When You Hedge Discretely: Optimization of Sharpe Ratio for Delta-Hedging Strategy under Discrete Hedging and Transaction Costs

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Abstract

We consider the delta-hedging strategy for a vanilla option under the discrete hedging and transaction costs, assuming that an option is delta-hedged using the Black-Scholes-Merton model with the log-normal volatility implied by the market price of the option. We analyze the expected profit-and-loss (P&L) of the delta-hedging strategy assuming the four possible dynamics of asset returns under the statistical measure: the log-normal diffusion, the jump-diffusion, the stochastic volatility and the stochastic volatility with jumps. For all of the four models, we derive analytic formulas for the expected P&L, expected transaction costs, and P&L volatility assuming hedging at fixed times. Using these formulas, we formulate the problem of finding the optimal hedging frequency to maximize the Sharpe ratio of the delta-hedging strategy. Also, we show that the Sharpe ratio of the delta-hedging strategy can be improved by incorporating the price and delta bands for the rebalancing of the delta-hedge and provide analytical approximations for computing the optimal bands in our optimization approach. As illustrations, we show that our method provides a very good approximation to the actual Sharpe ratio obtained by Monte Carlo simulations under the time-based re-hedging. In contrary to Monte Carlo simulations, our analytic approach provide a fast and an accurate way to estimate the risk-reward characteristic of the delta-hedging strategy for real time computations.

Keywords: delta-hedging errors, profit & loss distribution, discrete trading, transaction costs, parameters misspecification, jump-diffusion model, stochastic volatility, Sharpe ratio

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1 Introduction

As opposed to the risk-neutral valuation in the Black-Scholes-Merton (1973) model (BSM), in the real world, the delta-hedging strategy is subject to the following key risks: first, mis-specification of model parameters (El Karoui-Jeanblanc-Shreve (1998)); second, transaction costs (Leland (1985), Avellaneda-Paras (1994), Toft (1996)); third, discrete hedging (Derman (1999)). The first factor leads to the residual P&L variance that cannot be eliminated by increasing the hedging frequency. The second one implies that the expected P&L will decrease when the hedging frequency increases. The third one results in the inverse relationship between the P&L variance and the hedging frequency. These considerations imply a trade-off between the hedging frequency, the expected P&L and its variance.

The following interesting questions arise from the these considerations:

1. If there exist good approximations for expected P&L, transaction costs, and its variance assuming discrete hedging assuming different dynamics for the under-

lying dynamics so that one can get more insight into the trade-off between the risk and reward?

2. What is the break-even level of the implied volatility at which an option can be sold so that option seller expects to make money delta-hedging this option with specified risk tolerance measured by the volatility or expected Sharpe ratio of the delta-hedging strategy?

3. How can one optimize the Sharpe ratio of the delta-hedging strategy?

To comment on the second question, we note that, in practice, selling options (being short gamma) is a risky business and the risk-reward profile and timing of losses is different for sellers and buyers. On average, an option seller expects to make money as a compensation for occasional and potentially large losses. For example, Broadie-Chernov-Johannes (2009) use sample period from 1987 (including the 1987 crash) to 2005 to find that the average *monthly* return (for long position in a put) is -30% for at-the-money puts and -57% for 6% out-of-the-money puts. However, losses (especially for equity index options) will inevitably occur at bad times during market declines and risk-aversion periods. Thus, in addition to expected transaction costs, the break-even implied volatility should include a premium for the timing of losses. Our approach helps to quantify the premium using a level of the Sharpe ratio that the option seller finds appropriate in the long-term.

To remark on the third question, we emphasize that the Sharpe ratio is a favorite tool for analysis trading strategies. In our setting, on one hand, rebalancing the delta-hedge as little as possible increases expected profit. On the other hand, infrequent rebalancing increases P&L volatility. As a result, the Sharpe ratio enables to balancing the reward of the delta-hedging strategy (hedge less) and risk (hedge more). Conventional utility-based approaches (Hodges-Neuberger (1989), Zakamouline (2006)) provide little insight into profitability of hedging strategies and their risks.

It is well-known that trading in options results in a skewed distribution (compared with the normal distribution) of the P&L for a delta-hedging strategy so that a single measure of risk given by P&L volatility may be not adequate. However, our approach is best suited for the profitability analysis of a single trade. It is the risk-management of options book that should account for tails of the distribution of its P&L given aggregated positions in many options. In addition to the Sharpe ratio, we can also look at some asymmetry measures such as the skew of the P&L. However, by means of the Monte-Carlo analysis we show that the P&L skew is too sensitive to the price dynamics and assumed hedging strategy, in contrast to the expected P&L and its volatility. As a result, employing higher moments can be too noisy and informative for establishment of the optimal hedging frequency.

A typical approach to answer the above questions is to run Monte-Carlo simulations of the delta-hedging strategy under different assumptions for the underlying dynamics and hedging strategy. While realistic, Monte-Carlo analysis is time-consuming and provides little insight into analytical relationship between model and strategy parameters and optimizing of the hedging strategy.

Sepp (2012b) shows the connection between the realized P&L of the delta-hedging P&L and the realized variance of the asset price. As a result, the distribution of

P&L can be well approximated by chi-squared distribution under diffusion and jump-diffusion dynamics and this approach can be used for analysis of the tails of the P&L distribution and risk-management. In this article, we extend this set-up to provide analytic tools to answer the questions raised above for delta-hedging strategy of vanilla at-the-money call and put options under diffusions with stochastic volatility and jumps. For brevity, we assume drift-less dynamics with zero interest and dividend rates, and time-independent models parameters, as generalization is straightforward.

2 Preliminary analysis

Our analysis rests on the following assumptions.

2.1 Assumptions

1) We assume that, given an underlying with spot price $S(t)$, a vanilla option on $S(t)$ with value function $U(t, S; \sigma_i)$ is valued under the pricing measure $\mathbb{Q}$ using the BSM PDE with implied volatility σ_i:

$$U_t + \frac{1}{2}\sigma_i^2 S^2 U_{SS} = 0, \tag{2.1}$$

and appropriate terminal condition. Option delta $\Delta(t, S; \sigma_i) = U_S(t, S; \sigma_i)$ is also computed by solving equation (2.1). We assume that volatility σ_i is implied from a market quote for $U(0, S; \sigma_i)$ and remains fixed up to maturity time T. This assumption states that the option is priced and delta-hedged at implied volatility. This is a common approach because, when hedging at a volatility different to the implied, the P&L is exposed to directional changes in the underling price (we will discuss this point in Section 2.3 in more details).

2) We assume that the dynamics of underlying price $S^*(t)$ under the objective measure $\mathbb{P}$ are specified according to the SDE:

$$dS^*(t)/S^*(t) = \sigma(\cdot)dW(t), \;\; S^*(0) = S, \tag{2.2}$$

that can include jumps and stochastic volatility. We assume that model parameters of this SDE are estimated for trading and risk-management purposes and that the expected P&L and its variance are computed under the objective measure $\mathbb{P}$.

3) We assume that transaction costs are proportional. These can be interpreted as the bid-ask spread, which can be estimated empirically based on the stock liquidity as follows:

$$k = 2\frac{S_{\text{ask}} - S_{\text{bid}}}{S_{\text{ask}} + S_{\text{bid}}},$$

where S_{ask} (S_{bid}) is the quoted ask (bid) price. Thus, $k/2$ is the average percentage loss per trade amount and, approximately, $S_{\text{ask}}(t) = (1+k/2)S(t)$ ($S_{\text{bid}}(t) = (1-k/2)S(t)$).

4) We assume that the delta-hedging strategy for short position in U is held up to maturity time T and re-balanced at uniform times $\{t_n\}$, $t_n = n\delta t$, $n = 1, .., N$,

$\delta t = T/N$, $t_0 = 0$, $t_N = T$ with total number of trades N. Time-based hedging strategy is required for our analytical developments. In addition, we consider the spot and delta-based hedging strategies and show how to use them in our analysis.

5) We exclude the drift from the objective dynamics (2.2) because its impact on the P&L of a delta-hedged position is small and has the order of δt^2 which is smaller than that of the realized variance, δt (for example, see Sepp (2012b), equation 44). In fact, as we discuss in Section 2.3, the impact of the drift for the delta-hedged position can become apparent if the option delta is computed using the volatility that is different from the market implied volatility. However, this impact is also of the second order effect.

2.2 Profit-and-Loss

It is well-known that if the delta-hedge for short position in U is computed using implied volatility σ_i then the realized P&L in the absence of transaction costs is approximately given by:

$$P(N) = \sum_{n=1}^{N} \left(\sigma_i^2 \delta t - \Sigma_n^2\right) \Gamma(t_{n-1}, T, S^*; \sigma_i), \tag{2.3}$$

where Σ_n^2 is the variance realized under the objective measure $\mathbb{P}$:

$$\Sigma_n^2 = \left(\frac{S^*(t_n) - S^*(t_{n-1})}{S^*(t_{n-1})}\right)^2$$

and $\Gamma(t, T, S; \sigma_i)$ is the option cash-gamma defined by:

$$\begin{aligned}\Gamma(t, T, S; \sigma_i) &\equiv \frac{1}{2} S^2(t) U_{SS}(t, S; \sigma_i)\\ &= \frac{K}{2\sqrt{2(T-t)\pi\sigma_i^2}} \exp\left\{-\frac{1}{2}\left(\frac{\ln\frac{S}{K} - \frac{1}{2}(T-t)\sigma_i^2}{\sqrt{(T-t)\sigma_i^2}}\right)^2\right\}.\end{aligned} \tag{2.4}$$

A continuous-time version of equation (2.3) is obtained by El Karoui-Jeanblanc-Shreve (1998). Various extensions of it are analyzed in Carr (2005), Davis (2010), Sepp (2012a) who show that this formula is independent from the assumption about the dynamics of $S^*(t)$ and serves as an accurate approximation for the realized P&L.

Finally, following Leland (1985) and Toft (1996), we approximate the realized transaction costs by:

$$\begin{aligned}C(N) &= \frac{1}{2} k \sum_{n=1}^{N} S^*(t_n) \left|\Delta(t_n, S^*) - \Delta(t_{n-1}, S^*)\right|\\ &\approx k \sum_{n=1}^{N} \frac{S^*(t_n)}{S^*(t_{n-1})} \left|\frac{S^*(t_n) - S^*(t_{n-1})}{S^*(t_{n-1})}\right| \Gamma(t_{n-1}, T, S^*; \sigma_i).\end{aligned} \tag{2.5}$$

We note that Leland (1985) and Toft (1996) consider transaction costs under the diffusion model of BSM. While equation (2.5) applies for general dynamics of the spot price, the computation of the expected transaction cost is model dependent. Nevertheless, we obtain accurate approximations for the expected transaction cost under different dynamics for the asset price.

2.3 Option delta and P&L

We note that in practice, the convenient way to quote option price for strike K and maturity T is to use BSM implied volatility for this option even though the underlying dynamics for option pricing and underlying price are, in fact, different under the objective measure. It is typical to calibrate model parameters of the pricing dynamics so that the pricing model produces the same BSM implied volatility as observed in the market. Thus, equation (2.1) can be used irregardless of what dynamics is assumed for option pricing because, as we show in this section, the difference arising from hedging at different volatility has higher order effect and can be ignored in our analysis. More important question is what implied volatility needs to be used for computing option delta (see also section 4.4 in Sepp (2012b) for related discussion).

If we assume that the option is delta-hedged, with delta $\Delta(t, S; \sigma_h)$, using BSM volatility σ_h, which is different from option implied volatility σ_i observed in the market, then we can show that the realized P&L (2.3) will have an extra term denoted by $H(N)$ related to the difference in option delta-hedges:

$$H(N) = \sum_{n=1}^{N} \left(\Delta(t_{n-1}, S^*; \sigma_h) - \Delta(t_{n-1}, S^*; \sigma_i)\right)\left(S^*(t_n) - S^*(t_{n-1})\right). \tag{2.6}$$

Now we see that the realized P&L has exposure to the realized price path in addition to the realized variance of the price path. To compute the expected P&L and its volatility, now requires to estimate the drift of the asset price, which is more difficult to estimate compared to estimation of the volatility of the asset price. To analyze it further, we take $\sigma_h = \sigma_i + h$ so that

$$\begin{aligned}\Delta(t, S^*; \sigma_h) - \Delta(t, S^*; \sigma_i) &\approx h \frac{\partial}{\partial \sigma_i} \Delta(t, S^*; \sigma_i) \\ &= \frac{2h\sqrt{T-t}}{S} \left(\frac{\ln \frac{S}{K} - \frac{1}{2}(T-t)\sigma_i^2}{\sqrt{(T-t)\sigma_i^2}} \right) \Gamma(t, T, S; \sigma_i).\end{aligned} \tag{2.7}$$

and, for brevity, assuming that $S = K$, (2.6) becomes:

$$H(N) = -h\sigma_i \sum_{n=1}^{N} (T - t_n)\Gamma(t_n, T, S; \sigma_i) \left(\frac{S^*(t_n) - S^*(t_{n-1})}{S^*(t_{n-1})} \right) \tag{2.8}$$

so that the realized P&L is exposed to the realized return weighted by option cash-gamma.

To find the expected value of this term, we assume that the log-normal diffusion model for the price dynamics under objective measure has drift μ and volatility σ_r.

Then, we obtain

$$\overline{H} \equiv \mathbb{E}^{\mathbb{P}}[H(N)] \approx -h\sigma_i \left(\frac{1}{2}T\Gamma(T/2,T,S;\sigma_i)\right) \sum_{n=1}^{N} \mathbb{E}^{\mathbb{P}}\left[\frac{S^*(t_n)-S^*(t_{n-1})}{S^*(t_{n-1})}\right]$$
$$= -\frac{1}{2}h\sigma_i \mu T^2 \Gamma(0,T/2,S;\sigma_i).$$

Thus, $h > 0$, the contribution to the P&L is negative for a positive drift and vice versa. We note that the expected P&L in the diffusion model is given by (4.3) and is proportional to $(\sigma_i^2 - \sigma_r^2)T$ so that $\overline{H}$ only contributes in second order in T.

Under the diffusion model with volatility σ_r, the variance of H is approximated by:

$$\mathbb{V}^{\mathbb{P}}[H(N)] \approx \frac{1}{4}h^2\sigma_i^2\sigma_r^2 T^3 \Gamma^2(0,T/2,S;\sigma_i). \tag{2.9}$$

which does not depend on the hedging frequency and needs to be contrasted with irreducible variance f_{df} given in (4.4).

To summarize, unless the implied volatility is very small while the drift is very large, the total impact of H on the expected P&L is insignificant. Also, unless h^2 is very large, irreducible variance of the P&L will be dominated by f_{df} term. Although, hedging under different delta may significantly affect day-to-day variations in the P&L , the impact on the total realized P&L is expected to be insignificant so that, for our analysis, we can safely assume that the hedging is done using implied BSM volatility.

Finally, the pricing model is important for computation of option values and hedges, however, the delta-hedging P&L is realized under the objective measure. As a result, the expected P&L and its variance need to be computed using model parameters under the objective measure.

2.4 Option vega and P&L

Option vega can serve as an additional source of the P&L volatility. Accounting for vega risk will increase the P&L volatility so that, typically, it will result in lower Sharpe ratios. In assumption 4) in Section 2.1, we assume that the option is held to maturity so that variations in the implied volatility have an impact of smaller order than variations in the underlying price. A proper modeling of the volatility risk requires a model for the joint dynamics of the implied volatility and the underlying price, which we leave for future research.

3 Delta-hedging strategy

We consider the three approaches to delta-hedging to specify the sequence of hedging times $\{\tau_n\}$ (see Whalley-Wilmott (1997) and Zakamouline (2006) for analysis of these hedging strategies):
1) time-based with rebalancing at fixed times with deterministic $\{\tau_n\}$;

2) price-based with rebalancing at the first time that the change in the spot price exceeds specified level α_S:

$$\tau_n = \min_{\tau_{n-1}<t\leq T} \left\{ t : \left| \frac{S^*(t)}{S^*(\tau_{n-1})} - 1 \right| \geq \alpha_S \right\}; \tag{3.1}$$

3) delta-based with rebalancing at the first time that the change in the delta exceeds specified level α_Δ:

$$\tau_n = \min_{\tau_{n-1}<t\leq T} \{ t : |\Delta(t, S^*(t)) - \Delta(\tau_{n-1}, S^*(\tau_{n-1}))| \geq \alpha_\Delta \}. \tag{3.2}$$

Although our analysis assumes deterministic hedging times with the hedging frequency being the key optimization variable, we show how to extend our results to price- and delta-based hedging in a simple and accurate way. As a result, we can formulate the problem of optimizing the Sharpe ratio in term of finding the optimal price- and delta-bands to balance risk and reward. In practice, using price- or delta-bands is more efficient than deterministic hedging as it allows to save on the transaction cost and reduce P&L volatility if bands are chosen wide enough.

The well-known result for the diffusion model (4.1) with volatility σ_r states that, for the price-based strategy, the expected rebalancing interval $\overline{\tau}$ is given by:

$$\overline{\tau} \equiv \mathbb{E}^{\mathbb{P}}[\tau] = \frac{\alpha_S^2}{\sigma_r^2}. \tag{3.3}$$

Thus, a rough approximation to the number of rebalancing, $\overline{N}$, when following the price-based strategy is given by:

$$\overline{N} \approx \frac{T}{\overline{\tau}} = \frac{T\sigma_r^2}{\alpha_S^2}. \tag{3.4}$$

Alternatively, given specified frequency N we can imply the corresponding band α_S by:

$$\alpha_S = \frac{\sqrt{T}\sigma_r}{\sqrt{N}}. \tag{3.5}$$

The delta-based rebalancing is more robust because it accounts for option gamma as can be seen from the following approximation:

$$\begin{aligned} |\Delta(t, S^*(t)) - \Delta(\tau_{n-1}, S^*(\tau_{n-1}))| &\approx |(S^*(t) - S^*(\tau_{n-1}))U_{SS}(\tau_{n-1}, S^*(\tau_{n-1}))| \\ &= \left| \left(\frac{S^*(t)}{S^*(\tau_{n-1})} - 1 \right) \right| |S^*(\tau_{n-1})U_{SS}(\tau_{n-1}, S^*(\tau_{n-1}))| . \end{aligned} \tag{3.6}$$

Thus, for at-the-money options with high gamma, the rebalancing is expected to be more frequent, while, for out-of-the money options with low gamma, the re-hedging is only applied for big moves in the spot price.

It is difficult to find the exact correspondence between the spot and delta bands α_S and α_Δ because the delta-based hedging is dependent on both the realized price path and implied volatility. By means of Monte-Carlo analysis, we have found the following heuristic rule that approximately equates the expected number of rebalancing under the two strategies:

$$\alpha_\Delta = \alpha_S \left| S U_{SS}(0,S) \right| \ , \ U_{SS}(0,S) = \frac{2}{S^2}\Gamma\left(0, T/2, S; \sqrt{(\sigma_r^2 + \sigma_i^2)/2}\right). \tag{3.7}$$

Our justification is based on using (3.6) along with using the expected gamma (4.2) in the diffusion model. Similarly to (3.5), given specified frequency N we can imply the corresponding band α_Δ:

$$\alpha_\Delta = \frac{\sqrt{T}\sigma_r}{\sqrt{N}} \left| S U_{SS}(0,S) \right|. \tag{3.8}$$

The price- and delta-based bands, specified by equations (3.5) and (3.8), respectively, are obtained for the diffusion model. For models with stochastic volatility and jumps we use the same equations with σ_r^2 computed as follows (this heuristic is based on approximation of the Poisson jump process by the Brownian motion with the same quadratic variance, which can be justified because the bands are not wide):
for jump-diffusion, $\sigma_r^2 = \vartheta_{jd}$ with ϑ_{jd} specified by (4.9);
for stochastic volatility, $\sigma_r^2 = \vartheta_{sv}$ with ϑ_{sv} specified by (4.13);
for jump-diffusion with stochastic volatility, $\sigma_r^2 = \vartheta_{svj}$ with ϑ_{svj} specified by (4.18).

3.1 Mean-variance analysis

We analyze the moments of the P&L using the P&L given by formula (2.3). For this purpose, we need to specify the dynamics (2.2) and analyze the realized variance Σ_n^2 in (2.3). Importantly, we want to estimate the expected P&L and its variance, and expected transaction costs. In general, this problem is highly path-dependent since as cash gamma $\Gamma(t_{n-1}, T, S^*; \sigma_i)$ depends on the path of S^*, as does the realized variance. However, we derive closed-form approximations which allow to compute these quantities under the diffusion, jump-diffusion, and stochastic volatility with jumps models. We assume the time-based hedging strategy with price- and delta-based strategies can be analyzed in this framework using (3.5) and (3.8), respectively.

In particular, we show that the expected P&L under objective measure $\mathbb{P}$:

$$\overline{U} \equiv \mathbb{E}^{\mathbb{P}}\left[P(N)\right] = u, \tag{3.9}$$

is independent of the hedging frequency N.

The expected transaction costs $\overline{C}(N)$ can be approximated by:

$$\overline{C}(N) \equiv \mathbb{E}^{\mathbb{P}}\left[C(N)\right] = c\sqrt{N}, \tag{3.10}$$

where c^2 is the expected transaction cost per trade.

Finally, the variance $\overline{V}(N)$ can be approximated by:

$$\overline{V}(N) = \mathbb{V}^{\mathbb{P}}\left[P(N)\right] = \frac{p}{N} + f. \tag{3.11}$$

where p is the variance rate inversely proportional to hedging frequency and f is the irreducible variance. We emphasize that the presence of f in the P&L variance means that part of the P&L risk, arising from mis-specification of model parameters, jumps and stochastic volatility (or any combination of these factors), cannot be eliminated by increasing the delta-hedging frequency. We note that the variance of expected transaction costs is of order k^2 so that its contribution to the P&L variance $\overline{V}(N)$ can be ignored.

3.2 Sharpe ratio

The above results enable us to represent the Sharpe ratio $\overline{S}(N)$ of the delta-hedging strategy by:

$$\overline{S}(N) = \frac{1}{\sqrt{T}} \frac{u - c\sqrt{N}}{\sqrt{\frac{p}{N} + f}}, \tag{3.12}$$

where constants u, c, v, f are all positive. The specification of these constants depends on the underlying dynamics under $\mathbb{P}$ and will be considered in the following section. Factor $\frac{1}{\sqrt{T}}$ arises from normalization of the expected P&L and its variance by T. For illustrations of $\overline{S}(N)$ we refer to Figures 2 and 3.

Our objective is to maximize the Sharpe ratio $\overline{S}(N)$. We note that equation (3.12) has a unique global maximum. The stationary point N^*, so that the derivative of $\overline{S}(N)$ at point N^* is zero, $\overline{S}'(N^*) = 0$, solves the following equation:

$$-\frac{2cp}{N^*} + \frac{up}{N^*\sqrt{N}^*} - cf = 0. \tag{3.13}$$

Making substitution $m = 1/\sqrt{N^*}$, we obtain the cubic equation for m which has one real and two complex roots. The real-valued root is given by (using formulas from Section 5.6 in Press *et al* (1992)):

$$\begin{aligned} &m = A + \frac{Q}{A} + \frac{2c}{3u},\ A = \left(|R| + \sqrt{D}\right)^{1/3} \\ &Q = \frac{4}{9}\left(\frac{c}{u}\right)^2,\ R = -\left(\frac{1}{3}\left(\frac{c}{u}\right)^3 + \frac{1}{2}\frac{cf}{up}\right),\ D = R^2 - Q^3 > 0, \end{aligned} \tag{3.14}$$

where, while m is real, we get integer $N^* = 1/m^2$ by rounding. The obtained value of N^* yields the optimal hedging frequency.

4 Analysis

In this section we consider some specifications of the $\mathbb{P}$-dynamics (2.2) and derive constants to use in estimation of the Sharpe ratio (3.12). For convenience, we assume unit notional.

4.1 Diffusion model

Now we assume that the underlying price $S^*(t)$ under the objective measure $\mathbb{P}$ is driven by log-normal diffusion with volatility σ_r:

$$dS^*(t)/S^*(t) = \sigma_r dW(t),\ S^*(0) = S, \tag{4.1}$$

where $W(t)$ is standard Brownian motion.

First we introduce the expected cash gamma under $\mathbb{P}$, $\overline{\Gamma}(t_n, T; \sigma_r^2)$, $n = 0, .., N-1$, which is given by:

$$\overline{\Gamma}(t_n, T; \sigma_r^2) \equiv \mathbb{E}^{\mathbb{P}}\left[\Gamma(t_n, T, S; \sigma_i) \mid S = S^*(t_n)\right] = \Gamma(0, T, S; \zeta(t_n)), \tag{4.2}$$

where $\zeta(t_n) = \sqrt{(t_n\sigma_r^2 + (T - t_n)\sigma_i^2)/T}$. If volatility parameters under $\mathbb{Q}$ and $\mathbb{P}$ are equal, we obtain that $\overline{\Gamma}(t_n, T; \sigma_r) = \Gamma(0, T, S; \sigma_r)$, so that the expected option gamma at rebalancing times $\{t_n\}$ equals to its value at time $t_0 = 0$.

In Appendix A, we obtain equation (7.3) for an approximation of the expected P&L under $\mathbb{P}$:

$$\overline{U}_{df} = (\sigma_i^2 - \sigma_r^2)T\overline{\Gamma}_{df}, \tag{4.3}$$

where $\overline{\Gamma}_{df}$ is defined by (7.4). The coefficient u_{df} to use in the Sharpe ratio (3.12), is adjusted by the initial cost to enter the delta-hedge:

$$u_{df} = (\sigma_i^2 - \sigma_r^2)T\overline{\Gamma}_{df} - \frac{k}{2}S(t_0)\left|\Delta(t_0, S)\right|.$$

In Appendix A, we also obtain equation (7.8) for an approximation of the P&L variance which is represented using equation (3.11) with

$$p_{df} = 2q\sigma_r^4 T^2 \overline{\Gamma^2}_{df},\ f_{df} = \left(\sigma_i^2 - \sigma_r^2\right)^2 T^2 \left(\overline{\Gamma^2}_{df} - (\overline{\Gamma}_{df})^2\right), \tag{4.4}$$

with $\overline{\Gamma^2}_{df}$ defined by equation (7.6) and $q = \pi\sqrt{3}/4$.

We use formula (2.5) to approximate the expected realized transaction costs as follows:

$$\begin{aligned}
\overline{C}(N) &\approx k\mathbb{E}^{\mathbb{P}}\left[\sum_{n=1}^{N}\left|\left\{\frac{S^*(t_n)}{S^*(t_{n-1})}\frac{S^*(t_n) - S^*(t_{n-1})}{S^*(t_{n-1})}\right\}\right|\Gamma(t_{n-1}, T, S^*; \sigma_i)\right] \\
&\approx \mathbb{E}^{\mathbb{P}}\left[k\sum_{n=1}^{N}\left|\sqrt{\sigma_r^2\delta t}\epsilon_n\right|\Gamma(t_{n-1}, T, S^*; \sigma_i)\right], \\
&\approx k\overline{\Gamma}_{df}\sqrt{\sigma_r^2\delta t}\sum_{n=1}^{N}\mathbb{E}^{\mathbb{P}}\left[|\epsilon_n|\right] \\
&= c_{df}\sqrt{N},\ c_{df} = kT\sqrt{\frac{2\sigma_r^2}{\pi T}}\overline{\Gamma}_{df},
\end{aligned} \tag{4.5}$$

where we apply a normally distributed return $\sqrt{\sigma_r^2\delta t}\epsilon_n$, where ϵ_n is a standard normal random variable, to approximate the term in curly brackets in the first line.

The optimal value of hedging frequency N^*, so that the Sharpe ratio is maximized, is given by equation (3.14). To gain some intuition (for real-time applications we always solve the solution given by equation (3.14)), we consider a simplified case when the implied volatility is close to the realized: $\sigma_i \approx \sigma_r$. As a result, $f \approx 0$ and the Sharpe ratio becomes:

$$\overline{S}_{df} = \frac{1}{\sqrt{T}} \frac{(\sigma_i^2 - \sigma_r^2)T\overline{\Gamma}_{df} - kT\sqrt{\frac{2\sigma_r^2}{\pi T}}\overline{\Gamma}_{df}\sqrt{N}}{\sqrt{\frac{2q\sigma_r^4 T^2 \overline{\Gamma}^2{}_{df}}{N}}}$$
$$\approx \frac{1}{\sqrt{T}} \frac{(\sigma_i^2 - \sigma_r^2) - k\sqrt{\frac{2\sigma_r^2}{\pi T}}\sqrt{N}}{\sqrt{2q\sigma_r^4 \frac{1}{\sqrt{N}}}}.$$

The Sharpe ratio is maximized by choosing:

$$N^* = \left(\frac{u}{2c}\right)^2 = \frac{\pi T}{8}\left(\frac{\sigma_r}{k}\right)^2 \left(\frac{\sigma_i^2}{\sigma_r^2} - 1\right)^2, \tag{4.6}$$

with the optimal Sharpe ratio given by:

$$\overline{S} = \frac{1}{8(3)^{1/4}} \frac{\sigma_r}{k} \left(\frac{\sigma_i^2}{\sigma_r^2} - 1\right)^2.$$

It is interesting to conclude that with this choice of N^*, the expected transaction costs equal to the half of the expected P&L. The optimal frequency is inversely proportional to the square of the transaction costs, so if transaction costs rate k halves, the optimal frequency increases four fold and the Sharpe ratio doubles. Also, the Sharpe ratio does not depend on the expected cash-gamma and therefore is barely dependent on option strike and maturity.

To get more insight, we assume that the implied volatility trades at premium ϵ so that $\sigma_i = (1+\epsilon)\sigma_r$ and:

$$N^* \approx \frac{\pi T}{2}\left(\frac{\sigma_r}{k}\right)^2 \epsilon^2 \ , \ \overline{S} \approx \frac{3}{8}\frac{\sigma_r}{k}\epsilon^2. \tag{4.7}$$

Accordingly, the optimal frequency is proportional to the premium of the implied to realized volatility: the higher is the premium, the more frequent hedging is possible and, as a result, the higher Sharpe ratio is attainable. Finally, the Sharpe ratio is proportional to the ratio of realized volatility to transaction costs. Higher costs relative to realized volatility, reduce the Sharpe ratio and vice versa.

4.2 Jump-diffusion model

Now we assume that the underlying dynamics are driven by a jump-diffusion process:

$$dS^*(t)/S^*(t) = \sigma_r dW(t) + (e^\nu - 1)\, dN(t), \ S^*(0) = S, \tag{4.8}$$

where $N(t)$ is Poisson process with intensity λ. Given a jump in $N(t)$, the jump in log-price is constant with magnitude ν (it is easy to incorporate the volatility of the jump, but for brevity we do not consider it here).

In appendix A, we show that the expected P&L can be approximated by:

$$\overline{U}_{jd} = (\sigma_i^2 - \vartheta_{jd})T\overline{\Gamma}_{jd},$$

where $\overline{\Gamma}_{jd}$ is defined by equation (7.10) and ϑ_{jd} is the expected realized (quadratic) variance of $\ln S^*(t)$ under (4.8):

$$\vartheta_{jd} = \sigma_r^2 + \lambda\nu^2. \tag{4.9}$$

Similarly to (4.5), we approximate the expected transaction costs by

$$\overline{C}_{jd} \approx c_{jd}\sqrt{N} + k\lambda T|\nu|\overline{\Gamma}_{jd}, \ c_{jd} = kT\sqrt{\frac{2\sigma_r^2}{\pi T}}\overline{\Gamma}_{jd}, \tag{4.10}$$

where the additional term represent the expected number of jumps and related rebalancing costs so that u_{jd} is decreased by this term and initial transaction costs:

$$u_{jd} = (\sigma_i^2 - \vartheta_{jd})T\overline{\Gamma}_{jd} - k\lambda T|\nu|\overline{\Gamma}_{jd} - \frac{k}{2}S(t_0)\left|\Delta(t_0, S)\right|.$$

Using equation (7.12), the P&L variance can be represented be means of equation (3.11) with

$$\begin{aligned} p_{jd} &= q\overline{\Gamma}_{jd}^2 T^2\left(2\sigma_r^4 + \lambda\sigma_r^2\nu^2\right) \\ f_{jd} &= q\overline{\Gamma}_{jd}^2 T\lambda\nu^4 + \left((\sigma_i^2 - \vartheta_{jd})T\right)^2\left(\overline{\Gamma^2}_{jd} - (\overline{\Gamma}_{jd})^2\right). \end{aligned}$$

where $\overline{\Gamma^2}_{jd}$ is defined by (7.13).

The Sharpe ratio is given by equation (3.12) with optimal solution specified by (3.14). To get some insight, we assume that the product cf is small. So by analogy to (4.6), we obtain:

$$N^* = \left(\frac{u}{2c}\right)^2 = \frac{\pi T}{8}\left(\frac{\sigma_r}{k}\right)^2\left(\frac{\sigma_i^2}{\sigma_r^2} - 1 - \frac{\lambda(\nu^2 + k|\nu|)}{\sigma_r^2}\right)^2, \tag{4.11}$$

Thus, increasing jump intensity and jump magnitude will decrease the optimal frequency. As a result, in the presence of jumps, the optimal frequency is expected to be smaller than in the diffusion model. As in the diffusion model, the optimal frequency is proportional to the maturity time.

4.3 Stochastic volatility model

Now we assume that the underlying dynamics are driven by Heston (1993) stochastic volatility model with stochastic variance $V(t)$:

$$\begin{aligned} dS^*(t)/S^*(t) &= \sqrt{V(t)}dW(t), \ S^*(0) = S, \\ dV(t) &= \kappa(\theta - V(t))dt + \varepsilon\sqrt{V(t)}dZ(t), V(0) = V, \end{aligned} \tag{4.12}$$

where $W(t)$ and $Z(t)$ are correlated Brownians with correlation parameter ρ.

Similarly to the jump-diffusion model, if the option is delta-hedged using the implied-volatility σ_i, then the realized P&L is given by equation (2.3) with realized variance Σ_n^2 sampled using the dynamics of (4.12). The expected realized (quadratic) variance of $\ln S^*(t)$ under (4.12), $\vartheta_{sv}(T)$, is given by:

$$\vartheta_{sv}(T) = \theta + \frac{1}{\kappa T}\left(1 - e^{-\kappa T}\right)(V - \theta). \tag{4.13}$$

We approximate the expected P&L by:

$$\overline{U}_{sv} = (\sigma_i^2 - \vartheta_{sv}(T))T\overline{\Gamma}_{sv} - \overline{L}_{sv}, \tag{4.14}$$

where $\overline{\Gamma}_{sv}$ is defined by (7.15) and $\overline{L}_{sv}$ is the auto-correlation correction defined by equation (8.3). Thus:

$$u_{sv} = (\sigma_i^2 - \vartheta_{sv}(T))T\overline{\Gamma}_{sv} - \overline{L}_{sv} - \frac{k}{2}S(t_0)\left|\Delta(t_0, S)\right|.$$

We note that $\overline{L}_{sv}$ is negative so that the stochastic volatility contributes positively to the expected P&L of a short volatility position. This can be explained by the fact that, given a large value of realized variance Σ_n, because of the positive correlation between the instantaneous variance and the realized variance, it is expected that the realized variance, realized up to time time t_n, is large and thus the option is likely to be out of the money with a small cash-gamma. Therefore, even though the n-th contribution to the P&L is expected to be negative, its magnitude will be mitigated by a small cash-gamma. In opposite, if realized variance Σ_n is small, it is more likely that the option is still at-the-money thus the positive contribution is magnified by a larger value of the cash-gamma. In a diffusion model, the instantaneous variances is deterministic so that such effect is not observed.

Expected transaction costs are approximated by:

$$\overline{C}_{sv} = c_{sv}\sqrt{N}, \; c_{sv} = kT\sqrt{\frac{2\vartheta_{sv}(T)}{\pi T}}\overline{\Gamma}_{sv}; \tag{4.15}$$

while the P&L variance is approximated using (7.18) and given by equation (3.11) with

$$\begin{aligned} &p_{sv} = 2qV^2T^2\overline{\Gamma^2}_{sv}, \; f_{sv} = f_1 + f_2 \\ &f_1 = \overline{\Gamma^2}_{sv}V_{sv}, \; f_2 = \left((\sigma_i^2 - \vartheta_{sv}(T))T\right)^2\left(\overline{\Gamma^2}_{sv} - (\overline{\Gamma}_{sv})^2\right), \end{aligned} \tag{4.16}$$

where $\overline{\Gamma^2}_{sv}$ is defined by (7.19) and V_{sv} is defined by (7.21). By analogy, the Sharpe ratio is given by (3.12) using coefficients as defined above.

4.4 Stochastic volatility model with jumps

Now we consider stochastic volatility model (4.12) augmented with jumps in price as in dynamics of (4.8). This model is analyses by aggregating available results. First, we approximate the expected P&L by:

$$\overline{U}_{svj} = (\sigma_i^2 - \vartheta_{svj}(T))T\overline{\Gamma}_{svj} - \overline{L}_{sv}, \tag{4.17}$$

where $\vartheta_{svj}(T)$ is the quadratic variance:

$$\vartheta_{svj}(T) = \vartheta_{sv}(T) + \lambda\nu^2 \tag{4.18}$$

and $\overline{\Gamma}_{svj}$ is computed using (7.15) and applying (7.17) for computing (7.16). Transaction costs are approximated by:

$$\overline{C}_{svj} = c_{svj}\sqrt{N} + k\lambda T|\nu|\overline{\Gamma}_{svj}, \; c_{svj} = kT\sqrt{\frac{2\vartheta_{sv}(T)}{\pi T}}\overline{\Gamma}_{svj}; \tag{4.19}$$

and the P&L variance is given by equation (3.11) with

$$\begin{aligned}
&u_{svj} = (\sigma_i^2 - \vartheta_{svj}(T))T\overline{\Gamma}_{svj} - \overline{L}_{sv} - k\lambda T|\nu|\overline{\Gamma}_{svj} - \frac{k}{2}S(t_0)\,|\Delta(t_0, S)| \\
&p_{svj} = q\overline{\Gamma^2}_{svj}\left(2V^2T^2 + \lambda\sigma_r^2\nu^2\right), \; f_{svj} = f_1 + f_2 + f_3, \\
&f_1 = \overline{\Gamma^2}_{svj}V_{sv}, \; f_2 = q\overline{\Gamma^2}_{svj}T\lambda\nu^4, \; f_3 = \left((\sigma_i^2 - \vartheta_{svj}(T))T\right)^2\left(\overline{\Gamma^2}_{svj} - (\overline{\Gamma}_{svj})^2\right).
\end{aligned}$$

5 Illustrations

In this section, we provide some illustrations using diffusion (4.1) (DF), the jump-diffusion (4.8) (JD), stochastic volatility (4.12) (SV) and stochastic volatility with jumps (SVJ)[2].

5.1 Specification

Model parameters are given in Table 1 and are specified as follows: Case I) corresponds to an option on a liquid index or an ETF (such as the S&P 500, QQQQ, etc); Case II) corresponds to an option on a high-beta stock (such as CAT, APPL, etc). In the first case, the implied volatility trades at 10% premium to the expected realized volatility, with the spread of the implied volatility to the realised volatility being 1.5% and transaction costs being $k = 0.001$. In the second case, the implied volatility trades at 20% premium with the spread of 5% and transaction costs $k = 0.004$. The option under consideration is call option with $S = K = 1$, $T = 1$. The trade notional is $1000/\Gamma(0, T, S; \sigma_i)$, where Γ is option cash-gamma defined by (2.4). Thus, the notional equals to 830 for case I and 1521 for case II.

For models with jumps, the volatility of the Brownian motion, σ_r, is adjusted by the contribution from jumps, $\lambda\nu^2$. For models with stochastic volatility, we set $\sigma_r^2 = V(0) = \theta^2$. As a result, the expected quadratic variance, $\sigma_r^2 + \lambda\nu^2$, is the same for all four models.

We analyze delta-hedging based on:
1) time-based hedging with fixed frequency N;
2) price-based hedging (3.1);
3) delta-based hedging (3.2).

[2]Matlab files with sample code for these computations are available at http://www.mathworks.co.uk/matlabcentral/fileexchange/54345

For the price-based strategy, given N, the corresponding band α_S is determined by (3.5), while for delta-based hedging, the corresponding band α_Δ is specified by (3.7). Frequency (Freq) corresponding to N, $N = \{15, 30, 60, 120, 240, 480, 960, 1920\}$, is interpreted respectively as (approximately) 1 per m (one rebalancing per month; $N = 15$), 1 per w (once per week; $N = 60$), 1 per d (once per day; $N = 240$), and so forth.

The realized P&L is computed by:

$$P(\{t_n\}) = \sum_{n=0} \Delta\Pi_n, \tag{5.1}$$

where $\Delta\Pi_0 = -(k/2)S(t_0)\,|\Delta(t_0, S)|$ and

$$\begin{aligned}\Delta\Pi_n &= [\Pi_n - e^{r_n}\Pi_{n-1}] - S(t_n)[\Delta(t_n, S) - \Delta(t_{n-1}, S)] \\ &\quad - (\kappa/2)S(t_n)\,|\Delta(t_n, S) - \Delta(t_{n-1}, S)|\end{aligned}$$

where $\Pi(t_n, S) = S(t_n)\Delta(t_n, S) - U(t_n, S)$, r_n is the accrual rate for interval $(t_{n-1}, t_n]$ (we assume that r_n is zero in this analysis), $\{t_n\}$ are rebalancing times. Rebalancing times are fixed for the time-based strategy $t_n = n\Delta T$ with $\Delta T = T/N$ and N being the total number of rebalancing. For price- and delta-based hedging, rebalancing times are specified by (3.1) and (3.2), respectively, with the total number of rebalancing being a random variable. Here, $\Delta(t, S)$ is the option delta at time t. At the option maturity, the value of the option, $U(t_N, S)$, is specified by pay-off function $u(S(t_N), K)$, while the option delta, $\Delta(t_N, S)$, is (minus) one if the (put) call option is in-the-money and zero otherwise. The terms in the first line of equation (5.1) represent borrowing and delta-hedge rebalancing costs, respectively; the term in the second line represent transaction costs.

We note that in our analysis, the price- and delta-based bands are determined using formulas (3.5) and (3.7) with the specified hedging frequency N and volatility σ_r. These formulas are derived so that the expected P&L is approximately equal among all three strategies. If our reasoning is correct, MC results should be approximately equal for all three strategies and the analytic approximation. As a result, the optimal hedging frequency for the time-based strategy can be converted to the optimal hedging bands for price- and delta-based strategies.

For MC simulations, $2,000$ paths are used. We note that the MC error estimate is given by the MC P&L volatility divided by $\sqrt{2,000}$. However, the confidence bounds for the Sharpe ratio are even larger because the estimated P&L volatility has the same MC error estimate. For price- and delta-based strategies, we report only MC results. For the price- and delta-based strategies and use $10,000$ periods per year (which corresponds to observation frequency of about 10 minutes) to simulate the spot price and check the rebalancing condition.

5.2 Results

In Table 2, we provide coefficients for the Sharpe ratio and the P&L statistics computed using specified model parameters assuming time-based hedging.

In Figure 1, we plot the expected P&L and its volatility (on left scale) and the corresponding Sharpe ratio (on right scale) as functions of N for the diffusion model with parameters from case II. In Figure 2, we plot the analytical Sharpe ratio for all four models. We observe that models with stochastic volatility imply smaller Sharpe ratios that peak at smaller values of N.

In Figure 3, we plot the optimal Sharpe ratio as function of option maturity T for model with parameters from case II. We observe that the optimal Sharpe ratio peaks for options with maturities of one and two month and declines as maturity increases because expected transaction costs increase. For model with stochastic volatility and jumps, the obtainable optimal Sharpe ratio is smaller than that for the diffusion model.

In Figure 4, we plot the optimal re-hedging period, $250T/N^*$ for model with parameters from case II, using annualization factor of 250, as function of option maturity T, where N^* is the optimal hedging frequency for this maturity. The optimal re-hedging period increases for longer maturities from as low as hedging once per 3 days for short-term maturities (up to 2 month) to 7-8 days for longer-term maturities (above 2 years). The diffusion model implies shorter re-hedging periods.

In Tables 3 and 5, we report the expected P&L (P&L), transaction costs (Costs), the P&L volatility (Vol), and corresponding Sharpe ratio (Sharpe) obtained using analytic results (Analytic) and Monte Carlo (MC) simulations of the diffusion model for the time-based delta-hedging strategy using parameters from case I and case II, respectively. In Tables 4 and 6, we report the same quantities for the price- and delta-based strategies with bands defined using (3.5) and (3.8), respectively. In addition, in Tables 4 and 6, we report the expected number of rebalancing (Exp N) and its standard deviation (Std N) obtained from MC simulations. In Tables 7-10, 11-14 and 15-18, we report the same results for jump-diffusion model, stochastic volatility model and stochastic volatility model with jumps, respectively.

In Figures 5 and 6, we plot the realized P&L and its volatility obtained by analytical formula and MC simulations for case II. In Figure 7, we plot the corresponding Sharpe ratio. Finally, in Figure 8, we illustrate the skew of the P&L.

5.3 Discussion

First, we notice that for the time-based strategy, results obtained by our analytical approximation are very close to the MC estimates. We observe that, for the diffusion case, the optimal hedging frequency leads to expected transaction costs that are about half of the expected upside, in line with equation (4.6).

We note that the optimal frequency under jump-diffusion and stochastic volatility is much smaller than that in the diffusion case, in line with conclusion from equation (4.11). The expected P&L for the stochastic volatility is higher because of the auto-correlation as we have discussed in Section 3.3 and Appendix B. Importantly, while the expected P&L is about the same for the diffusion model and models with jump and stochastic volatility, the latter models imply much higher P&L volatility and, as a result, lower Sharpe ratios.

Importantly, we conclude that the optimal hedging frequency implied by our analysis is very close to MC results and it can be served as a tool to estimate the hedging frequency or required implied volatility level to reach a specified Sharpe ratio. We note that, when using the price- and delta-based strategies with bands defined using (3.5) and (3.8), respectively, we obtain Sharpe ratios close to those from the time-based strategy. Thus, our analysis can serve to estimate the expected P&L and its volatility as well as to imply the optimal hedging frequency for the price- and delta-based strategies.

In most of cases, both the price and delta-based strategies produce approximately equal Sharpe ratios and, in turn, are close to the time-based strategy using our proposed formulas (3.12) and (3.8). The optimal band that maximizes the Sharpe ratio for the delta-based strategy can be obtained using our formulation (3.12) along with (3.8). We see that indeed the optimal choice of hedging frequency and price- and delta-bands obtained from our analysis produces the highest Sharpe ratio possible.

An additional insight can be obtained from the analysis of the skew of the realized P&L. In Figure 6, we plot the realized skew of the delta-hedging P&L corresponding to the case II. We see that the realized skew is very sensitive to the price dynamics. For the diffusion model, the large spread between implied and realized volatility leads to large positive skew unless the hedging is too frequent (more than 4 times a day). Under the jump-diffusion, the realized skew declines significantly especially for time-based hedging, because of jumps. Under the stochastic volatility, the skew is negative, because the P&L is short the realized variance, which has a higher skew under the stochastic volatility (more extreme values of the realized variance are possible compared to the diffusion model). Under the stochastic volatility with jumps, the skew is almost flat when the hedging is not frequent enough (less than 4 times a day) so transaction costs are small. Because the volatility of the Brownian part is less than that in pure stochastic volatility model (as part of volatility is due to jumps), the larger spread between implied and realized volatility compensates for jumps and negative skew from stochastic volatility. We note that a jump leads to a large realized loss in a short option position only if the option is near at-the-money.

Noticeably, optimal hedging frequency also leads to almost optimal skew (large if it is positive or small if it is negative). The price-based hedging results in the highest skew. The delta-based hedging may result in too frequent rebalancing if the option remains near at-the-money with its delta changing too frequently compared to the price-based re-hedging. Overall, we see that the realized skew is too sensitive to the price dynamics and assumed hedging strategy, unlike the expected P&L and its volatility and, as a result, can be too noisy to be applied in determination of the optimal hedging frequency.

6 Conclusions

We have described a quantitative approach to improve the performance of the delta-hedging for volatility trading strategies. The key to our approach is to consider possible dynamics for price returns under the statistical measure and derive analytical formulas for the expected P&L of the delta-hedging strategy, transaction costs, and the P&L volatility. We have proposed an analytic method to maximize the Sharpe ratio of the hedging strategy and to find an optimal hedging frequency. We have shown how to apply our results to price- and delta-based hedging strategies.

For illustrations, we have shown that our method provides a very good approximation to the actual Sharpe ratio obtained by Monte Carlo simulations under the time-based re-hedging. Also, our approximations to convert the time based rule into the price and delta bands provide a reliable estimate for the Sharpe ratio. We remind that, under the strategy with the price and delta bands, the actual number of re-hedging times is different across different realized price paths. While this analysis can also be performed using Monte Carlo simulations, our analytic approach provide a fast and an accurate way to estimate the risk-reward characteristic of a delta-hedging strategy and our method can be implemented in a system for real time computations.

Our current framework is best suited to analyzing at-the-money options for which the impact of the skew is limited. A more general analysis that takes into account the skew and the dynamics of the implied volatility is left for future work.

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7 Appendix A. Mean and Variance of the P&L

We compute the expected value and the variance of the P&L function defined by (2.3). To start with, let us emphasize the difficulty of computing the first two moments of the P&L as defined by equation (2.3).
First, Σ_n and Γ both depend on return $(S^*_n - S^*_{n-1})$.
Second, Σ_n is auto-correlated.
Third, Γ depends on the path of S^*.

The problem therefore involves computing of expectation of a non-linear function over N correlated random variables. Analytically, this can be handled only by an approximation of the function. If option cash gamma were deterministic (as for variance swaps), our results would be exact. In general, it is impossible to find a norm in which the convergence takes place as the tools of continuous-time stochastic analysis are not applicable to handle this problem. We illustrate with Monte Carlo analysis that our approximations work well for the four considered models are reasonable for application.

While the P&L is a path-dependent function as option gamma and the realized variance depend on the same path as the underlying price, the dependence is mild for vanilla options. In our analysis we assume independence between the two and apply the following formula for variance of the product of two independent random variables X and Y:

$$\begin{aligned}\mathbb{V}[XY] &= (\mathbb{E}[X])^2\mathbb{V}[Y] + (\mathbb{E}[Y])^2\mathbb{V}[X] + \mathbb{V}[X]\mathbb{V}[Y] \\ &= \mathbb{E}[Y^2]\mathbb{V}[X] + (\mathbb{E}[X])^2\mathbb{V}[Y].\end{aligned} \tag{7.1}$$

For our developments, we define the quadratic moment generating function of a normal random variable with mean μ and variance ς as follows:

$$\begin{aligned}Z(q_2, q_1; \mu, \varsigma) &\equiv \frac{1}{\sqrt{2\pi\varsigma}} \int_{-\infty}^{\infty} \exp\left\{-q_2 x^2 - q_1 x - \frac{(x-\mu)^2}{2\varsigma}\right\} dx \\ &= \frac{1}{\sqrt{2B+1}} \exp\left\{\frac{\frac{1}{2}C^2}{2B+1} - A\right\},\end{aligned} \tag{7.2}$$

where $B = q_2\varsigma$, $C = (2\mu q_2 + q_1)\sqrt{\varsigma}$, $A = (q_2\mu^2 + q_1\mu)$, $q_2 > 0$.

7.1 Log-normal model

First, we consider the log-normal model with dynamics of (2.2):

$$\begin{aligned}\overline{U}_{df} \equiv \mathbb{E}^{\mathbb{P}}[P(N)] &\approx \sum_{n=1}^{N} \mathbb{E}^{\mathbb{P}}\left[\left(\sigma_i^2\delta t - \Sigma_n^2\right)\right] \mathbb{E}^{\mathbb{P}}\left[\Gamma(t_{n-1}, T, S^*; \sigma_i)\right] \\ &\approx \left(\sum_{n=1}^{N}(\sigma_i^2\delta t - \sigma_r^2\delta t)\right) \mathbb{E}^{\mathbb{P}}\left[\Gamma(T/2, T, S^*; \sigma_i)\right] \\ &= (\sigma_i^2 - \sigma_r^2)T\overline{\Gamma}_{df},\end{aligned} \tag{7.3}$$

where we apply the mid-point rule to approximate the sum and

$$\overline{\Gamma}_{df} = \Gamma\left(0, T, S; \sqrt{(\sigma_r^2 + \sigma_i^2)/2}\right). \tag{7.4}$$

To compute the P&L variance, we use (7.1) to obtain:

$$
\begin{aligned}
\overline{V}_{df} \equiv \mathbb{V}^{\mathbb{P}}\left[P(N)\right] &\approx \mathbb{V}^{\mathbb{P}}\left[\left(\sum_{n=1}^{N}\left(\sigma_i^2\delta t - \Sigma_n^2\right)\right)\Gamma(T/2,T,S^*;\sigma_i)\right] \\
&\approx \mathbb{E}^{\mathbb{P}}\left[\Gamma^2(T/2,T,S^*;\sigma_i)\right]\mathbb{V}^{\mathbb{P}}\left[\sum_{n=1}^{N}\left(\sigma_i^2\delta t - \Sigma_n^2\right)\right] \\
&+ \left(\mathbb{E}^{\mathbb{P}}\left[\sum_{n=1}^{N}\left(\sigma_i^2\delta t - \Sigma_n^2\right)\right]\right)^2 \mathbb{V}^{\mathbb{P}}\left[\Gamma(T/2,T,S^*;\sigma_i)\right].
\end{aligned}
\tag{7.5}
$$

Using formula (7.2), we get:

$$
\begin{aligned}
\overline{\Gamma^2_{df}} &\equiv \mathbb{E}^{\mathbb{P}}\left[\Gamma^2(T/2,T,S^*;\sigma_i)\right] = \Gamma^2(0,T/2,S;\sigma_i)\mathbb{E}^{\mathbb{P}}\left[e^{-q_2X^2(T/2)-q_1X(T/2)}\right] \\
&= \Gamma^2(0,T/2,S;\sigma_i)Z\left(q_2,q_1;-\frac{T\sigma_r^2}{4},\frac{T\sigma_r^2}{2}\right),
\end{aligned}
\tag{7.6}
$$

where $X(T) = \ln(S^*(T)/S^*(0))$ with $S^*(t)$ driven by dynamics (4.1) and

$$
q_2 = 1/(T\sigma_i^2/2)\ ,\ q_1 = 2(\ln(S/K) - T\sigma_i^2/4)/(T\sigma_i^2/2)
$$

After simplification and omitting the exponent:

$$
\overline{\Gamma^2_{df}} \approx \frac{\Gamma^2(0,T/2,S;\sigma_i)}{\sqrt{2\frac{\sigma_r^2}{\sigma_i^2}+1}}. \tag{7.7}
$$

The variance of the realized variance in the log-normal model, taking log-returns, is given by:

$$
\mathbb{V}^{\mathbb{P}}\left[\sum_{n=1}^{N}\Sigma_n^2\right] \approx \mathbb{V}^{\mathbb{P}}\left[\sum_{n=1}^{N}\left(\ln\frac{S^*(t_n)}{S^*(t_{n-1})}\right)^2\right] = \frac{2\sigma_r^4T^2}{N}.
$$

As a result, the P&L variance is given by:

$$
\overline{V}_{df} = \overline{\Gamma^2}\frac{\pi\sqrt{3}}{4}\frac{2\sigma_r^4T^2}{N} + \left(\sigma_i^2-\sigma_r^2\right)^2T^2\left(\overline{\Gamma^2} - (\overline{\Gamma}_{df})^2\right), \tag{7.8}
$$

where multiplier $\pi\sqrt{3}/4$ arises from normalizing the exact value of integral:

$$
\int_0^T \frac{T^2}{\sqrt{T^2-t^2}}dt = \pi/2
$$

by the value obtained by the mid-point approximation $2/\sqrt{3}$. In this way, equation (7.8) coincides with equation (1) in Derman (1999) if $\sigma_i = \sigma_r$.

7.2 Jump-diffusion model

Similarly to (7.6), under the jump-diffusion model (4.8) we obtain:

$$\overline{U}_{jd} \equiv \mathbb{E}^{\mathbb{P}}\left[P(N)\right] \approx (\sigma_i^2 - \vartheta_{jd}) T \overline{\Gamma}_{jd}, \tag{7.9}$$

where ϑ_{jd} is defined by (4.9) and

$$\overline{\Gamma}_{jd} \equiv \mathbb{E}^{\mathbb{P}}\left[\Gamma(T/2, T, S^*; \sigma_i)\right] = \Gamma(T/2, T, S^*; \sigma_i)\overline{Q}_{jd}(T/2; q_2/2, q_1/2), \tag{7.10}$$

and q_2, q_1 are defined as in (7.6). Here $\overline{Q}_{jd}(T; q_2, q_1)$ is computed by conditioning on the number of jumps as follows:

$$\begin{aligned} \overline{Q}_{jd}(T; q_2, q_1) &\equiv \mathbb{E}^{\mathbb{P}}\left[e^{-q_2 X^2(T) - q_1 X(T)}\right] \\ &= \sum_{m=0}^{\infty} \frac{e^{-\lambda T}(\lambda T)^m}{m!} Z\left(q_2, q_1; -\sigma_r^2 T/2 + m\nu, \sigma_r^2 T\right). \end{aligned} \tag{7.11}$$

where $X(T) = \ln(S^*(T)/S^*(0))$ with $S^*(t)$ driven by (4.8).

Using (7.5) for the P&L variance we get:

$$\begin{aligned} \overline{V}_{jd} &\approx \mathbb{V}^{\mathbb{P}}\left[\left(\sum_{n=1}^{N}\left(\sigma_i^2 \delta t - \Sigma_n^2\right)\right)\Gamma(T/2, T, S^*; \sigma_i)\right] \\ &\approx \overline{\Gamma^2}_{jd}\mathbb{V}^{\mathbb{P}}\left[\sum_{n=1}^{N}\Sigma_n^2\right] + \left((\sigma_i^2 - \vartheta_{jd})T\right)^2\left(\overline{\Gamma^2}_{jd} - (\overline{\Gamma}_{jd})^2\right), \end{aligned} \tag{7.12}$$

where

$$\overline{\Gamma^2}_{jd} = \Gamma^2(0, T/2, S; \sigma_i)\overline{Q}_{jd}(T/2; q_2, q_1) \tag{7.13}$$

and the variance is computed using equation (57) in Sepp (2012b):

$$\mathbb{V}^{\mathbb{P}}\left[\sum_{n=1}^{N}\Sigma_n^2\right] \approx \frac{(2\sigma_r^4 + \lambda\sigma_r^2\nu^2)T^2}{N} + \lambda T \nu^4. \tag{7.14}$$

7.3 Stochastic volatility model

Similarly to (7.6), under the stochastic volatility dynamics (4.12) we get the approximate expected P&L using equation (4.14) where

$$\overline{\Gamma}_{sv} \equiv \mathbb{E}^{\mathbb{P}}\left[\Gamma(T/2, T, S^*; \sigma_i)\right] = \Gamma(T/2, T, S^*; \sigma_i)\overline{Q}_{sv}(T/2; q_2/2, q_1/2) \tag{7.15}$$

and $\overline{Q}_{sv}(T; q_2, q_1)$ is computed as follows:

$$\begin{aligned} &\overline{Q}_{sv}(T; q_2, q_1) \equiv \mathbb{E}^{\mathbb{P}}\left[e^{-q_2 X^2(T) - q_1 X(T)}\right] \\ &= \frac{1}{\sqrt{\pi q_2}}\int_0^{\infty} \exp\left\{\frac{(ik - 1/2 - q_1)^2}{4q_2} + \alpha^{(\mathrm{SV})}(T, k) + (k^2 + 1/4)\beta^{(\mathrm{SV})}(T, k)V\right\} dk, \end{aligned} \tag{7.16}$$

where $X(T) = \ln(S^*(T)/S^*(0))$ with $S^*(t)$ driven by (4.12) and $\alpha^{(\mathrm{SV})}(T,k)$ and $\beta^{(\mathrm{SV})}(T,k)$ are defined by equation (7) in Lipton (2001):

$$\alpha^{(\mathrm{SV})}(T,k) = -\frac{\kappa\theta}{\varepsilon^2}\left[\psi_+ T + 2\ln\left(\frac{\psi_- + \psi_+ e^{-\zeta T}}{2\zeta}\right)\right],\ \beta^{(\mathrm{SV})}(T,k) = -\frac{1-e^{-\zeta T}}{\psi_- + \psi_+ e^{-\zeta T}}$$
$$\psi_\pm = \mp(ik\rho\varepsilon + \hat{\kappa}) + \zeta,\ \zeta = \sqrt{k^2\varepsilon^2(1-\rho^2) + 2ik\varepsilon\rho\hat{\kappa} + \hat{\kappa}^2 + \varepsilon^2/4},\ \hat{\kappa} = \kappa - \rho\varepsilon/2.$$

Under the stochastic volatility with jumps we use $\alpha^{(\mathrm{SVJ})}(T,k)$ augmented as follows:

$$\alpha^{(\mathrm{SVJ})}(T,k) = \alpha^{(\mathrm{SV})}(T,k) + \lambda T\left(e^{(-ik+1/2)\nu} - 1\right). \tag{7.17}$$

Using (7.5), for the P&L variance we get:

$$\begin{aligned}\mathbb{V}_{sv} &\approx \mathbb{V}^{\mathbb{P}}\left[\left(\sum_{n=1}^{N}\left(\sigma_i^2\delta t - \Sigma_n^2\right)\right)\Gamma(T/2,T,S^*;\sigma_i)\right]\\ &\approx \overline{\Gamma^2}_{sv}\mathbb{V}^{\mathbb{P}}\left[\sum_{n=1}^{N}\Sigma_n^2\right] + \left((\sigma_i^2 - \vartheta_{sv})T\right)^2\left(\overline{\Gamma^2}_{sv} - (\overline{\Gamma}_{sv})^2\right),\end{aligned} \tag{7.18}$$

where

$$\overline{\Gamma^2}_{sv} = \Gamma^2(0,T/2,S;\sigma_i)\overline{Q}_{sv}(T/2;q_2,q_1) \tag{7.19}$$

and (see equation (9) in Sepp (2012b)):

$$\mathbb{V}^{\mathbb{P}}\left[\sum_{n=1}^{N}\Sigma_n^2\right] \approx \frac{2V^2T^2}{N} + V_{sv} \tag{7.20}$$

$$V_{sv} = \frac{\varepsilon^2}{2\kappa^3}\left((\theta - 2V)e^{-2\kappa T} + 4(\theta\kappa T - V\kappa T + \theta)e^{-\kappa T} + (-5\theta + 2\theta\kappa T + 2V)\right). \tag{7.21}$$

8 Appendix B. Auto-covariance

As we have discussed in Sections 4.3, under the stochastic volatility dynamics, it is important to account for the autocorrelation of the variance dynamics. Here we derive an approximation assuming $\rho = 0$ and leaving non-zero case for future developments. For negative correlation, the correction to the expected P&L turns out to be higher than our estimate but not significantly.

We consider a simplified discrete version of the dynamics of (4.12):

$$S(t_n) = S(t_{n-1})e^{\sqrt{V_{n-1}\delta t}\epsilon_n}, \tag{8.1}$$

where $V_n = V(t_n)$. Then the P&L function (2.3) becomes:

$$\begin{aligned} P(N) &\approx \sum_{n=1}^{N} \left(\sigma_i^2 \delta t - V_n \delta t \epsilon_n^2\right) \Gamma(t_{n-1}, T, S_0 e^{\sum_{m=1}^{n-1} \sqrt{V_{m-1}\delta t}\epsilon_m}; \sigma_i) \\ &= \sum_{n=1}^{N} \left(\sigma_i^2 \delta t - \overline{V}_n \delta t \epsilon_n^2\right) \Gamma(t_{n-1}, T, S_0 e^{\sum_{m=1}^{n-1} \sqrt{V_{m-1}\delta t}\epsilon_m}; \sigma_i) \\ &- \sum_{n=1}^{N} \left(V_n \delta t \epsilon_n^2 - \overline{V}_n \delta t \epsilon_n^2\right) \Gamma(t_{n-1}, T, S_0 e^{\sum_{m=1}^{n-1} \sqrt{V_{m-1}\delta t}\epsilon_m}; \sigma_i), \end{aligned} \tag{8.2}$$

where $\overline{V}_n = \mathbb{E}[V(t_n)]$. The expectation of the first part is approximated using equation (4.14). For the second part, we consider:

$$\begin{aligned} L(N) &\equiv \sum_{n=1}^{N} \left(V_n - \overline{V}_n\right) \delta t \epsilon_n^2 \Gamma(t_{n-1}, T, S_0 e^{\sum_{m=1}^{n-1} \sqrt{V_{m-1}\delta t}\epsilon_m}; \sigma_i) \\ &= \sum_{n=1}^{N} \left(V_n - \overline{V}_n\right) \delta t \epsilon_n^2 C_{n-1} \exp\left\{ -\frac{1}{2} \left(\frac{\sum_{m=1}^{n-1} \sqrt{V_{m-1}\delta t}\epsilon_m}{\sqrt{(T-t_{n-1})\sigma_i^2}} + y_{n-1} \right)^2 \right\} \\ &\approx \sum_{n=1}^{N} \left(V_n - \overline{V}_n\right) \delta t \epsilon_n^2 C_{n-1} \exp\left\{ -\frac{1}{2} \frac{\left(\sum_{m=1}^{n-1} \sqrt{V_{m-1}\delta t}\epsilon_m\right)^2}{(T-t_{n-1})\sigma_i^2} - \widetilde{y}_{n-1} \frac{\sum_{m=1}^{n-1} \sqrt{V_{m-1}\delta t}\epsilon_m}{(T-t_{n-1})\sigma_i^2} - \frac{1}{2} y_{n-1}^2 \right\}, \end{aligned}$$

where $C_{n-1} = \frac{K}{2\sqrt{2(T-t_{n-1})\pi\sigma_i^2}}$, $y_{n-1} = \frac{\ln \frac{S_0}{K} - \frac{1}{2}(T-t_{n-1})\sigma_i^2}{\sqrt{(T-t_{n-1})\sigma_i^2}}$, $\widetilde{y}_{n-1} = \frac{-\frac{1}{2}(T-t_{n-1})\sigma_i^2}{\sqrt{(T-t_{n-1})\sigma_i^2}} = -\frac{1}{2}\sqrt{(T-t_{n-1})\sigma_i^2}$.

First computing the first-order expectation with respect to ϵ_n first and taking

$$I(t_{n-1}) \equiv \sum_{m=1}^{n-1} V_{m-1}\delta t \approx \int_0^{t_{m-1}} V(s) ds$$

we obtain:

$$\begin{aligned} \overline{L}_{sv} \equiv \mathbb{E}^{\mathbb{P}}[L(N)] &\approx \sum_{n=1}^{N} \mathbb{E}^{\mathbb{P}} \left[\left(V_n - \overline{V}_n\right) \delta t C_{n-1} e^{-\frac{1}{2} y_{n-1}^2} e^{-\Psi_{n-1} I(t_{n-1})} \right] \\ &\approx \sum_{n=1}^{N} \delta t C_{n-1} e^{-\frac{1}{2} y_{n-1}^2} \widehat{G}(t_{n-1}, V, \Psi_{n-1}), \end{aligned} \tag{8.3}$$

where $\Psi_{n-1} = (1/2)/((T-t_{n-1})\sigma_i^2)$ and $\widehat{G}(T, V, \Psi)$ with $\overline{V}(T) = \mathbb{E}[V(T)]$ is computed using equation (40) in Sepp (2011b) as follows:

$$\begin{aligned} \widehat{G}(T, V, \Psi) &\equiv \mathbb{E}\left[\left(V(T) - \overline{V}(T)\right) e^{-\Psi I(T)} \right] \\ &= -\frac{\partial}{\partial \Theta} \widehat{G}^{(VI)}(\Theta, \Psi; 0, T, V) \mid_{\Theta=0} - \overline{V}(T) \widehat{G}^{(VI)}(\Theta, \Psi; 0, T, V) \mid_{\Theta=0} \\ &= -\left(A_\Theta(T, \Psi) + B_\Theta(T, \Psi) V\right) e^{A(T,\Psi) + B(T,\Psi) V}, \end{aligned} \tag{8.4}$$

$$
\begin{aligned}
&A(T,\Psi) = -\frac{\kappa\theta}{\varepsilon^2}\left[\psi_+ T + 2\ln\left(\frac{\psi_- + \psi_+ e^{-\zeta T}}{2\zeta}\right)\right],\\
&B(T,\Psi) = -2\Psi\frac{1-e^{-\zeta T}}{\psi_- + \psi_+ e^{-\zeta T}},\\
&A_\Theta(T,\Psi) = -2\kappa\theta\frac{1-e^{-\zeta T}}{\psi_- + \psi_+ e^{-\zeta T}} + (1-e^{-\kappa T})\theta,\\
&B_\Theta(T,\Psi) = -\frac{4\zeta e^{-\zeta T}}{\left(\psi_- + \psi_+ e^{-\zeta T}\right)^2} + e^{-\kappa T},\\
&\psi_\pm = \mp\kappa + \zeta,\ \ \zeta = \sqrt{\kappa^2 + 2\varepsilon^2\Psi}.
\end{aligned}
$$

Model Parameters

Table 1: Model Parameters

	Case I				Case II			
	DF	JD	SV	SVJ	DF	JD	SV	SVJ
σ_i	16.50%	16.50%	16.50%	16.50%	30.00%	30.00%	30.00%	30.00%
k	0.001	0.001	0.001	0.001	0.004	0.004	0.004	0.004
σ_r	15%	14.14%			25%	22.91%		
λ		1.00		1.00		1.00		1.00
ν		-5%		-5%		-10%		-10%
V			$15\%^2$	$14.14\%^2$			$25\%^2$	$22.91\%^2$
θ			$15\%^2$	$14.14\%^2$			$25\%^2$	$22.91\%^2$
κ			4.00	4.00			4.00	4.00
ε			0.25	0.25			0.50	0.50
ρ			-0.50	-0.50			-0.50	-0.50

Notations: DF is the diffusion model, JD is the jump-diffusion model, SV is the Heston stochastic volatility model, SVJ is the Heston stochastic volatility model with jumps, σ_i is the option implied BSM volatility, k is the proportional transaction costs, σ_r is the realized volatility.

Table 2: Sharpe ratio parameters

	Case I				Case II			
	DF	JD	SV	SVJ	DF	JD	SV	SVJ
u	0.5692%	0.5510%	0.6703%	0.6550%	1.8558%	1.7628%	2.0876%	1.9995%
c	0.0151%	0.0139%	0.0153%	0.0142%	0.0571%	0.0506%	0.0581%	0.0518%
p	0.2461%	0.2006%	0.2524%	0.2085%	0.5998%	0.4423%	0.6214%	0.4634%
f	0.0004%	0.0020%	0.0107%	0.0110%	0.0040%	0.0119%	0.0403%	0.0418%
N^*	239	138	74	68	129	75	49	43
$\overline{C}(N^*)$	0.2334%	0.1633%	0.1316%	0.1171%	0.6485%	0.4382%	0.4067%	0.3397%
$u - \overline{C}(N^*)$	0.3358%	0.3877%	0.5387%	0.5379%	1.2073%	1.3246%	1.6809%	1.6598%
$\sqrt{\overline{V}(N^*)}$	0.3781%	0.5877%	1.1879%	1.1860%	0.9300%	1.3341%	2.3018%	2.2930%
$\overline{S}(N^*)$	0.89	0.66	0.45	0.45	1.30	0.99	0.73	0.72

Notations: u, c, p, f are parameters for the Sharpe ration in equation (3.12), N^* is the optimal Sharpe ratio, $\overline{C}(N^*)$ is the transaction costs at the optimal hedging frequency, $u - \overline{C}(N^*)$ is the expected P&L, $\sqrt{\overline{V}(N^*)}$ is the volatility if the P&L, $\overline{S}(N^*)$ is the Sharpe ratio at optimal hedging frequency.

Diffusion model

Table 3: Diffusion model, case I, time-based hedging

Time-based		DF	Analytic	Time-	based	DF	MC	Time-	based
Freq	N	P &L	Vol	Costs	Sharpe	P &L	Vol	Costs	Sharpe
Optimal	239	2.79	3.18	1.94	0.88	2.80	3.07	1.98	0.91
1 per m	15	4.24	10.77	0.49	0.39	4.31	10.47	0.51	0.41
2 per m	30	4.04	7.72	0.69	0.52	4.04	8.14	0.72	0.50
1 per w	60	3.75	5.59	0.97	0.67	3.73	5.78	1.01	0.65
2 per w	120	3.35	4.14	1.37	0.81	3.32	4.19	1.36	0.79
1 per d	240	2.78	3.18	1.94	0.88	2.62	3.02	1.95	0.87
2 per d	480	1.98	2.56	2.74	0.77	1.99	2.19	2.74	0.91
4 per d	960	0.84	2.19	3.88	0.39	0.84	1.51	3.89	0.55
8 per d	1920	-0.76	1.98	5.49	-0.39	-0.75	1.05	5.47	-0.71

Table 4: Diffusion model, case I, price- and delta-based hedging

	Range-	based	DF	MC	price-	based	DF	MC	Delta-	based	price-	based	Delta-	based
N	Price	Delta	P &L	Vol	Costs	Sharpe	P &L	Vol	Costs	Sharpe	Exp N	Std N	Exp N	Std N
239	0.0097	0.0245	2.51	2.23	2.23	1.13	2.58	2.31	2.19	1.12	201	11	136	78
15	0.0387	0.0977	4.41	7.59	0.63	0.58	4.34	8.10	0.65	0.54	15	3	11	7
30	0.0274	0.0691	3.73	5.21	0.85	0.72	3.78	6.04	0.87	0.63	29	4	21	13
60	0.0194	0.0488	3.54	4.00	1.18	0.88	3.71	4.38	1.20	0.85	56	6	40	25
120	0.0137	0.0345	3.07	2.97	1.63	1.03	3.46	4.25	1.20	0.82	107	8	40	25
240	0.0097	0.0244	2.49	2.27	2.22	1.10	2.64	2.36	2.24	1.12	202	12	139	78
480	0.0068	0.0173	1.71	1.65	3.03	1.04	1.76	1.74	2.95	1.01	375	16	248	136
960	0.0048	0.0122	0.69	1.18	4.06	0.59	0.79	1.36	3.89	0.58	678	21	443	225
1920	0.0034	0.0086	-0.62	0.88	5.39	-0.70	-0.28	1.49	5.02	-0.19	1184	27	763	366

Table 5: Diffusion model, case II, time-based hedging

Time-based		DF	Analytic	Time-	based	DF	MC	Time-	based
Freq	N	P &L	Vol	Costs	Sharpe	P &L	Vol	Costs	Sharpe
Optimal	129	18.36	14.12	9.86	1.30	18.28	13.98	10.03	1.31
1 per m	15	24.86	31.89	3.36	0.78	25.19	34.76	3.61	0.72
2 per m	30	23.47	23.54	4.76	1.00	24.62	24.97	5.02	0.99
1 per w	60	21.50	17.97	6.73	1.20	22.04	18.78	6.93	1.17
2 per w	120	18.71	14.40	9.51	1.30	18.95	14.10	9.65	1.34
1 per d	240	14.77	12.23	13.45	1.21	14.48	10.51	13.56	1.38
2 per d	480	9.20	10.98	19.02	0.84	9.40	7.55	19.15	1.25
4 per d	960	1.32	10.30	26.90	0.13	1.24	4.82	26.99	0.26
8 per d	1920	-9.82	9.95	38.05	-0.99	-9.81	4.40	38.02	-2.23

Table 6: Diffusion model, case II, price- and delta-based hedging

	Range-	based	DF	MC	price-	based	DF	MC	Delta-	based	price-	based	Delta-	based
N	Price	Delta	P &L	Vol	Costs	Sharpe	P &L	Vol	Costs	Sharpe	Exp N	Std N	Exp N	Std N
129	0.0220	0.0448	15.95	10.49	11.57	1.52	16.73	10.12	11.76	1.65	114	9	79	46
15	0.0645	0.1313	24.01	24.77	4.39	0.97	24.13	25.60	4.48	0.94	15	3	12	7
30	0.0456	0.0928	22.09	19.29	5.96	1.14	22.21	20.75	6.06	1.07	29	4	21	13
60	0.0323	0.0656	20.26	14.15	8.29	1.43	19.80	14.23	8.35	1.39	56	6	40	24
120	0.0228	0.0464	17.36	11.49	11.35	1.51	17.00	10.33	11.19	1.65	107	8	73	43
240	0.0161	0.0328	12.97	8.38	15.58	1.55	13.04	7.84	14.82	1.66	202	11	134	76
480	0.0114	0.0232	7.18	5.87	21.19	1.22	7.96	6.36	20.81	1.25	375	16	253	133
960	0.0081	0.0164	0.10	3.84	28.08	0.02	1.41	7.05	27.03	0.20	678	21	445	217
1920	0.0057	0.0116	-8.63	3.91	37.08	-2.21	-5.97	9.23	34.18	-0.65	1184	28	753	349

Notations: N is the hedging frequency, P&L is the expected/realized P&L of the delta-hedging strategy, Vol is the volatility of the P&L, Costs is the transaction costs, Sharpe is the Sharpe ratio, Exp N and Std N is the average and the standard deviation, respectively, of the hedging frequency under the price- and delta-based re-balancing.

Jump-diffusion model

Table 7: Jump-diffusion model, case I, time-based hedging

Time-based		JD,	Analytic,	Time-	based	JD,	MC,	Time-	based
Optimal	138	3.22	4.84	1.41	0.66	3.20	4.89	1.41	0.66
1 per m	15	4.13	10.28	0.50	0.40	4.10	11.01	0.50	0.37
2 per m	30	3.94	7.72	0.68	0.51	3.93	8.45	0.68	0.46
1 per w	60	3.68	6.04	0.95	0.61	3.60	6.45	0.94	0.56
2 per w	120	3.31	5.00	1.32	0.66	3.30	5.21	1.31	0.63
1 per d	240	2.78	4.38	1.84	0.63	2.78	4.34	1.83	0.64
2 per d	480	2.04	4.04	2.58	0.51	2.14	3.81	2.59	0.56
4 per d	960	0.99	3.86	3.63	0.26	1.04	3.51	3.57	0.29
8 per d	1920	-0.49	3.77	5.11	-0.13	-0.51	3.27	5.11	-0.16

Table 8: Jump-diffusion model, case I, price- and delta-based hedging

	Range-	based	JD,	MC,	price-	based	JD,	MC,	Delta-	based	price-	based	Delta-	based
N	Price	Delta	P &L	Vol	Costs	Sharpe	P &L	Vol	Costs	Sharpe	Exp N	Std N	Exp N	Std N
138	0.0128	0.0456	3.31	4.17	1.57	0.79	3.21	4.26	1.56	0.75	112	8	76	45
15	0.0387	0.1384	4.10	8.02	0.58	0.51	3.98	8.79	0.60	0.45	14	3	10	7
30	0.0274	0.0978	4.03	6.16	0.77	0.65	3.81	6.63	0.77	0.57	27	4	18	12
60	0.0194	0.0692	3.49	5.12	1.07	0.68	3.56	5.54	1.08	0.64	51	6	35	22
120	0.0137	0.0489	3.19	4.32	1.44	0.74	3.21	4.29	1.47	0.75	96	8	66	40
240	0.0097	0.0346	2.70	3.81	1.95	0.71	2.79	3.89	2.03	0.72	181	11	125	74
480	0.0068	0.0245	2.00	3.57	2.73	0.56	2.08	3.45	2.62	0.60	343	15	221	124
960	0.0048	0.0173	1.28	3.25	3.64	0.39	1.33	3.29	3.48	0.40	615	20	397	208
1920	0.0034	0.0122	-0.01	3.08	4.75	0.00	0.19	3.12	4.47	0.06	1092	26	685	344

Table 9: Jump-diffusion model, case II, time-based hedging

Time-based		JD,	Analytic,	Time-	based	JD,	MC,	Time-	based
Optimal	75	20.15	20.31	7.08	0.99	20.30	21.20	7.13	0.96
1 per m	15	23.83	30.95	3.40	0.77	23.74	35.20	3.43	0.67
2 per m	30	22.60	24.84	4.63	0.91	22.47	27.86	4.56	0.81
1 per w	60	20.86	21.13	6.38	0.99	20.66	22.15	6.29	0.93
2 per w	120	18.39	19.00	8.84	0.97	18.43	19.45	8.81	0.95
1 per d	240	14.90	17.85	12.33	0.83	15.63	16.81	12.24	0.93
2 per d	480	9.96	17.24	17.27	0.58	10.79	14.85	17.31	0.73
4 per d	960	2.99	16.93	24.25	0.18	3.66	13.33	23.95	0.27
8 per d	1920	-6.88	16.77	34.11	-0.41	-5.51	11.88	34.31	-0.46

Table 10: Jump-diffusion model, case II, price- and delta-based hedging

	Range-	based	JD,	MC,	price-	based	JD,	MC,	Delta-	based	price-	based	Delta-	based
N	Price	Delta	P &L	Vol	Costs	Sharpe	P &L	Vol	Costs	Sharpe	Exp N	Std N	Exp N	Std N
75	0.0289	0.0587	20.58	19.08	7.96	1.08	19.90	18.54	7.70	1.07	61	6	41	25
15	0.0645	0.1313	22.30	28.58	3.86	0.78	23.27	28.54	3.88	0.82	14	3	10	6
30	0.0456	0.0928	22.41	23.94	5.24	0.94	21.93	23.44	5.09	0.94	25	4	18	11
60	0.0323	0.0656	20.88	20.21	7.07	1.03	20.35	19.64	7.02	1.04	48	5	33	20
120	0.0228	0.0464	18.75	17.68	9.51	1.06	18.51	16.93	9.62	1.09	91	8	63	37
240	0.0161	0.0328	14.75	16.07	13.00	0.92	14.85	14.97	12.70	0.99	173	10	115	66
480	0.0114	0.0232	9.73	14.62	17.49	0.67	11.03	13.91	16.72	0.79	323	14	205	115
960	0.0081	0.0164	4.42	13.19	23.77	0.34	5.51	13.50	22.06	0.41	586	19	370	193
1920	0.0057	0.0116	-3.09	12.88	31.33	-0.24	-1.04	13.17	29.60	-0.08	1034	26	661	319

Notations: N is the hedging frequency, P&L is the expected/realized P&L of the delta-hedging strategy, Vol is the volatility of the P&L, Costs is the transaction costs, Sharpe is the Sharpe ratio, Exp N and Std N is the average and the standard deviation, respectively, of the hedging frequency under the price- and delta-based re-balancing.

Stochastic volatility model

Table 11: Stochastic volatility model, case I, time-based hedging

Time-based		SV,	Analytic,	Time-	based	SV,	MC,	Time-	based
Freq	N	P &L	Vol	Costs	Sharpe	P &L	Vol	Costs	Sharpe
Optimal	74	4.47	9.84	1.09	0.45	4.50	9.99	1.05	0.45
1 per m	15	5.07	13.76	0.49	0.37	5.46	13.66	0.50	0.40
2 per m	30	4.87	11.46	0.70	0.42	4.94	11.68	0.69	0.42
1 per w	60	4.58	10.12	0.98	0.45	4.58	9.88	0.96	0.46
2 per w	120	4.17	9.37	1.39	0.45	4.44	9.25	1.32	0.48
1 per d	240	3.60	8.98	1.97	0.40	3.77	8.71	1.88	0.43
2 per d	480	2.78	8.78	2.78	0.32	2.78	8.55	2.66	0.33
4 per d	960	1.63	8.67	3.93	0.19	2.10	8.51	3.74	0.25
8 per d	1920	0.00	8.62	5.56	0.00	0.67	8.24	5.30	0.08

Table 12: Stochastic volatility model, case I, price- and delta-based hedging

	Range-	based	SV,	MC,	price-	based	SV,	MC,	Delta-	based	price-	based	Delta-	based
N	Price	Delta	P &L	Vol	Costs	Sharpe	P &L	Vol	Costs	Sharpe	Exp N	Std N	Exp N	Std N
74	0.0174	0.0623	4.40	8.99	1.25	0.49	4.33	8.92	1.26	0.49	67	22	46	30
15	0.0387	0.1384	5.21	10.85	0.61	0.48	5.10	11.26	0.62	0.45	15	6	11	7
30	0.0274	0.0978	4.85	9.64	0.84	0.50	4.74	10.22	0.87	0.46	29	10	21	14
60	0.0194	0.0692	4.57	8.82	1.15	0.52	4.45	9.22	1.18	0.48	55	18	39	26
120	0.0137	0.0489	4.38	9.01	1.58	0.49	4.22	8.86	1.63	0.48	104	33	73	47
240	0.0097	0.0346	3.46	8.92	2.21	0.39	3.65	8.90	2.12	0.41	199	61	131	81
480	0.0068	0.0245	2.86	8.62	2.96	0.33	3.05	8.58	2.90	0.35	368	108	243	140
960	0.0048	0.0173	1.58	9.13	4.00	0.17	1.91	8.84	3.71	0.22	672	183	421	231
1920	0.0034	0.0122	0.77	9.08	5.16	0.08	0.88	9.21	4.91	0.10	1144	290	738	380

Table 13: Stochastic volatility model, case II, time-based hedging

Time-based		SV,	Analytic,	Time-	based	SV,	MC,	Time-	based
Freq	N	P &L	Vol	Costs	Sharpe	P &L	Vol	Costs	Sharpe
Optimal	49	25.57	35.02	6.18	0.73	26.72	35.57	5.98	0.75
1 per m	15	28.34	43.49	3.42	0.65	29.07	46.22	3.46	0.63
2 per m	30	26.91	37.58	4.84	0.72	28.56	39.50	4.72	0.72
1 per w	60	24.91	34.25	6.84	0.73	26.78	34.80	6.57	0.77
2 per w	120	22.07	32.45	9.68	0.68	22.95	33.26	9.10	0.69
1 per d	240	18.06	31.51	13.69	0.57	19.50	32.83	13.05	0.59
2 per d	480	12.40	31.03	19.36	0.40	14.10	31.12	18.08	0.45
4 per d	960	4.38	30.79	27.38	0.14	6.91	32.52	25.87	0.21
8 per d	1920	-6.96	30.67	38.72	-0.23	-2.48	32.38	36.14	-0.08

Table 14: Stochastic volatility model, case II, price- and delta-based hedging

	Range-	based	SV,	MC,	price-	based	SV,	MC,	Delta-	based	price-	based	Delta-	based
N	Price	Delta	P &L	Vol	Costs	Sharpe	P &L	Vol	Costs	Sharpe	Exp N	Std N	Exp N	Std N
49	0.0357	0.0726	24.54	33.66	7.14	0.73	27.85	34.23	7.09	0.81	45	18	30	21
15	0.0645	0.1313	28.25	43.49	3.42	0.65	28.76	37.61	4.38	0.76	15	6	11	7
30	0.0456	0.0928	26.84	37.58	4.84	0.71	28.39	35.23	5.98	0.81	29	11	20	14
60	0.0323	0.0656	24.83	34.25	6.84	0.73	25.09	33.64	7.89	0.75	55	21	38	25
120	0.0228	0.0464	22.00	32.45	9.68	0.68	20.03	33.03	10.77	0.61	104	39	71	46
240	0.0161	0.0328	17.99	31.51	13.69	0.57	18.81	33.06	14.66	0.57	197	70	131	78
480	0.0114	0.0232	12.84	32.98	20.13	0.39	12.47	34.44	19.44	0.36	363	124	236	138
960	0.0081	0.0164	6.82	34.43	26.84	0.20	7.66	33.59	25.03	0.23	645	204	411	220
1920	0.0057	0.0116	-2.00	36.12	34.86	-0.06	1.62	35.73	31.84	0.05	1125	337	697	351

Notations: N is the hedging frequency, P&L is the expected/realized P&L of the delta-hedging strategy, Vol is the volatility of the P&L, Costs is the transaction costs, Sharpe is the Sharpe ratio, Exp N and Std N is the average and the standard deviation, respectively, of the hedging frequency under the price- and delta-based re-balancing.

Stochastic volatility model with jumps

Table 15: Stochastic volatility model with jumps, case I, time-based hedging

Time-based		SVJ,	Analytic,	Time-	based	SVJ,	MC,	Time-	based
Freq	N	P &L	Vol	Costs	Sharpe	P &L	Vol	Costs	Sharpe
Optimal	68	4.46	9.83	1.03	0.45	5.02	9.71	0.98	0.52
1 per m	15	4.98	13.09	0.51	0.38	5.24	14.10	0.49	0.37
2 per m	30	4.79	11.11	0.70	0.43	5.34	11.54	0.66	0.46
1 per w	60	4.52	9.97	0.97	0.45	4.67	9.89	0.91	0.47
2 per w	120	4.14	9.35	1.34	0.44	4.68	8.99	1.25	0.52
1 per d	240	3.61	9.03	1.88	0.40	4.23	8.90	1.75	0.47
2 per d	480	2.85	8.86	2.64	0.32	3.82	8.67	2.48	0.44
4 per d	960	1.78	8.77	3.71	0.20	2.64	8.43	3.51	0.31
8 per d	1920	0.27	8.73	5.22	0.03	1.20	8.57	4.82	0.14

Table 16: Stochastic volatility model with jumps, case I, price- and delta-based hedging

	Range-	based	SVJ,	MC,	price-	based	SVJ,	MC,	Delta-	based	price-	based	Delta-	based
N	Price	Delta	P &L	Vol	Costs	Sharpe	P &L	Vol	Costs	Sharpe	Exp N	Std N	Exp N	Std N
68	0.0182	0.0650	5.10	9.42	1.12	0.54	4.93	9.27	1.05	0.53	59	20	34	22
15	0.0387	0.1384	5.69	11.16	0.58	0.51	5.47	11.88	0.56	0.46	14	5	10	6
30	0.0274	0.0978	5.65	10.18	0.78	0.56	5.51	10.11	0.79	0.54	27	10	19	12
60	0.0194	0.0692	5.28	9.17	1.04	0.58	5.59	9.23	1.02	0.61	50	18	33	22
120	0.0137	0.0489	4.91	9.11	1.39	0.54	5.05	8.95	1.43	0.56	95	31	64	42
240	0.0097	0.0346	4.62	8.97	1.90	0.52	4.56	8.90	1.91	0.51	177	59	117	72
480	0.0068	0.0245	3.86	8.68	2.57	0.44	3.69	8.63	2.59	0.43	333	105	216	125
960	0.0048	0.0173	2.88	8.98	3.44	0.32	3.03	8.65	3.27	0.35	599	178	374	208
1920	0.0034	0.0122	1.35	8.72	4.57	0.15	1.80	8.81	4.28	0.20	1061	280	655	333

Table 17: Stochastic volatility model with jumps, case II, time-based hedging

Time-based		SVJ,	Analytic,	Time-	based	SVJ,	MC,	Time-	based
Freq	N	P &L	Vol	Costs	Sharpe	P &L	Vol	Costs	Sharpe
Optimal	43	25.25	34.89	5.59	0.72	28.32	38.08	5.27	0.74
1 per m	15	27.37	41.02	3.48	0.67	27.50	47.91	3.26	0.57
2 per m	30	26.10	36.40	4.74	0.72	28.24	38.77	4.35	0.73
1 per w	60	24.31	33.86	6.53	0.72	27.55	34.31	5.94	0.80
2 per w	120	21.79	32.51	9.05	0.67	25.28	33.23	8.23	0.76
1 per d	240	18.22	31.82	12.63	0.57	22.16	32.02	11.52	0.69
2 per d	480	13.16	31.47	17.68	0.42	17.48	31.31	15.95	0.56
4 per d	960	6.02	31.29	24.82	0.19	11.85	31.86	22.19	0.37
8 per d	1920	-4.08	31.20	34.93	-0.13	2.43	32.15	31.63	0.08

Table 18: Stochastic volatility model with jumps, case II, price- and delta-based hedging

	Range-	based	SVJ,	MC,	price-	based	SVJ,	MC,	Delta-	based	price-	based	Delta-	based
N	Price	Delta	P &L	Vol	Costs	Sharpe	P &L	Vol	Costs	Sharpe	Exp N	Std N	Exp N	Std N
43	0.0381	0.0775	28.37	36.12	5.87	0.79	28.15	35.20	5.92	0.80	37	16	25	17
15	0.0645	0.1313	31.42	40.00	3.78	0.79	31.51	39.92	3.77	0.79	13	6	10	6
30	0.0456	0.0928	29.89	36.99	5.11	0.81	28.66	36.49	4.99	0.79	26	11	17	11
60	0.0323	0.0656	28.39	35.11	6.69	0.81	28.37	32.61	6.52	0.87	47	19	31	20
120	0.0228	0.0464	24.75	34.16	9.10	0.72	25.13	32.98	9.09	0.76	91	37	59	38
240	0.0161	0.0328	21.65	32.90	12.20	0.66	21.58	32.87	12.25	0.66	170	67	109	70
480	0.0114	0.0232	18.28	33.82	16.13	0.54	18.87	32.48	15.37	0.58	312	123	190	112
960	0.0081	0.0164	12.33	34.37	22.04	0.36	13.52	32.71	21.31	0.41	563	205	352	197
1920	0.0057	0.0116	7.08	36.65	28.66	0.19	8.03	34.75	26.30	0.23	958	319	589	309

Notations: N is the hedging frequency, P&L is the expected/realized P&L of the delta-hedging strategy, Vol is the volatility of the P&L, Costs is the transaction costs, Sharpe is the Sharpe ratio, Exp N and Std N is the average and the standard deviation, respectively, of the hedging frequency under the price- and delta-based re-balancing.

Theoretical Sharpe ratios

Figure 1: The expected P&L and volatility (left scale) and corresponding Sharpe ratio (right scale) for the diffusion model using parameters from case II as functions of hedging frequency N

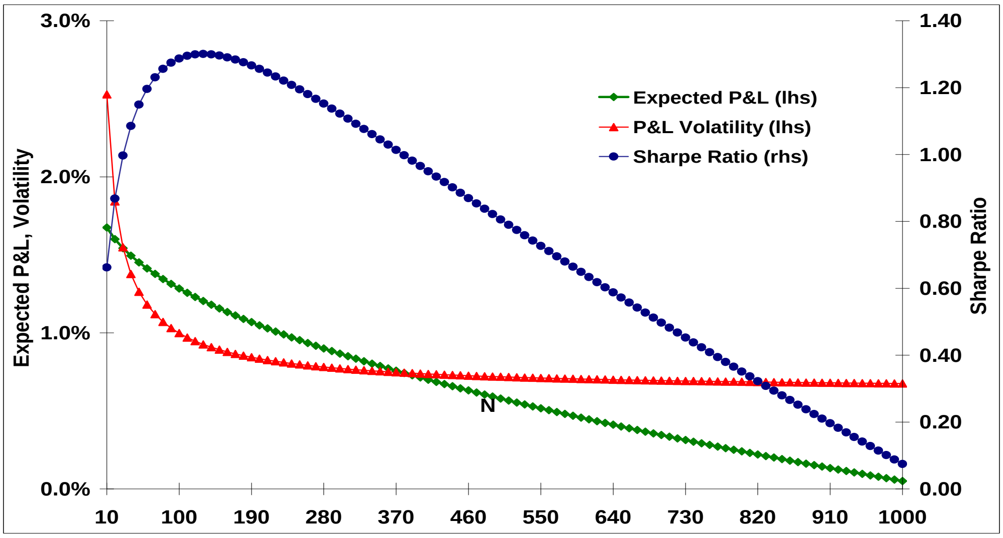

Figure 2: The Sharpe ratio for models using parameters from case II (right side) as functions of hedging frequency N

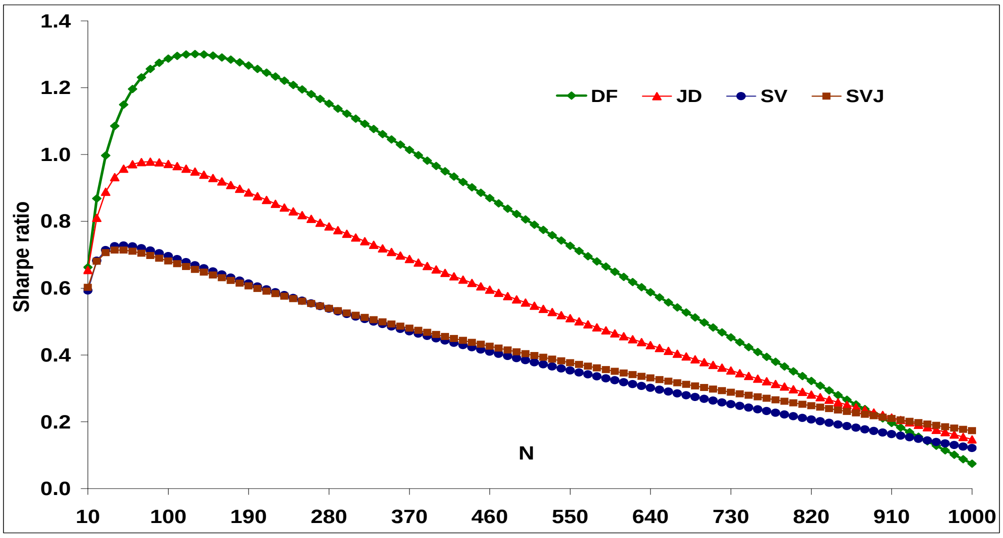

Figure 3: Term structure of Optimal Sharpe ratio, $S(N^*)$, for models using parameters from case II as functions of option maturity T

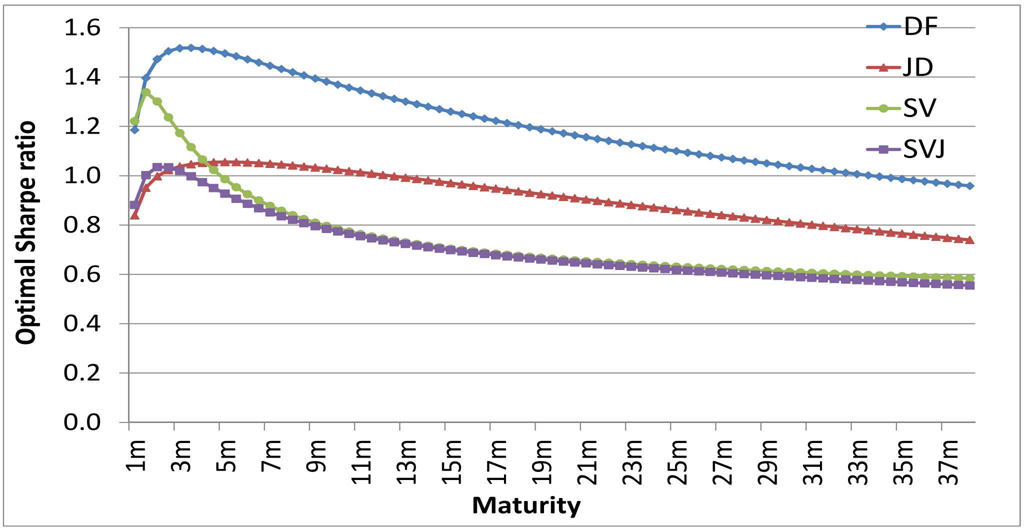

Figure 4: Term structure of optimal rehedging periods in days, $250T/N^*$, for models using parameters from case II as functions of option maturity T

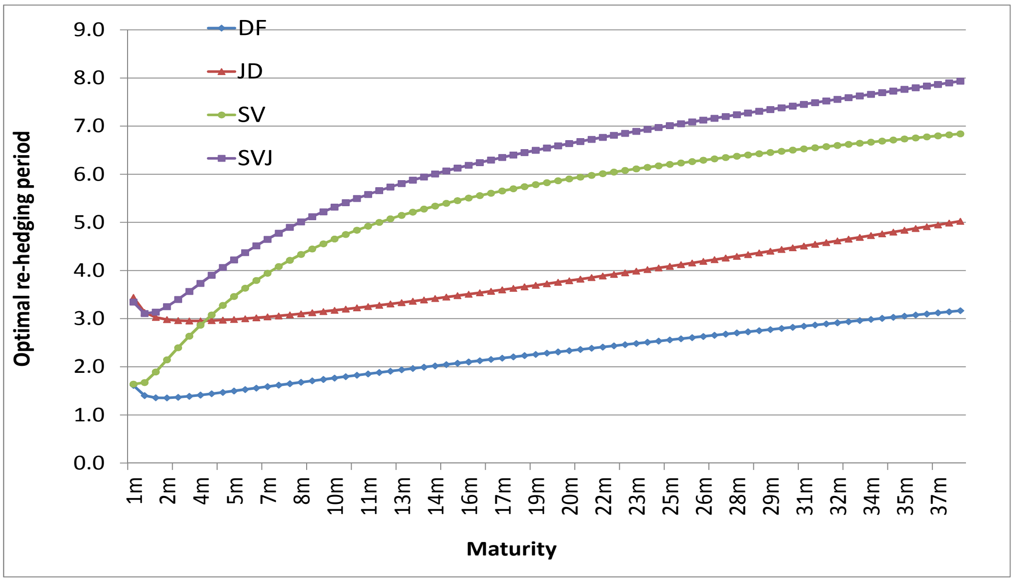

Figure 5: **Realized P&L.** The realized P&L of the delta-hedging using parameters from case II for the diffusion model (top left), jump-diffusion model (top right), stochastic volatility model (bottom left), the stochastic volatility model with jumps (bottom right)

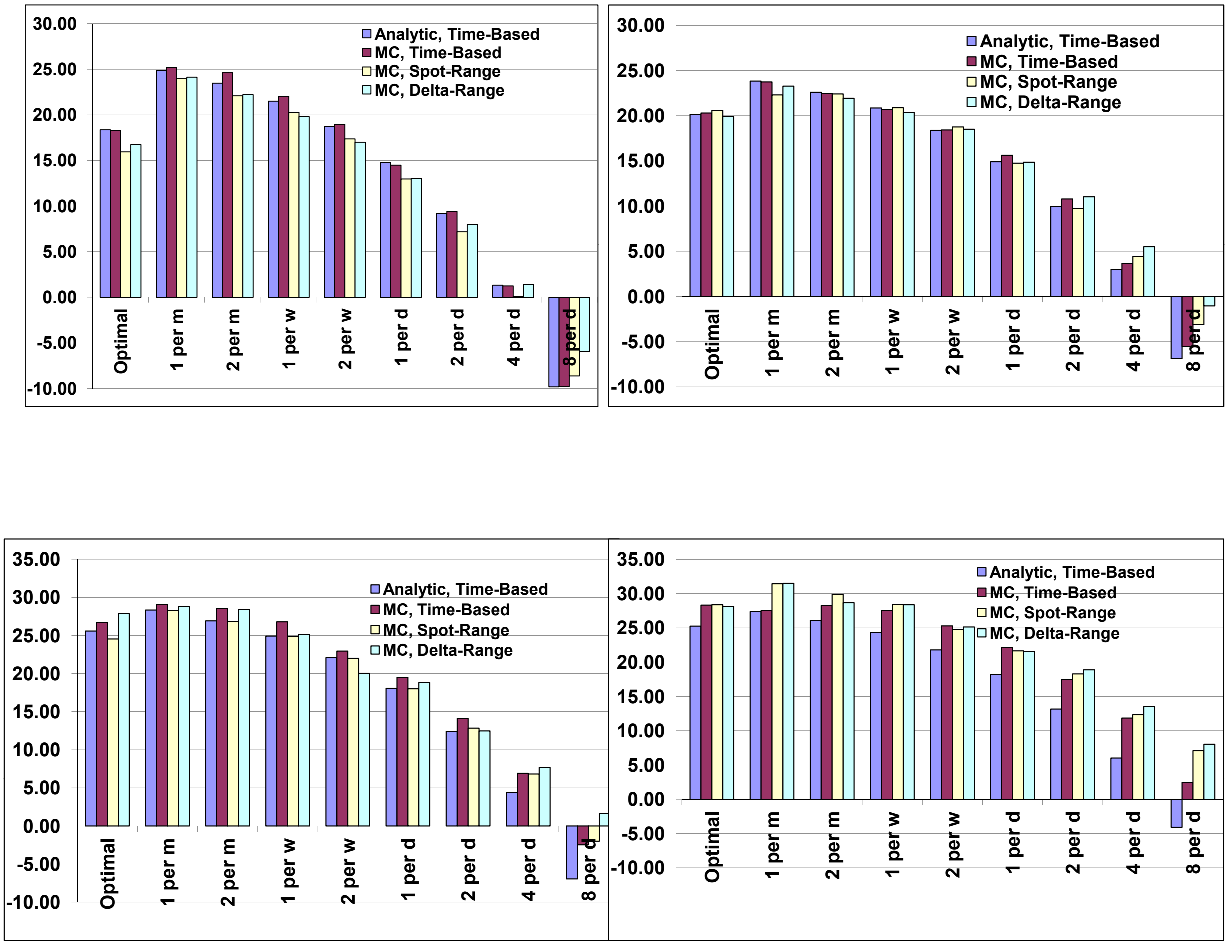

Figure 6: **Realized volatility of P&L.** The realized volatility of P&L of the delta-hedging using parameters from case II for the diffusion model (top left), jump-diffusion model (top right), stochastic volatility model (bottom left), the stochastic volatility model with jumps (bottom right)

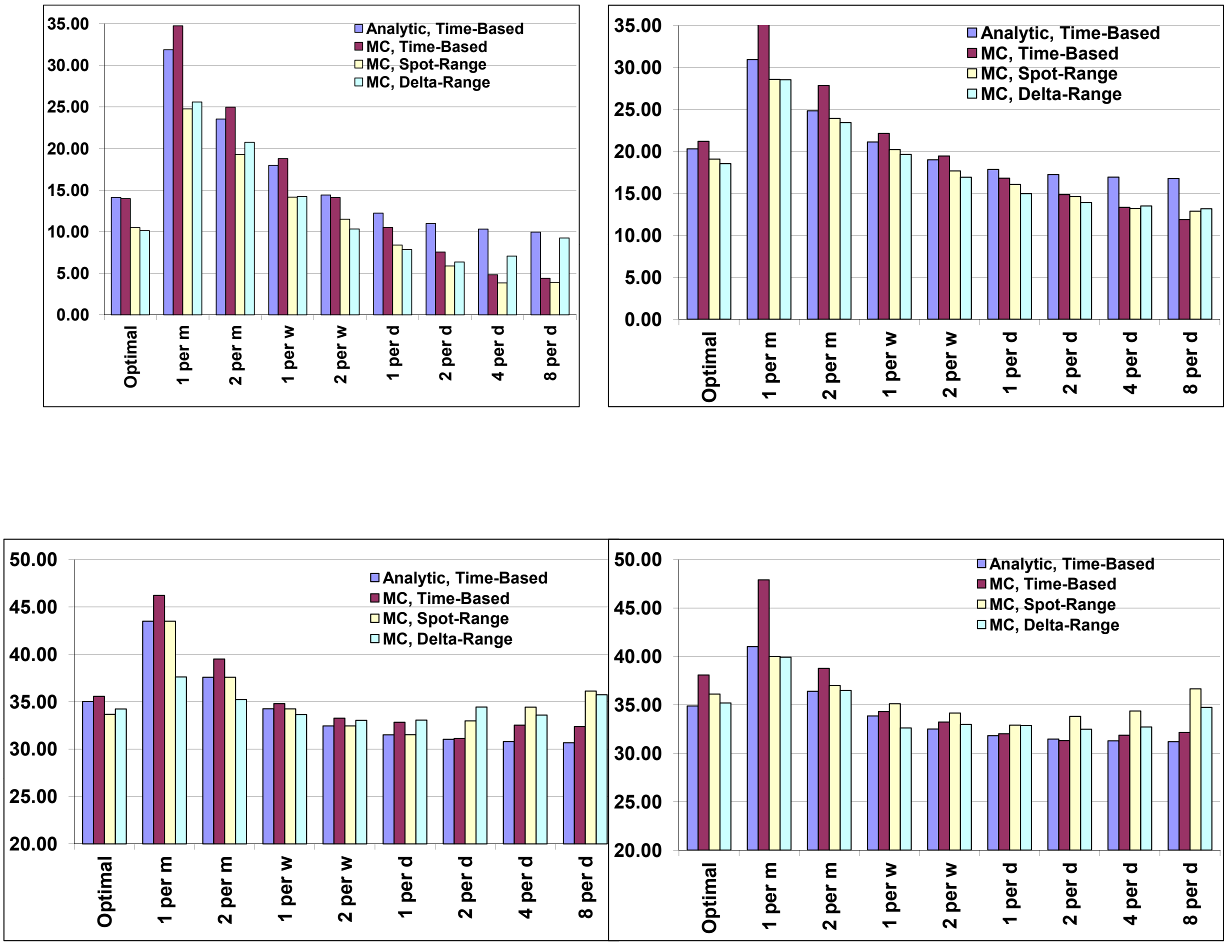

Figure 7: **Realized Sharpe ratio of P&L.** The realized Sharpe ratios of P&L of the delta-hedging using parameters from case II for the diffusion model (top left), jump-diffusion model (top right), stochastic volatility model (bottom left), the stochastic volatility model with jumps (bottom right)

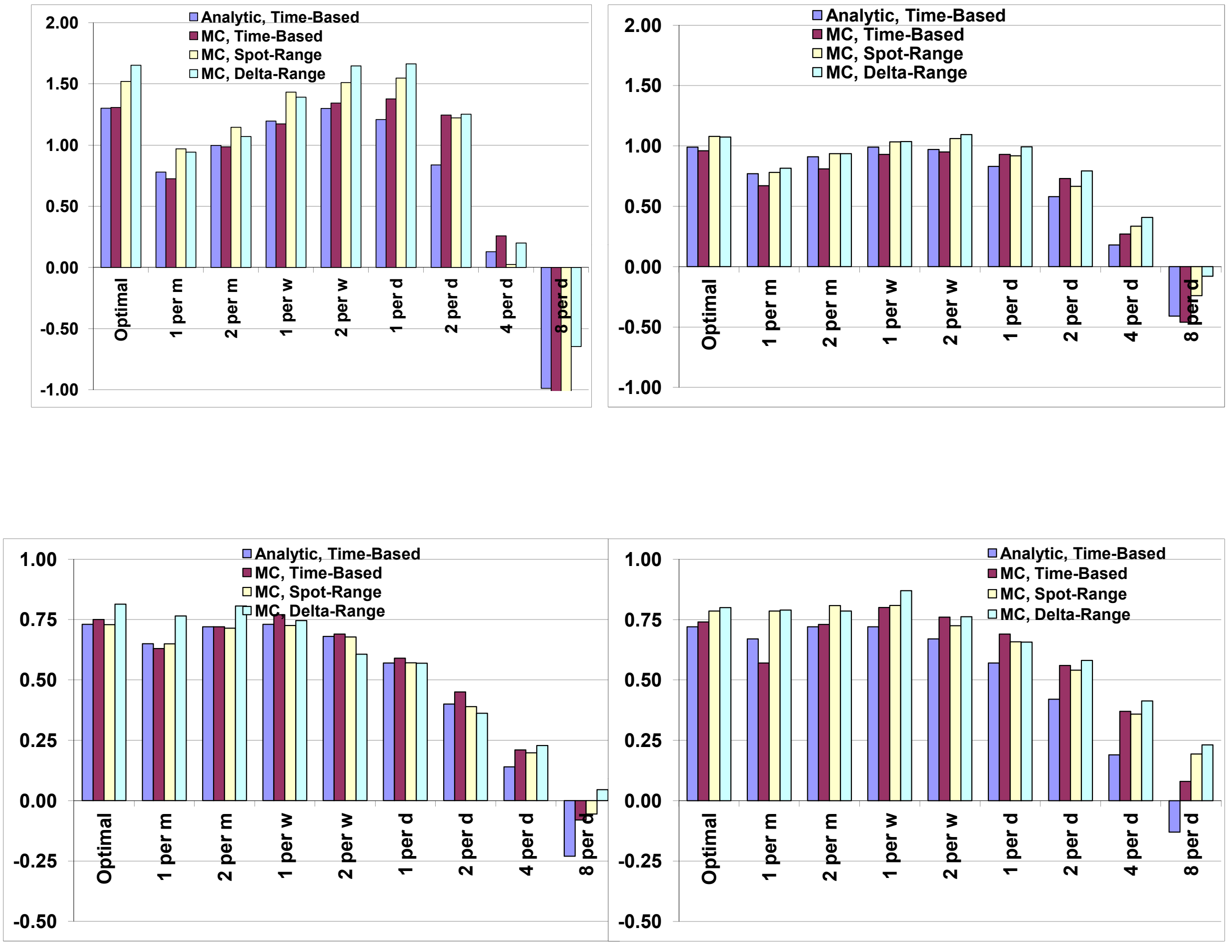

Figure 8: **Realized skew of P&L.** The realized skew of the delta-hedging using parameters from case II for the diffusion model (top left), jump-diffusion model (top right), stochastic volatility model (bottom left), the stochastic volatility model with jumps (bottom right)