Sticky varswaps

Jingyi Huang and Olaf Torné extend the definition of the skew-stickiness ratio (SSR) to handle covariance between the spot and the theoretical fair strike of a variance swap, and derive analytical approximations of this quantity in the local volatility and Heston models, analogous to known formulas for the classical at-the-money forward SSR

The skew-stickiness ratio (SSR), introduced by Bergomi (2015), is the industry standard metric for describing the joint dynamics of spot and implied volatility in a diffusion model. It is defined as:

$$\mathrm{SSR}_{\mathrm{atmf}} = \frac{1}{\mathcal{S}_T} \frac{\mathbb{E}[\mathrm{d}\hat{\sigma}_{F_T T}\, \mathrm{d}\ln S]}{\mathbb{E}[\mathrm{d}\ln S^2]} \tag{1}$$

where $\hat{\sigma}_{F_T T}$ is the at-the-money forward volatility (ATMF vol) and $\mathcal{S}_T$ is the volatility surface skew, ie, it is the regression coefficient of $\mathrm{d}\hat{\sigma}_{F_T T}$ onto $\mathrm{d}\ln S$ normalised by the skew. This metric is analysed in great depth in Bergomi (2015), and analytical approximations are provided within a variety of stochastic volatility models (see also El Aoud & Abergel 2014; Vargas *et al* 2015).

Systematic discrepancies between the realised and model-implied level of spot/implied volatility covariance appear as P&L in an option book (Bergomi 2017). Thus, an important application of the SSR is in the calibration of stochastic volatility models. A concrete implementation of this process in a production setting is described in Cohen (2019) for local stochastic volatility (LSV). Since LSV is guaranteed to calibrate to the vanilla surface, it affords flexibility in the specification of the parameters in the stochastic variance process. It is then possible to select parameters to match a target SSR level, typically selected to strike a balance between the statistically estimated realised level and a bid/ask level relevant to the risk profile of the book.

A limitation of $\mathrm{SSR}_{\mathrm{atmf}}$ is that in practice the realised dynamics of the volatility surface may not be adequately captured by ATMF vol alone. Figure 1 shows the realised term structure of the SSR of the Nikkei 225 (.N225), estimated over the second half of 2019, for ATMF vol, 60% strike vol and a variance swap (varswap) theoretical fair strike. ATMF vol exhibits an SSR close to sticky-strike behaviour, ie, corresponding to $\mathrm{SSR} = 1$. On the other hand, the 60% strike and varswap realised SSR term structures are both significantly beneath the sticky-strike level at longer tenors. This is a systematic discrepancy that results from structured product hedging activity in this market (see Ahallal & Torné (2018) and the references therein). Another limitation is that (1) is not well defined for vanishing skew. It is not uncommon for some volatility surfaces to have positive short-term skew, due to bullish sentiment, and to transition into a typical negative skew at longer tenors. Thus, some intermediate tenors will have either zero or very small skew, and undefined or numerically unstable $\mathrm{SSR}_{\mathrm{atmf}}$.

It is straightforward to extend (1) to arbitrary strikes, resulting in a grid of SSR metrics by strike and maturity. Alternatively, we could define SSR for higher-order parameters of the volatility surface. While these are valid approaches, it is appealing to have a compact and easily interpretable metric, ideally using only a single number per maturity, from a practical trading perspective as well as for use in model calibration, since production stochastic volatility models typically have only a few degrees of freedom. The variance swap fair strike has a unique status in that it encodes a wide range of volatility strikes into a single model-independent price. As such, it is a tool of choice for equity derivatives traders looking for a rough but compact assessment of the volatility surface. Therefore, it is natural to measure the joint dynamics of the spot and the theoretical fair strike of a variance swap as a means of extending $\mathrm{SSR}_{\mathrm{atmf}}$ to a wider strike range.[1]

The classical SSR definition carries over nicely to varswap fair strikes. Denote by S the current spot level and by $\hat{\sigma}_T^2(S)$ the corresponding varswap fair strike at maturity T, and note that the latter only depends on the spot via the implied volatility. Also, denote by $\hat{\sigma}_T^{2,\mathrm{SS}}(S)$ the varswap strike under the additional assumption of a sticky-strike vol regime (see the next section). Then varswap SSR is defined as:

$$\mathrm{SSR}_{\mathrm{vsw}} = \frac{1}{\Delta_T} \frac{\mathbb{E}[\mathrm{d}\hat{\sigma}_T^2\, \mathrm{d}\ln S]}{\mathbb{E}[\mathrm{d}\ln S^2]} \tag{2}$$

where normalisation is by the so-called varswap skew delta:

$$\Delta_T = \frac{\mathrm{d}\hat{\sigma}_T^{2,\mathrm{SS}}}{\mathrm{d}\ln S}$$

Similarly to (1), this defines $\mathrm{SSR}_{\mathrm{vsw}}$ as the regression coefficient of $\mathrm{d}\hat{\sigma}_T^2$ onto $\mathrm{d}\ln S$, appropriately normalised. Notice there is also a symmetry in the normalisation factors $\mathcal{S}_T$ and Δ_T because under the sticky-strike vol regime, denoted $\hat{\sigma}_{F_T T}^{\mathrm{SS}}$, $\mathcal{S}_T = \mathrm{d}\hat{\sigma}_{F_T T}^{\mathrm{SS}}/\mathrm{d}\ln S$ holds.

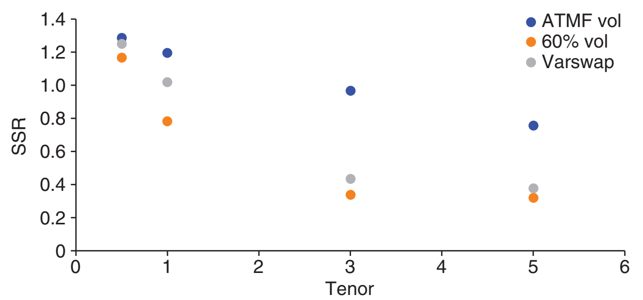
In the next section, (2) is derived directly from first principles and $\mathrm{SSR}_{\mathrm{vsw}}$ is interpreted as a weighting coefficient between sticky-strike and sticky-delta dynamics for $\mathrm{d}\hat{\sigma}_T^2$. In particular, $\mathrm{SSR}_{\mathrm{vsw}} = 1$ under the assumption of a sticky strike, and $\mathrm{SSR}_{\mathrm{vsw}} = 0$ under a sticky delta. Thus, both $\mathrm{SSR}_{\mathrm{atmf}}$ and $\mathrm{SSR}_{\mathrm{vsw}}$ are dimensionless quantities with a concrete interpretation in terms of market dynamics, and they can be compared directly with each other as well as across different underlyings.

The first aim of this work is to extend to the setting of varswaps some of the analytical results known for $\mathrm{SSR}_{\mathrm{atmf}}$. We consider two canonical stochastic volatility models – the local volatility model and the Heston model – and derive closed-form approximations of $\mathrm{SSR}_{\mathrm{vsw}}$ therein. We also describe an efficient and accurate numerical procedure to calculate (2) in an LSV model.

The closed-form expressions for $\mathrm{SSR}_{\mathrm{vsw}}$ are the following. First, in the local volatility model, assuming the skew and term structure are sufficiently

[1] *In this context, the varswap fair strike is simply defined by the standard option strip replication formula. Indeed, we are only interested in capturing the co-movement of spot and implied volatility within a diffusion model, and therefore higher-order corrections normally encapsulated in the varswap basis are not relevant.*

1 Term structure of realised SSR for ATMF vol, 60% strike vol and variance swap theoretical fair strike



The underlying is .N225, realised SSRs estimated over the second half of 2019

close to a flat Black-Scholes model in the sense defined below, its holds that:

$$\mathrm{SSR}_{\mathrm{vsw}} = \frac{\left(\mathcal{D}^1_T + \dfrac{1}{T}\displaystyle\int_0^T \mathcal{D}^1_t \,\mathrm{d}t\right) - \dfrac{\hat{\sigma}^2_{F_T T} T}{2}\left(\mathcal{D}^2_T + \dfrac{1}{T}\displaystyle\int_0^T \dfrac{t}{T}\mathcal{D}^2_t \,\mathrm{d}t\right)}{\mathcal{D}^1_T - \dfrac{\hat{\sigma}^2_{F_T T} T}{2}\mathcal{D}^2_T} \tag{3}$$

where $\mathcal{D}^1_T$ and $\mathcal{D}^2_T$ are the skew and curvature of the implied variance, respectively. This formula is quite similar to the skew-averaging formula for $\mathrm{SSR}_{\mathrm{atmf}}$, except that it involves a contribution from curvature as well as skew (see Bergomi 2015, (2.87)). Next, in the Heston model it holds that:

$$\mathrm{SSR}_{\mathrm{vsw}} = \frac{1-\mathrm{e}^{-\kappa T}}{\kappa T}\frac{\eta\rho}{\mathcal{D}^1_T - \frac{1}{2}\hat{\sigma}^2_{F_T T} T \mathcal{D}^2_T} \tag{4}$$

up to second order in the volatility of variance η, where κ denotes the mean-reversion rate and ρ is the spot-variance correlation. $\mathcal{D}^1_T$ and $\mathcal{D}^2_T$ are known in closed form (Bergomi & Guyon 2012, appendix A) and are given by (21).

The short tenor behaviour in (3) and (4) is such that:

$$\lim_{T\to 0} \mathrm{SSR}_{\mathrm{vsw}} = \lim_{T\to 0} \mathrm{SSR}_{\mathrm{atmf}} = 2$$

This is as expected since ATMF vol and varswap fair strike coincide in this limit. For tenors $T > 0$, $\mathrm{SSR}_{\mathrm{atmf}}$ and $\mathrm{SSR}_{\mathrm{vsw}}$ no longer coincide, the latter being driven by the dynamics of both ATMF vol and out-of-the-money (OTM) strikes.

Volatility regime weighting

This section derives (2) from first principles and offers further insights into its interpretation. Denote by S the current spot level and by $\hat{\sigma}^2_T(S)$ the corresponding varswap fair strike of maturity T. The aim is to describe the co-movement of the spot and the volatility implied by certain diffusion models. For this purpose, the varswap fair strike is merely a natural encoding of the volatility surface and it is sufficient to define it by the standard formula:

$$\hat{\sigma}^2_T(S) = 2\left(\int_0^S \frac{1}{K^2}P(K)\,\mathrm{d}K + \int_S^\infty \frac{1}{K^2}C(K)\,\mathrm{d}K\right) \tag{5}$$

where:

$$P(K) = KN(-d_2) - SN(-d_1), \qquad C(K) = SN(d_1) - KN(d_2)$$

and:

$$d_1(K) = \frac{\ln(S/K) + \hat{\sigma}^2_{KT}T/2}{\hat{\sigma}_{KT}\sqrt{T}}, \qquad d_2(K) = d_1(K) - \hat{\sigma}_{KT}\sqrt{T}$$

where for simplicity the risk-neutral drift is set to zero. Formula (5) is proved in Coulombe *et al* (2009, (1)).

Next consider a spot move from S to S', and a simultaneous shift of the implied volatility surface from $\hat{\sigma}_{KT}(S)$ to $\hat{\sigma}_{KT}(S')$. Under the sticky-strike regime:

$$\hat{\sigma}_{KT}(S') \equiv \hat{\sigma}^{\mathrm{SS}}_{KT}(S') = \hat{\sigma}_{KT}(S) \tag{6}$$

while under the sticky-delta regime:

$$\hat{\sigma}_{KT}(S') \equiv \hat{\sigma}^{\mathrm{SD}}_{KT}(S') = \hat{\sigma}_{\frac{K}{S'}S,T}(S) \tag{7}$$

Inserting either (6) or (7) into (5) produces the corresponding sticky-strike and sticky-delta dynamics for the varswap fair strike, denoted by $\hat{\sigma}^{2,\mathrm{SS}}_T(S')$ and $\hat{\sigma}^{2,\mathrm{SD}}_T(S')$ respectively. Also, note that $\hat{\sigma}^{2,\mathrm{SD}}_T(S') = \hat{\sigma}^2_T(S)$, which can be seen by making the substitution $\mathrm{d}K = S\,\mathrm{d}k$ in (5) and applying the identity (7).

In general, the simultaneous spot and implied volatility move may fall outside of the two canonical regimes, and in this case we can express the updated varswap fair strike as:

$$\hat{\sigma}^2_T(S') = \lambda\hat{\sigma}^{2,\mathrm{SS}}_T(S') + (1-\lambda)\hat{\sigma}^{2,\mathrm{SD}}_T(S') \tag{8}$$

where this equality determines the weighting parameter λ. This expression is a common way for traders to parameterise a general dynamic in terms of sticky strike and sticky delta. If $\lambda > 1$ ($\lambda < 1$), the dynamic is said to over-realise (under-realise) the skew.

Rearranging terms, (8) can be viewed as the regression of:

$$\Delta\hat{\sigma}^2_T = \hat{\sigma}^2_T(S') - \hat{\sigma}^2_T(S)$$

onto:

$$\Delta\hat{\sigma}^{2,\mathrm{SS}}_T = \hat{\sigma}^{2,\mathrm{SS}}_T(S') - \hat{\sigma}^2_T(S)$$

with regression coefficient λ. That is, $\Delta\hat{\sigma}^2_T = \lambda\Delta\hat{\sigma}^{2,\mathrm{SS}}_T$. The infinitesimal version of the regression coefficient is:

$$\lambda = \frac{\mathbb{E}[\mathrm{d}\hat{\sigma}^2_T\,\mathrm{d}\hat{\sigma}^{2,\mathrm{SS}}_T]}{\mathbb{E}[(\mathrm{d}\hat{\sigma}^{2,\mathrm{SS}}_T)^2]}$$

Using the fact that $\hat{\sigma}^{2,\mathrm{SS}}_T$ is a deterministic function of S, we can project onto $\mathrm{d}\ln S$, in analogy with (1), to obtain definition (2).

Lastly, Coulombe *et al* (2009) supply the following explicit formula for the skew delta:

$$\frac{\mathrm{d}\hat{\sigma}^{2,\mathrm{SS}}_T}{\mathrm{d}\ln S} = \frac{2}{\sqrt{T}}\int_0^\infty N'(z_K)\frac{\partial\hat{\sigma}_{KT}}{\partial K}\,\mathrm{d}K$$

where:

$$N'(z_K) = \hat{\sigma}_{KT}\sqrt{T}\rho_x, \quad z_K = \frac{x_K + \hat{\sigma}^2_{KT}T/2}{\hat{\sigma}_{KT}\sqrt{T}}$$

$$\rho_x = \frac{1}{\sqrt{2\pi\hat{\sigma}^2_{KT}T}}\exp\left\{-\frac{(x_K + \hat{\sigma}^2_{KT}T/2)^2}{2\hat{\sigma}^2_{KT}T}\right\}, \quad x_K = \ln\frac{K}{F_T}$$

In the following sections, the implied variance is given in the following form:

$$\hat{\sigma}^2_{KT} = \hat{\sigma}^2_{F_T T} + \mathcal{D}^1_T x_K + \tfrac{1}{2}\mathcal{D}^2_T x_K^2$$

Therefore:

$$\frac{\partial \hat{\sigma}_{KT}}{\partial K} = \frac{1}{2\hat{\sigma}_{KT} K}(\mathcal{D}^1_T + \mathcal{D}^2_T x_K)$$

Then:

$$\frac{\mathrm{d}\hat{\sigma}^{2,\mathrm{SS}}_T}{\mathrm{d}\ln S} = \int_{-\infty}^{\infty} (\mathcal{D}^1_T + \mathcal{D}^2_T x_K)\rho_K \,\mathrm{d}x_K$$

Keeping only the zeroth-order $\hat{\sigma}^2_{KT} = \hat{\sigma}^2_{F_T T}$ in ρ_x then yields:

$$\frac{\mathrm{d}\hat{\sigma}^{2,\mathrm{SS}}_T}{\mathrm{d}\ln S} = \mathcal{D}^1_T - \frac{\hat{\sigma}^2_{F_T T} T}{2}\mathcal{D}^2_T \tag{9}$$

Local volatility model

The local volatility model is given by:

$$\mathrm{d}x(t) = -\tfrac{1}{2}\sigma(t, x(t))^2\,\mathrm{d}t + \sigma(t, x(t))\,\mathrm{d}W(t)$$

where $x(t) = \ln S(t)$, and the drift has been set to zero for simplicity. Recall that the skew-averaging formula asserts that:

$$\mathrm{SSR}_{\mathrm{atmf}} = 1 + \frac{1}{T}\int_0^T \frac{\mathcal{S}_t}{\mathcal{S}_T}\,\mathrm{d}t \tag{10}$$

at first order in a perturbation of the local vol around a constant σ_0, where $\mathcal{S}_t$ is the skew of the implied volatility at maturity t (see Bergomi 2015, (2.87)). First, we follow the method of the proof of (10) described in Bergomi (2015) and adapt this argument to $\mathrm{SSR}_{\mathrm{vsw}}$ to obtain (3). Second, we describe a method for obtaining higher-order approximations.

As $\hat{\sigma}^2_T$ is a deterministic function of the spot, (2) simplifies to:

$$\mathrm{SSR}_{\mathrm{vsw}} = \frac{1}{\Delta_T}\frac{\mathrm{d}\hat{\sigma}^2_T}{\mathrm{d}\ln S}$$

Assume the local volatility has the form:

$$\sigma(t, x) = \bar{\sigma}(t) + \alpha(t)(x - x^*) + \tfrac{1}{2}\beta(t)(x - x^*)^2 \tag{11}$$

and assume that it is close enough to a constant σ_0, ie:

$$\sigma(t, x) = \sigma_0 + \delta\sigma(t, x) \tag{12}$$

with $\delta\sigma$ small. Here, $x^* = \ln S^*$, with S^* the reference spot value at which the local vol is calibrated. By definition:

$$\hat{\sigma}^2_T = \mathbb{E}_{\sigma(t,x)}\bigg[\frac{1}{T}\int_0^T \sigma(t, x(t))^2\,\mathrm{d}t\bigg]$$

Two simplifications can be made at first order in $\delta\sigma$. First:

$$\sigma(t, x(t))^2 = \sigma_0^2 + 2\sigma_0\delta\sigma(t, x_t)$$

and second:

$$\begin{aligned}\hat{\sigma}^2_T &= \mathbb{E}_{\sigma(t,x)}\bigg[\frac{1}{T}\int_0^T \sigma_0^2 + 2\sigma_0\delta\sigma(t, x_t)\,\mathrm{d}t\bigg]\\ &= \sigma_0^2 + \mathbb{E}_{\sigma_0}\bigg[\frac{1}{T}\int_0^T 2\sigma_0\delta\sigma(t, x_t)\,\mathrm{d}t\bigg]\end{aligned}$$

Then, by straightforward computation:

$$\begin{aligned}\frac{\mathrm{d}\hat{\sigma}^2_T}{\mathrm{d}x_0} &= \frac{\mathrm{d}}{\mathrm{d}x_0}\bigg(\sigma_0^2 + \mathbb{E}_{\sigma_0}\bigg[\frac{1}{T}\int_0^T 2\sigma_0\delta\sigma(t, x_t)\,\mathrm{d}t\bigg]\bigg)\\ &= 2\sigma_0\frac{\mathrm{d}}{\mathrm{d}x_0}\bigg(-\frac{\sigma_0}{2} + \frac{1}{T}\int_0^T \bar{\sigma}(t) + \alpha(t)\bigg(x_0 - x^* - \frac{\sigma_0^2 t}{2}\bigg)\,\mathrm{d}t\\ &\quad + \frac{1}{2}\frac{1}{T}\int_0^T \beta(t)\bigg(\bigg(x_0 - x^* - \frac{\sigma_0^2 t}{2}\bigg)^2 + \sigma_0^2 t\bigg)\,\mathrm{d}t\bigg)\end{aligned}$$

Therefore:

$$\frac{\mathrm{d}\hat{\sigma}^2_T}{\mathrm{d}x_0}\bigg|_{x_0=x^*} = \frac{2\sigma_0}{T}\int_0^T \alpha(t)\,\mathrm{d}t - \frac{\sigma_0^3}{T}\int_0^T t\beta(t)\,\mathrm{d}t \tag{13}$$

Choosing to expand (13) around the ATMF level $\sigma_0 = \hat{\sigma}_{F_T T}$ yields:

$$\frac{\mathrm{d}\hat{\sigma}^2_T}{\mathrm{d}x_0}\bigg|_{x_0=x^*} = \frac{2\hat{\sigma}_{F_T T}}{T}\int_0^T \alpha(t)\,\mathrm{d}t - \frac{\hat{\sigma}^3_{F_T T}}{T}\int_0^T t\beta(t)\,\mathrm{d}t \tag{14}$$

It remains to express this in terms of implied variance. In Bergomi (2015), an expression for $\hat{\sigma}_{KT}$ is derived in terms of the local volatility parameters $\bar{\sigma}$, α and β, valid at first order in $\delta\sigma$. Adjustments in the derivation yield the following formula for implied variance:

$$\hat{\sigma}^2_{KT} = \hat{\sigma}^2_{F_T T} + \mathcal{D}^1_T \ln\bigg(\frac{K}{F_T}\bigg) + \frac{\mathcal{D}^2_T}{2}\ln\bigg(\frac{K}{F_T}\bigg)^2 \tag{15}$$

where:

$$\left.\begin{aligned}\mathcal{D}^1_T &= \frac{\mathrm{d}\hat{\sigma}^2_{KT}}{\mathrm{d}\ln K}\bigg|_{K=F_T} = \frac{2\hat{\sigma}_{F_T T}}{T}\int_0^T \alpha(t)\frac{t}{T}\,\mathrm{d}t\\ \mathcal{D}^2_T &= \frac{\mathrm{d}^2\hat{\sigma}^2_{KT}}{\mathrm{d}\ln K^2}\bigg|_{K=F_T} = \frac{2\hat{\sigma}_{F_T T}}{T}\int_0^T \beta(t)\bigg(\frac{t}{T}\bigg)^2\,\mathrm{d}t\end{aligned}\right\} \tag{16}$$

Combining (14) and (16) gives:

$$\frac{\mathrm{d}\hat{\sigma}^2_T}{\mathrm{d}x_0} = \bigg(\mathcal{D}^1_T + \frac{1}{T}\int_0^T \mathcal{D}^1_t\,\mathrm{d}t\bigg) - \frac{\hat{\sigma}^2_{F_T T} T}{2}\bigg(\mathcal{D}^2_T + \frac{1}{T}\int_0^T \frac{t}{T}\mathcal{D}^2_t\,\mathrm{d}t\bigg) \tag{17}$$

Next, given the implied variance (15), the skew delta is given by (9). Combining (17) and (9) gives the desired result: equation (3).

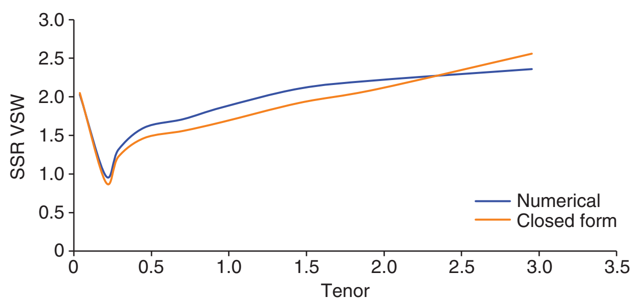
Figure 2 shows the term structure of $\mathrm{SSR}_{\mathrm{vsw}}$ calculated by a direct numerical evaluation, as well as using (3), for the Hang Seng China Enterprises Index (.HSCEI) on January 14, 2021. The change in monotonicity results from a change in the sign of the skew at longer tenors. The accuracy of (3) may vary depending on the underlying volatility surface. It is recommended to apply it for qualitative analysis, but to use an efficient numerically exact calculation for production, such as outlined in the 'Heston-LSV' section later.

Formula (3) is a useful rule of thumb, and it is sufficient for the purposes stated in the introduction. Nevertheless, there exists a more general method of proof for obtaining (14) that is not restricted to first order in $\delta\sigma$. To this end, define the Black-Scholes term structure as:

$$\sigma^{\mathrm{BS}}_{x_0}(t) = \sigma(t, x_0) \tag{18}$$

The results of Benhamou *et al* (2010) express the price of a European option in the local vol model as an expansion around its Black-Scholes price under (18), and up to arbitrary order in the local vol coefficients. These results can

2 Term structure of SSR_{vsw} in LV for .HSCEI on January 14, 2021



be applied directly to the log contract to obtain a closed-form approximation of $\hat{\sigma}_T^2$ in local vol. To illustrate, we apply the lowest order expansion from that paper (see Theorem 2.1 therein), but note that the argument below is also applicable to higher-order expansions. Using the notation of that paper, set $h(x(T)) = (-2/T)(x(T) - x_0)$ and:

$$\mathrm{Greek}_i^h(x(T)) = \begin{cases} -2/T & \text{if } i = 1 \\ 0 & \text{if } i \geqslant 2 \end{cases}$$

With this we obtain:

$$\begin{aligned} \hat{\sigma}_T^2 &= \mathbb{E}_{\sigma(t,K)}\left[\frac{1}{T}\int_0^T \sigma(t, x(t))^2 \, \mathrm{d}t\right] \\ &= \mathbb{E}_{\sigma(t,K)}[h(x(T))] \end{aligned}$$

Now a direct application of theorem 2.1 states that:

$$\begin{aligned} \hat{\sigma}_T^2 = {}& \frac{1}{T}\int_0^T \sigma(t, x_0)^2 \, \mathrm{d}t \\ & - \frac{1}{T}\int_0^T \sigma(t, x_0)^2 \int_t^T \sigma(s, x_0) \frac{\mathrm{d}\sigma}{\mathrm{d}x}(s, x_0) \, \mathrm{d}s \, \mathrm{d}t \end{aligned}$$

Inserting into this the expression of local vol (11) and simplifying then yields:

$$\begin{aligned} \left.\frac{\mathrm{d}\hat{\sigma}_T^2}{\mathrm{d}x_0}\right|_{x_0 = x^*} = {}& \frac{2}{T}\int_0^T \bar{\sigma}(s)\alpha(s) \, \mathrm{d}s - \frac{1}{T}\int_0^T \int_t^T \bar{\sigma}(t)^2 \bar{\sigma}(s)\beta(s) \, \mathrm{d}s \, \mathrm{d}t \\ & - \frac{1}{T}\int_0^T \int_t^T (2\bar{\sigma}(t)\bar{\sigma}(s)\alpha(t)\alpha(s) + \bar{\sigma}(t)^2 \alpha(s)^2) \, \mathrm{d}s \, \mathrm{d}t \end{aligned}$$

This can be recognised as generalising (13) to an expansion around a term structure instead of the constant σ_0.

Heston model

The Heston model is given by:

$$\left.\begin{aligned} \mathrm{d}x(t) &= -\tfrac{1}{2}v(t) \, \mathrm{d}t + \sqrt{v(t)} \, \mathrm{d}W^1(t) \\ \mathrm{d}v(t) &= -\kappa(v(t) - \bar{v}) \, \mathrm{d}t + \eta\sqrt{v(t)} \, \mathrm{d}W^2(t) \\ \langle \mathrm{d}W^1(t), \mathrm{d}W^2(t)\rangle &= \rho \, \mathrm{d}t \end{aligned}\right\} \tag{19}$$

where $x(t) = \ln S(t)$, and the drift has been set to zero for simplicity. The variance swap fair strike is:

$$\begin{aligned} \hat{\sigma}_T^2 &= \mathbb{E}\left[\frac{1}{T}\int_0^T v(s) \, \mathrm{d}s\right] \\ &= \bar{v} + (v_0 - \bar{v})\frac{1 - \mathrm{e}^{-\kappa T}}{\kappa T} \end{aligned}$$

The regression coefficient term in (2) then reads:

$$\begin{aligned} \frac{\mathbb{E}[\mathrm{d}\hat{\sigma}_T^2 \, \mathrm{d}\ln S]}{\mathbb{E}[\mathrm{d}\ln S^2]} &= \frac{\partial \hat{\sigma}_T^2}{\partial v_0} \frac{\mathbb{E}[\mathrm{d}v \, \mathrm{d}\ln S]}{\mathbb{E}[\mathrm{d}\ln S^2]} \\ &= \frac{1 - \mathrm{e}^{-\kappa T}}{\kappa T}\eta\rho \end{aligned}$$

It remains to calculate the skew delta using (9). The main ingredients for this formula are the derivatives:

$$\mathcal{D}_T^1 = \left.\frac{\mathrm{d}\hat{\sigma}_{KT}^2}{\mathrm{d}\ln K}\right|_{K = F_T} \quad \text{and} \quad \mathcal{D}_T^2 = \left.\frac{\mathrm{d}^2\hat{\sigma}_{KT}^2}{\mathrm{d}\ln K^2}\right|_{K = F_T}$$

The expression of the implied variance at second order in η is given in appendix A of Bergomi & Guyon (2012). Moreover, as $\mathcal{D}_T^2$ is already purely second order in η, we retain only the order-zero contribution of $\hat{\sigma}_{F_T T}^2$, which is just $\hat{\sigma}_T^2$. The skew delta is then:

$$\Delta_T = \mathcal{D}_T^1 - \frac{\hat{\sigma}_T^2 T}{2}\mathcal{D}_T^2$$

and finally:

$$\mathrm{SSR}_{\mathrm{vsw}} = \frac{1 - \mathrm{e}^{-\kappa T}}{\kappa T} \frac{\eta\rho}{\mathcal{D}_T^1 - \frac{1}{2}(\hat{\sigma}_T^2 T)\mathcal{D}_T^2} \tag{20}$$

where the implied variance skew and curvature can be expressed as follows:

$$\left.\begin{aligned} \mathcal{D}_T^1 T &= \frac{1}{\hat{\sigma}_T^2 T}C^{x\xi} - \frac{1}{2(\hat{\sigma}_T^2 T)^2}(C^{x\xi})^2 + \frac{1}{\hat{\sigma}_T^2 T}C^{\mu} \\ \tfrac{1}{2}\mathcal{D}_T^2 T &= \frac{1}{4(\hat{\sigma}_T^2 T)^2}C^{\xi\xi} - \frac{5}{4(\hat{\sigma}_T^2 T)^3}(C^{x\xi})^2 + \frac{1}{(\hat{\sigma}_T^2 T)^2}C^{\mu} \end{aligned}\right\} \tag{21}$$

with:

$$\begin{aligned} \xi(s) &= \hat{\sigma}_s^2 s \\ C^{x\xi} &= \frac{\rho\eta}{\kappa}\int_0^T \xi(s)(1 - \mathrm{e}^{-\kappa(T-s)}) \, \mathrm{d}s \\ C^{\xi\xi} &= \frac{\eta^2}{\kappa^2}\int_0^T \xi(s)(1 - \mathrm{e}^{-\kappa(T-s)})^2 \, \mathrm{d}s \\ C^{\mu} &= \frac{\rho^2\eta^2}{\kappa}\int_0^T \xi(s)\int_s^T \mathrm{e}^{-\kappa(u-s)}(1 - \mathrm{e}^{-\kappa(T-u)}) \, \mathrm{d}u \, \mathrm{d}s \end{aligned}$$

From (20) we may compute the short- and long-term limits of $\mathrm{SSR}_{\mathrm{vsw}}$:

$$\lim_{T \to 0} \mathrm{SSR}_{\mathrm{vsw}} = 2, \quad \lim_{T \to \infty} \mathrm{SSR}_{\mathrm{vsw}} = \eta\rho\left(\eta\rho - \frac{1}{4}\frac{\eta^2}{\kappa} + \frac{3}{4}\frac{\rho^2\eta^2}{\kappa}\right)^{-1} \tag{22}$$

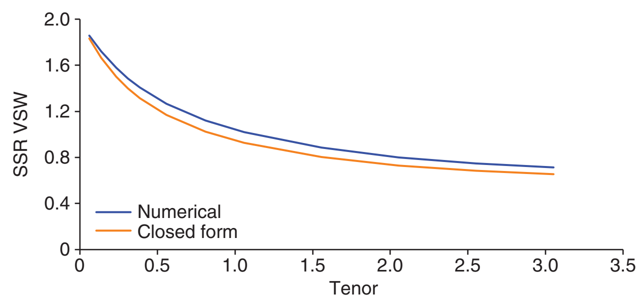
Figure 3 shows an example where the term structure of $\mathrm{SSR}_{\mathrm{vsw}}$ is calculated by numerical evaluation from first principles, as well as using (20).

Lastly, an important result is that $1 < \mathrm{SSR}_{\mathrm{atmf}}(T) \leqslant 2$ for all T (Bergomi 2015, (9.9)). This is a key difference with $\mathrm{SSR}_{\mathrm{vsw}}$, as from (22) it can be seen that we may have $\mathrm{SSR}_{\mathrm{vsw}}(T) < 1$ for some T. For example, this will be the case if $\rho^2 < \frac{1}{3}$.

Heston-LSV

The analytical formulas presented in the sections above are most useful for conducting a qualitative analysis of $\mathrm{SSR}_{\mathrm{vsw}}$, such as to explain the direction of its response to changes in market data or model parameters. However, to be

3 Term structure of varswap SSR in the Heston model, with parameters $\eta = 0.75$, $\rho = -0.1$, $\kappa = 2.0$, $v_0 = \bar{v} = 1.0$



The exact value obtained by purely numerical means is shown alongside the closed-form approximation

assured of production quality precision in a variety of market conditions, it is advisable to complement closed-form approximations with an exact numerical procedure for evaluating (2). We describe one such procedure in the case of Heston-LSV, defined by:

$$\begin{aligned} \mathrm{d}x(t) &= -\tfrac{1}{2}v(t)\sigma(t,x(t))^2\,\mathrm{d}t + \sqrt{v(t)}\sigma(t,x(t))\,\mathrm{d}W^1(t) \\ \mathrm{d}v(t) &= -\kappa(v(t)-\bar{v})\,\mathrm{d}t + \eta\sqrt{v(t)}\,\mathrm{d}W^2(t) \\ \langle \mathrm{d}W^1(t), \mathrm{d}W^2(t)\rangle &= \rho\,\mathrm{d}t \end{aligned}$$

As the varswap fair strike is a deterministic function of the initial spot and variance, S and v, $\hat{\sigma}_T^2 = \hat{\sigma}_T^2(S, v)$. Therefore, (2) takes the form:

$$\mathrm{SSR}_{\mathrm{vsw}} = \frac{1}{\Delta_T}\left(\frac{\mathrm{d}\hat{\sigma}_T^2}{\mathrm{d}\ln S} + \frac{\mathrm{d}\hat{\sigma}_T^2}{\mathrm{d}v}\frac{\sigma(0,S)\eta\rho}{\sqrt{v}}\right) \tag{23}$$

As previously, Δ_T is known in closed form. Next, the varswap fair strike is:

$$\hat{\sigma}_T^2 = \mathbb{E}\left[-2\ln\left(\frac{S(T)}{F(T)}\right)\right]$$

where $F(T)$ is the forward. This can be priced by a backward partial differential equation (PDE) using the finite-difference method. The solution grid at $t = 0$ contains $\hat{\sigma}_T^2(S, v)$ for all values of S and v, and therefore the derivatives appearing in (23) can be calculated immediately by finite difference. Notice this only requires a single pass of the PDE solver.

A. Examples of LSV calibration on September 1, 2021

Name	$\mathrm{SSR}_{\mathrm{atm}}$	$\mathrm{SSR}_{\mathrm{vsw}}$	κ	ρ	η (atm)	η (vsw)
BA.N	0.99	0.67	4.0	−0.85	2.29	3.99
UBER.N	1.61	1.08	4.0	−0.60	0.90	3.04
CCL.N	2.42	1.57	4.0	−0.60	0.53	1.81

$\mathrm{SSR}_{\mathrm{atm}}$ and $\mathrm{SSR}_{\mathrm{vsw}}$ denote realised values at the 2Y tenor, estimated with one year of data. κ and ρ are held arbitrarily fixed, while η is calibrated to match the corresponding SSR

Table A illustrates differences in parameter values that may arise from using (1) or (2) for calibration. Three single-stock LSV parameters were calibrated using market data on September 1, 2021. Boeing (BA.N), Uber (UBER.N) and Carnival (CCL.N) illustrate cases where the historical $\mathrm{SSR}_{\mathrm{vsw}}$, for the 2Y tenor, estimated using the previous year of data, respectively under-realises, realises and over-realises skew. In each row, the mean-reversion κ and correlation ρ are fixed, and the vol-of-var η is calibrated to match either $\mathrm{SSR}_{\mathrm{atmf}}$ or $\mathrm{SSR}_{\mathrm{vsw}}$ to its historical value, which is also reported in the table. This example is somewhat contrived in that a production calibration would also involve other targets and constraints and imperfectly fit all three parameters. However, it illustrates a typical way in which differences in the two realised SSRs ultimately influence the model calibration.

Conclusion

We introduced the varswap skew-stickiness ratio, $\mathrm{SSR}_{\mathrm{vsw}}$, defined so that it naturally complements the textbook $\mathrm{SSR}_{\mathrm{atmf}}$. This is a convenient metric to describe the spot and implied volatility dynamics at OTM strikes using a single number. Moreover $\mathrm{SSR}_{\mathrm{vsw}}$ shares several interesting properties with the classical $\mathrm{SSR}_{\mathrm{atmf}}$. Most notably, the short-tenor behaviour of the two metrics coincide, and several of the closed-form approximations known for $\mathrm{SSR}_{\mathrm{atmf}}$ have analogous expressions for $\mathrm{SSR}_{\mathrm{vsw}}$. Nevertheless, the two metrics have several key differences: most notably, $\mathrm{SSR}_{\mathrm{vsw}}$ may under-realise skew in the Heston model, and it is driven by curvature as well as skew at first order in the local vol model. Lastly, we described an efficient and accurate numerical procedure for calculating the exact value of $\mathrm{SSR}_{\mathrm{vsw}}$. ■

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