each σ_i is also continuous and that there exists an $\varepsilon > 0$ such that $\sigma_i(t) = \sigma_0 > 0$, for each t in $[0, \varepsilon]$ and $i = 1, \ldots, N$. Then, if we set*

$$\nu(t, S_t) = \sqrt{\frac{\sum_{i=1}^{N} w_i \sigma_i^2(t) \frac{1}{\Sigma_i(t)} \left\{ -\frac{1}{2\Sigma_i^2(t)} \left[ln \frac{S_t}{S_0} - (r-q)t + \frac{1}{2}\Sigma_i^2(t) \right]^2 \right\}}{\sum_{i=1}^{N} w_i \frac{1}{\Sigma_i(t)} \left\{ -\frac{1}{2\Sigma_i^2(t)} \left[ln \frac{S_t}{S_0} - (r-q)t + \frac{1}{2}\Sigma_i^2(t) \right]^2 \right\}}} \tag{2.6}$$

for $(t, S_t) > (0, 0)$ and $\nu(t, S_0) = \sigma_0$, the SDE

$$dS_t = (r-q)S_t dt + \nu(t, S_t) S_t dW_t \tag{2.7}$$

has a unique strong solution whose marginal density is given by the mixture of lognormals

$$p_t(s) = \sum_{i=1}^{N} w_i \frac{1}{s\Sigma_i(t)\sqrt{2\pi}} exp\left\{ -\frac{1}{2\Sigma_i^2(t)} \left[ln\frac{s}{S_0^i} - (r-q)t + \frac{1}{2}\Sigma_i^2(t) \right]^2 \right\} \tag{2.8}$$

(2.9)

Proposition 2.2 *Condiser a European option with maturity T, strike K and written on the asset. The call option value at the initial time $t = 0$ is then given by the following convex combination of Black-Shores prices*

$$O = \sum_{i=1}^{N} w_i \left[\xi_i S_0 \Phi \left(\frac{ln\frac{\xi_i S_0}{K} + (r - q + \frac{1}{2}\eta_i^2)T}{\eta_i\sqrt{T}} \right) - K\Phi \left(\frac{ln\frac{\xi_i S_0}{K} + (r - q - \frac{1}{2}\eta_i^2)T}{\eta_i\sqrt{T}} \right) \right] \tag{2.10}$$

where

$$\eta_i := \frac{\Sigma_i(T)}{\sqrt{T}} = \sqrt{\frac{\int_0^T \sigma_i^2(t)dt}{T}} \tag{2.11}$$

3 Implied Volatility Surface

In this section, we follow the steps in Bloomberg's documentation of implied volatility surface reported by Analytics (2017).

3.1 Calibration Data

In Chinese market, there is only one option traded in Shanghai Security Exchange, which is 50 ETF option. The option details are shown in table 1.

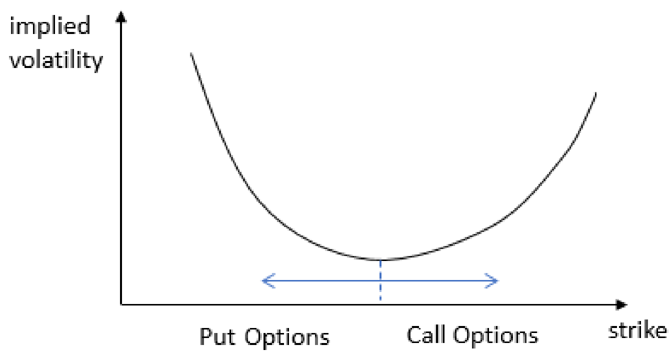
Table 1: Description of 50 ETF option.

Type	Maturity	Option Number
European	This month	8
	Next month	6
	This quarter	8
	Next quarter	6

1. **Continuous risk-free rate:** we take repo rate from our database as risk-free rate, and transfer it to continuous rate with formula $ln(1+r)$. In order to compare results with Bloomberg, we use Bloomberg simple risk-free rate as input. Bloomberg only provides rates on different maturities and we do linear interpolation between these points.

2. **Continuous dividend yield:** we use put-call-parity to implied continuous dividend yield with at-the-money option (or take the average of two options' implied dividend yields whose strikes lie on two sides of the forward price). In order to compare with Bloomberg, we get the implied dividend yield from forward price with formula: $q = r - ln(F)$. Bloomberg forecasts stock forward prices on option maturity dates and we linearly interpolate the implied dividend yields between the two consecutive maturities.

3. **Option data:** we use **mid price** to calibrate model parameters. Our IT team catches 50 ETF options' last bid and ask prices from Wind every day and we average the bid and ask price to get daily middle prices. We only use **out-of-money options** in calibration.



   - Bloomberg uses good end-of-day surface of previous business day as reference volatility surface, which sourced from broker/dealer quotes of the OTC market. For example, a small perturbation of bid/ask prices could lead to a large variation in the wings (small strikes and large strikes) if the valid option prices cover only a narrow range around the forward. Bloomberg augments the market quotes with wing samples from the reference surface to minimize the instability and mark the correct smile level.
   - We use the same filtered market option data but we do not know the sample data from reference volatility surface they use.

In summary, we can keep the **risk-free rate** and **dividend yield** consistent with Bloomberg in some way. However, we cannot get the exact **option data** used in Bloomberg calibration.

3.2 Calibration Method

When calibration the lognormal-mixture model, Bloomberg calibrates smile curves at each maturity independently and then do interpolation between these maturities. Also, Bloomberg modify this model by augmenting the state space with a "default" state where the sticj price drops to zero. Brigo and Mercurio (2002) mentioned that the model may be problematic to reproducing highly steep curves for very short maturities. Thus Bloomerbg added default probability in order to facilitate calibration to steep, short-term equity put skews.

3.2.1 Modified Model Brief

In the lognormal mixture model, the risk-neutral density of the spot price at a fixed maturity is modeled as a weighted sum of lognormal densities with different mean and variance as shown in formula(2.2).For convenience, we use the formula expression as

$$pdf(T, S) = \sum_{i=1}^{N} \omega_i(T) \cdot lognormalpdf(S; \xi_i(T)F(T), \Sigma_i(T)) \tag{3.12}$$

where

- N is the number of lognormals,
- $F(T)$ is the forward price,
- $0 \leqslant \omega_i(T) \leqslant 1$ is the time-dependent weight of the i-th lognormal
- $\xi_i(T) > 0$ is the time-dependent shift of the i-th lognormal
- $\Sigma_i(T)$ is the time-dependent standard deviation of the i-th lognormal

with constrains

$$\sum_{i}^{N} \omega_i(T)\xi_i(T) = 1$$

$$Q(T) + \sum_{i=1}^{N} \omega_i(T) = 1$$

where $Q(T)$ is the default probability that stock price drops to zero.

Once the model is modified by adding default probability, the price formulas of call option and put option are given as

$$C(T,K) = \sum_{i=1}^{N} \omega_i(T) \cdot BS(\xi_i(T)S_0, K, r, q, T, \frac{\Sigma_i(T)}{\sqrt{T}})$$
$$P(T,K) = \sum_{i=1}^{N} \omega_i(T) \cdot BS(\xi_i(T)S_0, K, r, q, T, \frac{\Sigma_i(T)}{\sqrt{T}}) + Q(T)Ke^{-rT} \tag{3.13}$$

3.2.2 Calibration Steps

The optimization problem is minimizing the 'distance' between model prices and market option prices at a given maturity. And the calibration is performed one maturity at a time.

Objective function: we take the number of lognormals N in density mixture as 4, then at maturity T the vector of optimization parameters $\mathbf{x} = (w_1, w_2, w_3, \xi_1, \xi_2, \xi_3, \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, Q)$ has 11 dimensionality.

Assume at maturity T, there are L out-of-the-money options, then the objective function is

$$obj = \sqrt{\frac{1}{L}\sum_{j=1}^{L}\left(\frac{P^{model}(T,K_j;\mathbf{x}) - P^{mkt}(T,K_j)}{P^{mkt}(T,K_j)}\right)^2}, \tag{3.14}$$

where $\mathbf{x}$ is the parameter vector, $P^{model}(T, K_j; \mathbf{x})$ denotes the model price with parameters $\mathbf{x}$ and $P^{mkt}(T, K_j)$ is the mid market price of European option price.

Optimization method: we use local search algorithm to get the optimal parameters. Specifically, we use Python package *scipy.optimize.fmin_slsqp*, which uses Sequential Least SQuares Programming to minimize a function of several variables with any combination of bounds, equality and inequality constraints.

However, the local search algorithm usually provide different answers according to initial guess of parameter values. So we use sobol sequential algorithm to sample 100 sets of initial guesses and take the optimal one.

Assume that there are m maturities in the market, and the calibration is performed independently. It is possible that there appears calendar arbitrage. In order to mitigate the possibility of calendar arbitrage arising from completely independent calibrations across different market maturities $T_1, \ldots, T_m$, we follow Bloomberg documentation and constrain the term structure of the parameters as

$$0 < \Sigma_i(T_1) < \cdots < \Sigma_i(T_m)$$
$$0 \leqslant Q(T_1) \leqslant \cdots \leqslant Q(T_m).$$

Thus we calibrate the first maturity at first, and take the calibrated parameters $(\Sigma_1(T_1), \Sigma_2(T_1), \Sigma_3(T_1), \Sigma_4(T_1), Q(T_1))$ as the lower bound when calibrate the second maturity parameters. Finally, we can get m sets of optimal parameters for corresponding m maturities.

3.2.3 Strike Interpolation and Extrapolation

After we calibrated the optimal parameter values on maturity T, the option price can be achieved according to formula (3.13) for any strike K. Then the implied volatility can be calculated easily using bisection algorithm from Black-Shores formula and we can get the whole smile curve.

Damp hazard rate: Bloomberg states *'whereas a positive hazard rate can help match steep put skews at short-term maturities, it may force a very steep skew for short-term extrapolation , so some damping of this hazard rate is needed for extremely low strikes'*.

Assuming a Poisson default process, we can imply the default intensity or hazard rate $\lambda(t)$ consistent with the survival probability $P(t) = 1 - Q(t)$ as

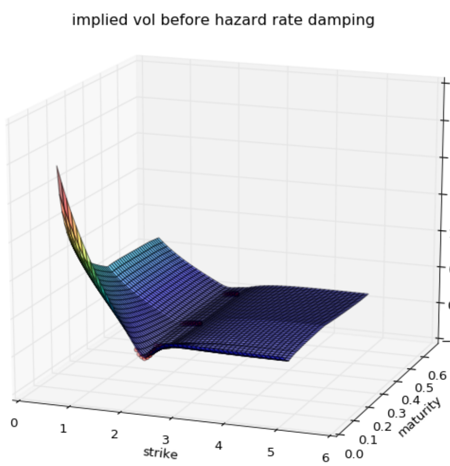
$$P(t) = 1 - Q(t) = \sum_i w_i(t) = e^{-\lambda(t)t}$$

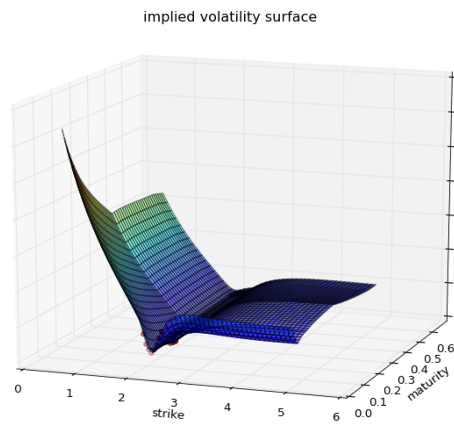
Thus, define k_{min} to be the minimum of all market strikes and 90% moneyness, for $k < k_{min}$, the hazard rate is damped as

$$\begin{aligned} x_m &= log(k_{min}^2/2T_i) \\ x &= log(k^2/2T_i) \\ \lambda_{new} &= \lambda e^{\frac{x_m^2 - x^2}{2T_i}}. \end{aligned} \tag{3.15}$$

For $k > k_{min}$, no hazard rate damping is applied. If λ is damped, the weights $\omega_i(t)$ is calculated by keeping the ratio $\omega_{i+1}(t)/\omega_i(t)$ the same. The put the parameters into formula (3.13) to get option price and corresponding implied volatility.

Below is the comparison of two implied-vol surfaces. The first one's hazard rate doesn't change, and the second one's hazard rate is damped according to formula (3.15):





Extrapolation on extreme strikes with Roger Lee's formula: the lognormal mixture model yields legitimate risk-neutral density, hence the implied volatility smile is arbitrage-free in strike. However, when strike is quite small or large, Bloomberg did not use the implied volatility generated by the model, instead, an extrapolation method based on work of Lee (2004) has been used.

The reason stated by Bloomberg as *'when the strike is extremely small or large, the option price is quite close to zero and the numerical error in the implied volatility calculation using bisection algorithm is big. To resolve this numerical difficulty, we extrapolate the implied volatility linearly with respect to the logarithm of forward moneyness* $log(k/F)$ *according to Roger Lee's formula.'*

The extrapolation range given by Bloomberg is $(0, k_{min}] \cup [k_{max}, +\infty)$, where

$$k_{min} = Fe^{-4V_{est}\sqrt{T}-\frac{1}{2}V_{est}^2 T},$$
$$k_{max} = Fe^{5V_{est}\sqrt{T}+\frac{1}{2}V_{est}^2 T},$$

where V_{est} is the 3 times at-the-money implied volatility, and the extrapolation formula is

$$\sigma_{imp} = a \cdot log(k/F) + b. \tag{3.16}$$

However this extrapolation formula cannot guarantee the second derivative is continuous, thus the local volatility surface may appear spikes and sink-holes shown as following.

Extrapolation on extreme strikes with BDK's formula: Another arbitrage-free extrapolation method was proposed by Benaim et al. (2008), which is continuous, twice differentiable and option prices converge to 0 as $K \to 0, \infty$. This method extrapolates on option price.

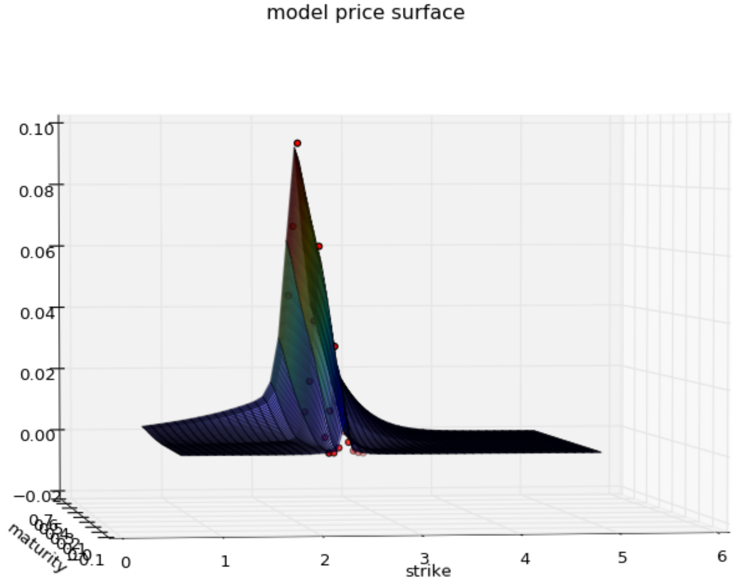
when strike is small, the put option price is extrapolated as

$$P(K) = K^{\mu} exp(a + bK + cK^2), \tag{3.17}$$

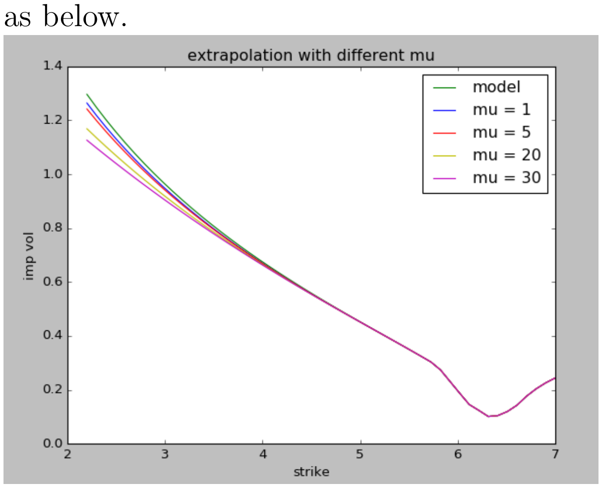
and when strike is large, the call option price is extrapolated as

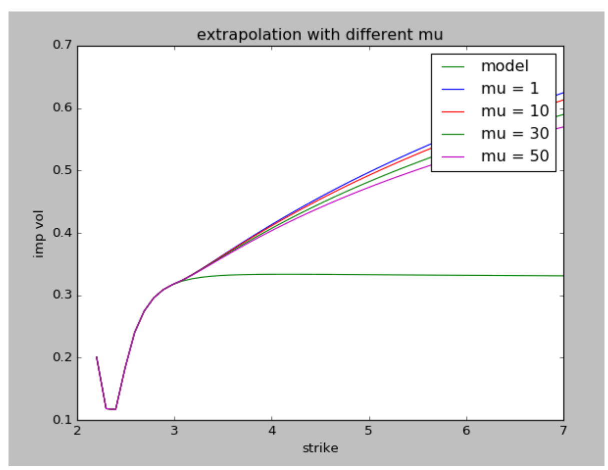
$$C(K) = K^{-\mu} exp(a + b/K + c/K^2) \tag{3.18}$$

where $\mu > 0$ is chosen by user. Below is the option price surface:



Different μ may lead to different shape of extrapolated implied volatility curve, shown as below.





The extrapolation point is 1.95 and 3.0. Noted that the two implied volatility extrapolation methods on extreme strikes cannot guarantee a nice local-vol surface.

3.2.4 Maturity Interpolation and Extrapolation

Bloomberg's Parameters interpolation and extrapolation: Following Bloomberg documentation, define

$$\alpha(t) = \frac{T_{i+1} - t}{T_{i+1} - T_i}$$
$$\eta_i(t) = log(\frac{\xi_{i+1}(t)}{\xi_i(t)}.$$

Together with $\sum_i w_i(t)\xi_i(t) = 1$, the $\eta_i(t)$ uniquely determine the $\xi_i(t)$.

For $T_j < t < T_{j+1}$, parameter's interpolation are:

- The weights

$$w_i(t) = \left(\frac{w_i(T_{j+1})}{\sum_{i=1}^N w_i(T_{j+1})} \frac{\sqrt{t} - \sqrt{T_j}}{\sqrt{T_{j+1}} - \sqrt{T_j}} + \frac{w_i(T_j)}{\sum_{i=1}^N w_i(T_j)} \frac{\sqrt{T_{j+1}} - \sqrt{t}}{\sqrt{T_{j+1}} - \sqrt{T_j}} \right)$$

- The variances

$$\Sigma_i^2(t) = (1 - \alpha(t))\Sigma_i^2(T_{j+1}) + \alpha(t)\Sigma_i^2(T_j)$$
$$\alpha(t) = \frac{T_{j+1} - t}{T_{j+1} - T_j}$$

- The shift

$$log^2 \left(\frac{\xi_{i+1}(t)}{\xi_i(t)} \right) = (1 - \alpha(t))log^2 \left(\frac{\xi_{i+1}(T_{j+1})}{\xi_i(T_{j+1})} \right) + \alpha(t)log^2 \left(\frac{\xi_{i+1}(T_j)}{\xi_i(T_j)} \right)$$

Piecewise cubic Hermite polynomial interpolation: The implied volatility surface generated by lognormal-mixture model can guarantee arbitrage-free in strike, but cannot guarantee no arbitrage in maturity due to the interpolation method. Thus we get the implied volatilities on market maturities from the model, and then do piecewise cubic Hermite polynomial interpolation on the total variances for each fixed strike level for each fixed strike level. Suppose $T_i < T < T_{i+1}$ where T_i and T_{i+1} are two maturities, let

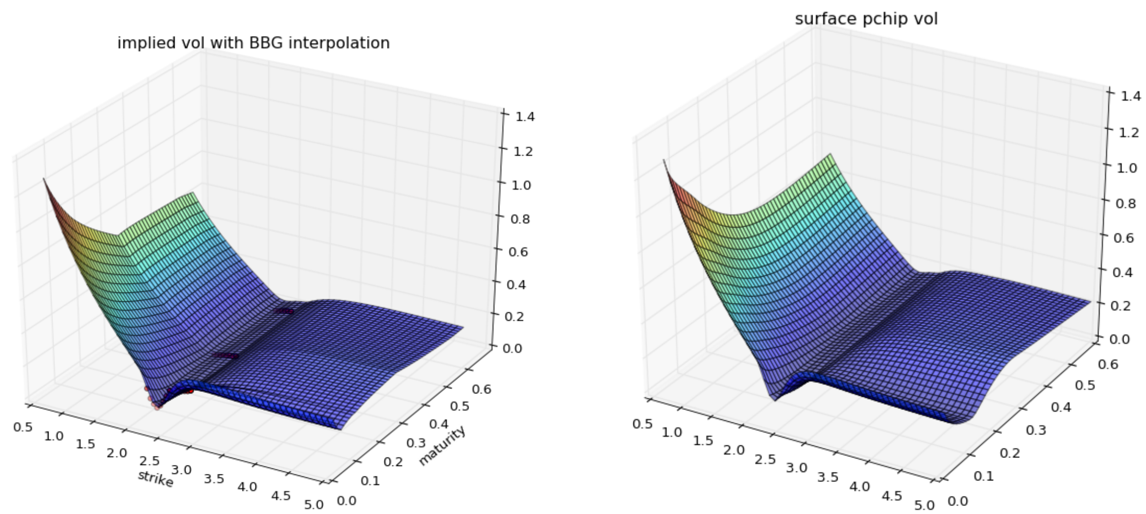
$$\widehat{\sigma}(T, K) = \sqrt{\frac{a_0 + a_1T + a_2T^2 + a_3T^3}{T}} \tag{3.19}$$

where a_i's are the Hermite interpolation parameters for the interval $[T_i, T_{i+1}]$ at strike level K.

Then the partial derivatives in the formula are approximated as

$$\frac{\partial \sigma_{imp}}{\partial T} = \frac{\widehat{\sigma}(T^+, K) - \widehat{\sigma}(T^-, K)}{2\triangle t}$$
$$\frac{\partial \sigma_{imp}}{\partial K} = \frac{\widehat{\sigma}(T, K^+) - \widehat{\sigma}(T, K^-)}{2\triangle k}$$
$$\frac{\partial^2 \sigma_{imp}}{\partial K^2} = \frac{\widehat{\sigma}(T, K^+) + \widehat{\sigma}(T, K^-) - 2\widehat{\sigma}(T, K)}{(\triangle k)^2}$$

Below is the comparison of two interpolation methods on maturity:



Short-term extrapolation: we do flat extrapolation, which assumes the implied volatility remains constant before the first maturity. The flat extrapolation can guarantee there is no calendar arbitrage in short term, $\frac{\partial(\sigma^2 T)}{\partial T} > 0$.

3.3 Calibration Surface

4 Local Volatility Surface

4.1 Dupire's Formula

Bruno (1994) gives the formula of local volatility in terms of option price and price derivatives. Gatheral (2011) furthermore gives the local volatility formula in terms of implied volatility and its derivatives as

$$\sigma^2(T, K) = \frac{\sigma_{imp}^2 + 2\sigma_{imp}T(\frac{\partial \sigma_{imp}}{\partial T} + (r-q)K\frac{\partial \sigma_{imp}}{\partial K})}{1 + 2d_1 K\sqrt{T}\frac{\partial \sigma_{imp}}{\partial K} + K^2 T\left(d_1 d_2(\frac{\partial \sigma_{imp}}{\partial K})^2 + \sigma_{imp}\frac{\partial^2 \sigma_{imp}}{\partial K^2}\right)} \quad (4.20)$$

where

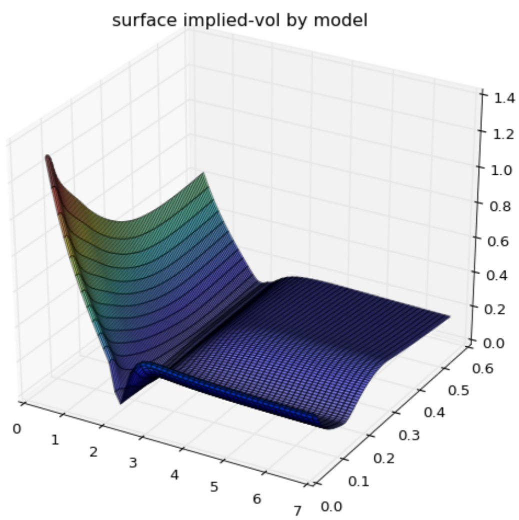
$$
\begin{aligned}
d_1 &= \frac{-ln(K/S) + \frac{1}{2}\sigma^2}{\sigma\sqrt{T}} \\
d_2 &= d_1 - \sigma\sqrt{T} \\
\overline{r} &:= \frac{1}{T}\int_0^T r(t)dt \\
\overline{q} &:= \frac{1}{T}\int_0^T q(t)dt
\end{aligned}
$$

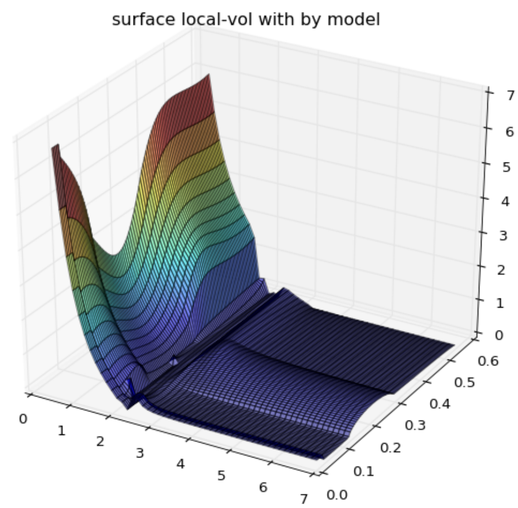
4.2 Address Spikes and Sink-holes

In practice, the right-hand-side of Dupire's formula (4.20) is not necessarily positive due to various reasons such as bad market data, not very smooth implied volatility surface, numerical issues, etc. So we floor the local volatility values to 1%.

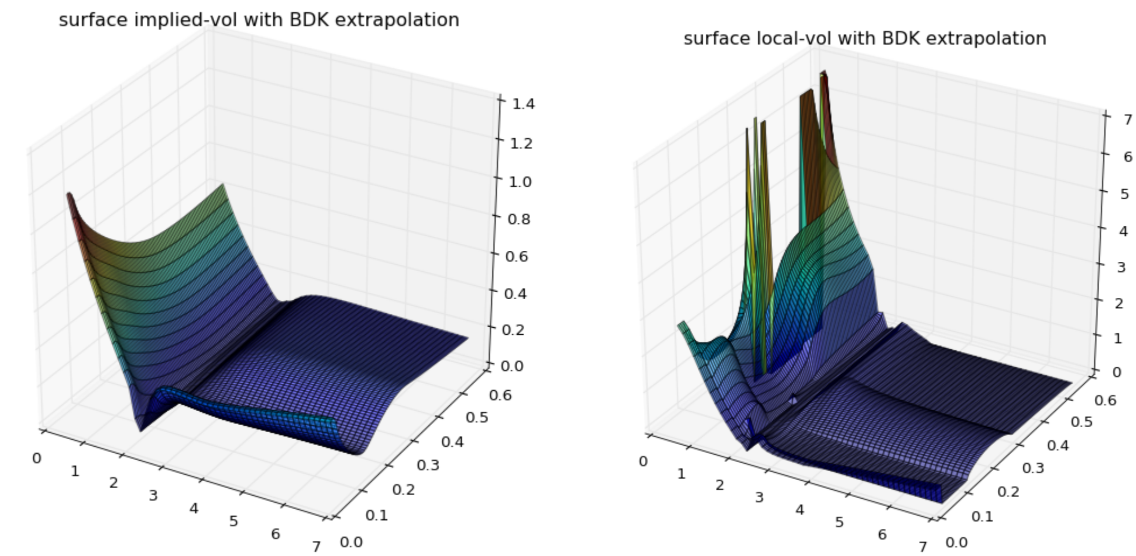
4.3 Local-vol Surface On 2017-Mar-13

Surface generated by model with hazard rate damping:

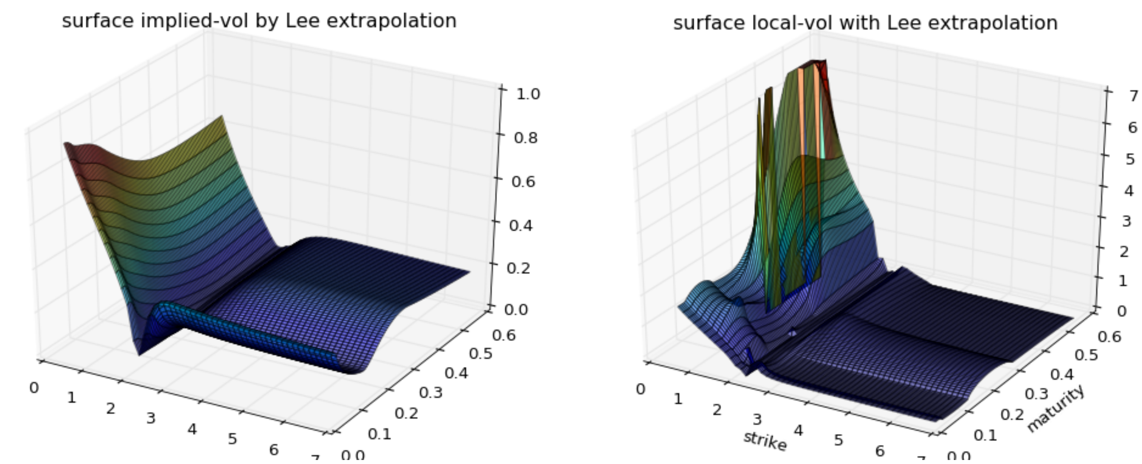




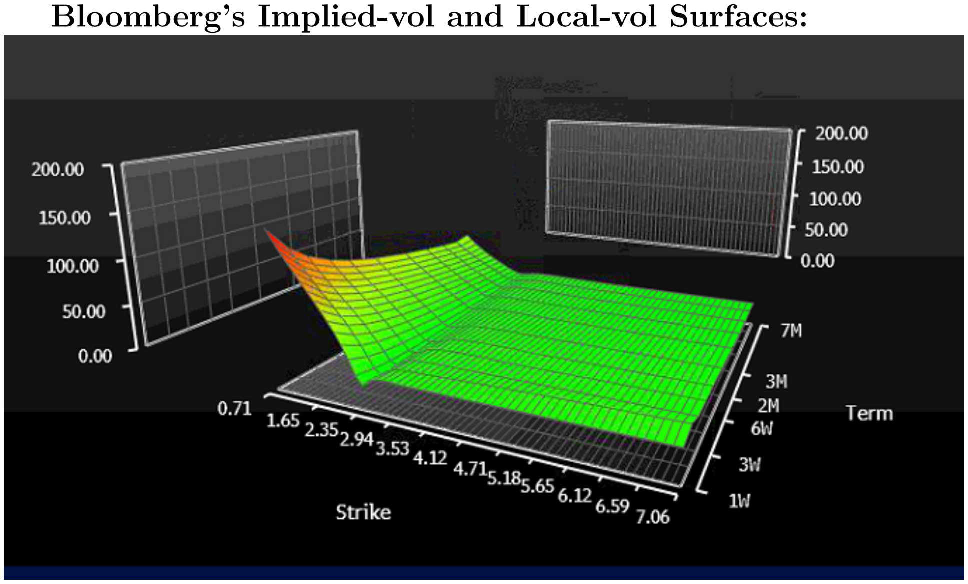
Using BDK's extrapolation:

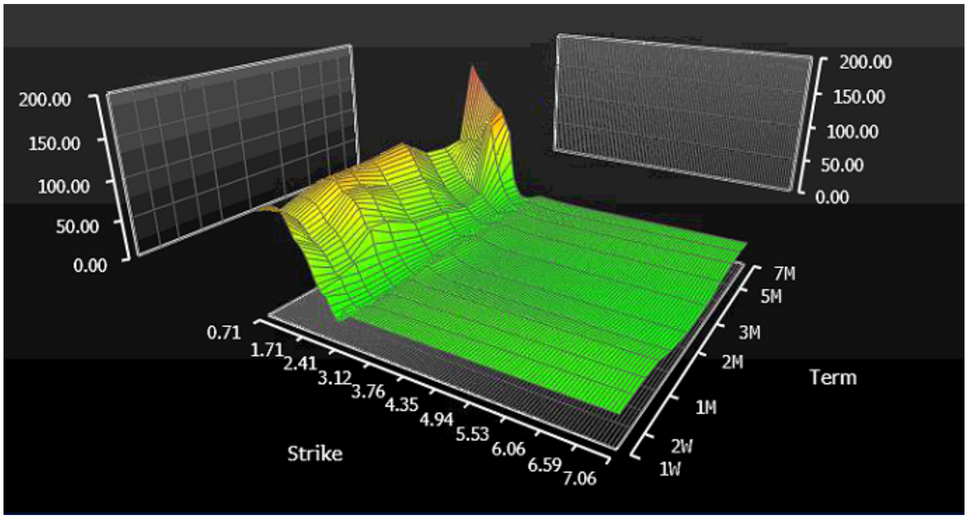


Using Roger Lee's extrapolation:



Bloomberg's Implied-vol and Local-vol Surfaces:





5 PDE Pricing

5.1 PDE

After getting the local volatility surface, we use local-vol PDE to get the reprice all the European options on the market.

Assuming underlying stocks follow the geometric Brownian motion under risk-neutral measure

$$\frac{dS}{S} = (r_t - q_t)dt + \sigma(t, S)dW_t$$

option prices $V(t, S)$ satisfy the standard Black-Shores local-vol PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2(t, S)S^2\frac{\partial^2 V}{\partial S^2} + (r_t - q_t)S\frac{\partial V}{\partial S} - r_t V = 0 \tag{5.21}$$

where

- r_t: instantaneous risk-free rate at time t
- q_t: continuous dividend yield at time t

Parameters: currently we assume r_t and q_t are constant of the value at maturity. We will continue to modify the constant parameters into time-dependent ones in the next step.

PDE grid: we set

$$S_{min} = min\{Fe^{-5\bar{\sigma}\sqrt{T}}, 0.8 \times S_0\}$$
$$S_{max} = max\{S_0, K\} \times e^{(r-q-\frac{1}{2}\bar{\sigma})T+4\bar{\sigma}\sqrt{T}}$$

Boundary conditions: use a mixture of Dirichlet and second-order boundary condition:

$$\frac{\partial^2 V}{\partial S^2}(S = S_{min}) = 0, \quad V(S_{max}) = 0. \tag{5.22}$$

Then at S_{min}, PDE formula (5.21) can be simplify as

$$\frac{\partial V}{\partial t} + (r_t - q_t)S\frac{\partial V}{\partial S} - r_t V = 0$$

Discretization: use Crank-Nicolson scheme with Rannarcher smoothing. At last time step, use fully-implicit scheme with 1/4 time step, and use Crank-Nicolson scheme at previous time steps.

5.2 Pricing Results

We use local-vol PDE with calibrated local-vol surface to price all market European options. Since we only use out-of-the-money options to calibrated surface, thus if the pricing results are good, the calibrated surface is good too to some extend. We also compare our pricing results with Bloomberg results with the same inputs in Table 2.

Here are some analysis based on the pricing results:

- The absolute average relative error of our model is 37.054%, while the average relative error of Bloomberg is 6.704%.
- The relative error of the first maturity is the largest one, which implies that the short-term extrapolation before the first maturity may not be good.
- The relative error of ITM options is smaller than OTM options.
- Almost all the model price is larger than the market mid price, which means the local volatility values we calibrated may be a bit larger than the actual ones. Also, there may appear spikes in the local-vol surface and misprice the European options.

6 Calibrate Whole Implied volatility Surface with One Set of Parameters

6.1 Problems of Bloomberg's Method

Bloomberg calibrated implied volatility surface with lognormal-mixture model, however there are some problematic things:

Table 2: **Pricing relative error and compare with Bloomberg**: the error is in percent format. The upper row are our model pricing relative errors, the lower row are the Bloomberg pricing relative error

strike	K	
maturity	call	put
T	Our model relative error Bloomberg relative error	

strike	2.2		2.25		2.3		2.35	
maturity	call	put	call	put	call	put	call	put
Mar-22	0.98 -0.04	402.11 10.20	2.42 -0.12	641.70 52.20	3.49 -1.43	142.94 2.27	36.88 -3.37	63.64 7.85
Apr-26			3.13 -0.40	84.95 -8.55	2.65 -2.19	26.91 1.26	2.84 -3.67	5.99 -0.85
Jun-28	-0.76 -1.01	3.68 -0.35	-3.29 -1.02	-7.14 7.08	-1.20 -1.70	1.57 0.38	2.52 -0.86	5.79 2.28
Sep-27			-5.20 -1.34	-11.77 0.64	-3.63 -0.99	-4.11 0.49	-2.44 -1.40	-0.50 0.55

strike	2.4		2.45		2.5		2.55	
maturity	call	put	call	put	call	put	call	put
Mar-22	66.65 -22.70	9.51 2.79	60.29 -31.36	0.73 -0.24	166.02 -65.49	1.06 0.20	-59.12 -89.03	0.13 0.08
Apr-26	23.01 -4.40	5.02 -1.11	8.42 -11.13	0.57 -0.49	121.15 -13.47	2.11 -0.06		
Jun-28	9.36 -1.95	5.40 -0.04	-2.26 -2.45	0.32 0.29	17.36 0.04	2.15 0.18	21.06 -8.54	1.53 -0.28
Sep-27	5.15 -2.12	3.86 -0.54	1.87 -0.77	0.84 -0.14	4.08 0.17	1.76 0.84		

- Bloomberg calibrated implied volatility curves for each maturities and then do interpolation between these maturities, which cannot guarantee that no calendar arbitrage exists.
- Bloomberg modified the original model by adding default probability $Q(T)$ of underlying stock price, in order to fit the steep put skew in short term. However, the default probability $Q(T)$ may lead to over-steep skew out of the market strikes range. Thus the default probability is decreased with a subjective formula.
- On the very small or large strikes, Bloomberg did strike extrapolation with a subjective formula.

Thus we would like to calibrate the whole implied volatility surface with only one set of parameters.

6.2 Calibration Steps

Following work of Brigo and Mercurio (2002), assume the integrated volatility of each lognormal $\eta_i(T) = \frac{\Sigma_i(T)}{\sqrt{T}}$ is time-dependent and has term structure of Nelson-Siegel format:

$$\eta_i(T) = \eta(T; a_i, b_i, c_i, \tau_i) = a_i + b_i \left[1 - exp\left(-\frac{T}{\tau_i}\right)\right] \frac{\tau_i}{T} + c_i exp\left(-\frac{T}{\tau_i}\right), \tag{6.23}$$

where a_i, b_i, c_i, τ_i are the coefficient that needs to be calibrated.

Calibrated parameters: we still take the number of lognormals N in the density function as 4, then the vector of optimization parameters for the whole surface is $\mathbf{x} = (w_1, w_2, w_3, \xi_1, \xi_2, \xi_3, a_1, \ldots, a_4, b_1, \ldots, b_4, c_1, \ldots, c_4, \tau_1, \ldots, \tau_4,)$, which has 22 dimensionality.

Objective function: Assume there are total M maturities in the market, and each maturity has L out-of-the-money options, then the objective function is

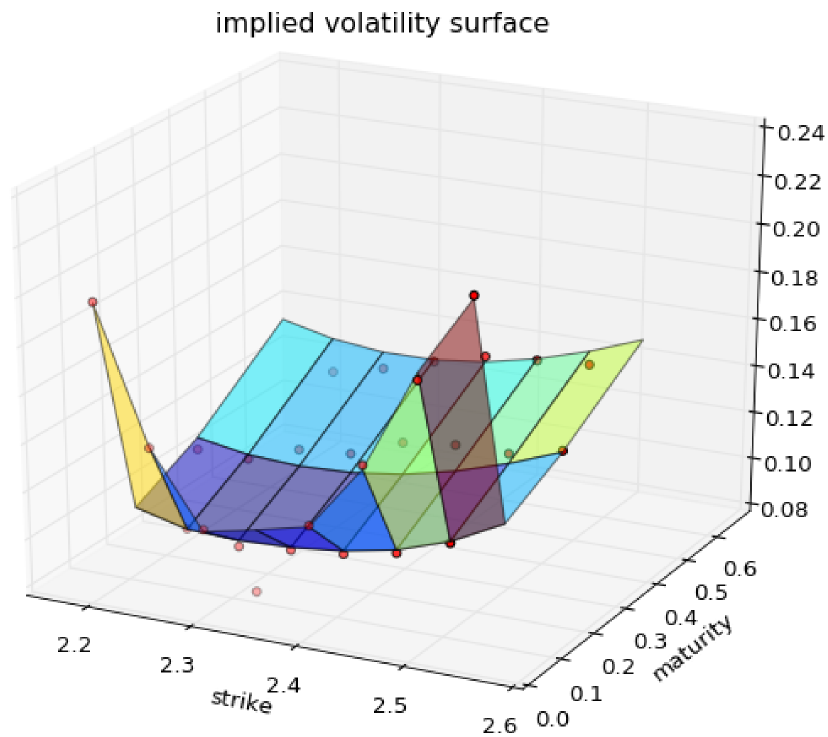
$$obj = \sqrt{\frac{1}{ML} \sum_{i=1}^{M} \sum_{j=1}^{L} \left(\frac{P^{model}(T_i, K_j; \mathbf{x}) - P^{mkt}(T_i, K_j)}{P^{mkt}(T_i, K_j)} \right)^2}, \tag{6.24}$$

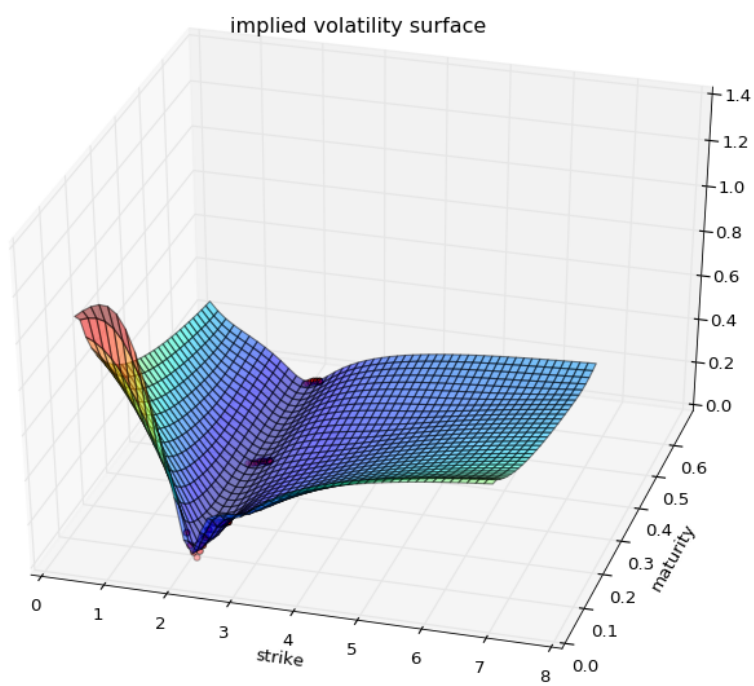
where $\mathbf{x}$ is the parameter vector, $P^{model}(T, K_j; \mathbf{x})$ denotes the model price with parameters $\mathbf{x}$ and $P^{mkt}(T, K_j)$ is the mid market price of European option price.

Optimization method: we still use Python package *scipy.optimize.fmin_slsqp* to get optimal calibrated parameters.

6.3 Calibrated Implied-vol Surface

The calibrated implied volatility surface is shown as below. Noted that the calibration method is hard to fit the short-term smile well.

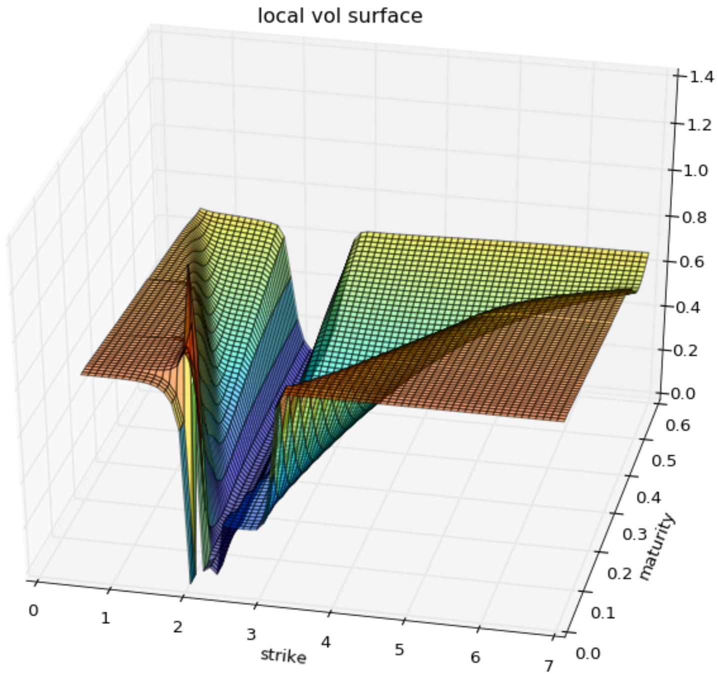




The calibration method is hard to fit the short-term smile well.

6.4 Implied-vol Surface

Through Dupire's formula, the local volatility surface is shown as



6.5 Pricing Results

The pricing result is shown is Table 3.

- The absolute average relative error with our model local-vol surface is 11.186%, and the average relative error with Bloomberg's local-vol surface is 6.704%.
- The large errors cluster in short-term maturity and OTM options. This may due to
  - the calibrated implied-vol surface cannot fit the market data exactly, especially in short term maturity;
  - the extrapolation before the first maturity is set as simple flat extrapolation;
  - when transform to local-vol surface with Dupire's formula, there is some volatile in short-term time.

6.6 Next Step

- We will change the constant parameters r and q in PDE into time-dependent r_t and q_t, as in formula (5.21);

Table 3: **Pricing relative errors and compare with Bloomberg**: the error is in **percent format**. The upper row are our model pricing relative errors, the lower row are the Bloomberg pricing relative error

strike	K	
maturity	call	put
T	Our model relative error Bloomberg relative error	

strike	2.2		2.25		2.3		2.35	
maturity	call	put	call	put	call	put	call	put
Mar-22	-0.21 -0.04	-53.34 10.20	-0.50 -0.12	-31.51 52.20	0.72 -1.43	63.51 2.27	23.80 -3.37	45.54 7.85
Apr-26			0.15 -0.40	6.08 -8.55	0.21 -2.19	13.95 1.26	-1.70 -3.67	1.21 -0.85
Jun-28	0.03 -1.01	16.58 -0.35	-1.77 -1.02	2.38 7.08	-4.98 -1.70	-7.29 0.38	-9.43 -0.86	-6.61 2.28
Sep-27			4.10 -1.34	18.59 0.64	2.85 -0.99	7.42 0.49	-0.16 -1.40	1.96 0.55

strike	2.4		2.45		2.5		2.55	
maturity	call	put	call	put	call	put	call	put
Mar-22	-4.23 -22.70	4.16 2.79	-83.04 -31.36	-0.81 -0.24	-36.15 -65.49	0.30 0.20	-73.45 -89.03	0.10 0.08
Apr-26	5.87 -4.40	1.18 -1.11	-9.98 -11.13	0.43 -0.49	17.50 -13.47	-0.43 -0.06		
Jun-28	-13.28 -1.95	-5.51 -0.04	-18.91 -2.45	-3.63 0.29	-6.00 0.04	-0.52 0.18	1.06 -8.54	0.30 -0.28
Sep-27	-1.43 -2.12	-0.02 -0.54	-0.48 -0.77	0.02 -0.14	8.24 0.17	2.80 0.84		

- Figure out a good short-term extrapolation method;

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