Riding on a Smile

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Constructing binomial tree models that are consistent with the volatility smile effect

The Black-Scholes theory has two important but independent features. The primary feature is that it is preference-free – the values of contingent claims do not depend on investors' risk preferences. Therefore, an option can be valued as though the underlying stock's expected return is riskless. This risk-neutral valuation is allowed because the option can be hedged with stock to create an instantaneously riskless portfolio.

A secondary feature of the theory is its assumption that stock prices evolve lognormally with constant local volatility σ at any time and market level. This stock price evolution over an infinitesimal time dt is described by the stochastic differential equation

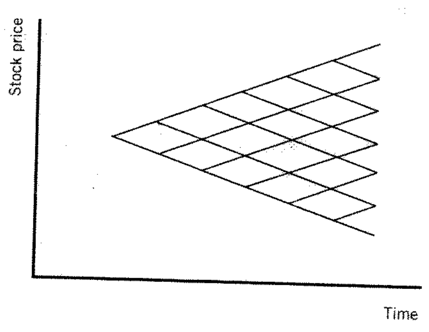
$$\frac{dS}{S} = \mu dt + \sigma dZ \qquad (1)$$

where S is the stock price, μ its expected return and dZ a Wiener process with a mean of zero and a variance equal to dt.

The Black-Scholes formula for a call with strike K and time to expiration t, when the riskless rate is r, follows from applying the general method of risk-neutral valuation to a stock whose evolution is specifically assumed to follow equation (1).

In the Cox-Ross-Rubinstein binomial implementation of the process in equation (1), the stock evolves along a risk-neutral binomial tree with constant logarithmic stock price spacing, corresponding to constant volatility, as illustrated in Figure 1.

1. Risk-neutral stock tree with constant volatility



The binomial tree corresponding to the risk-neutral stock evolution is the same for all options on that stock, irrespective of their strike level or time to expiration. The stock tree cannot "know" about which option we are valuing on it.

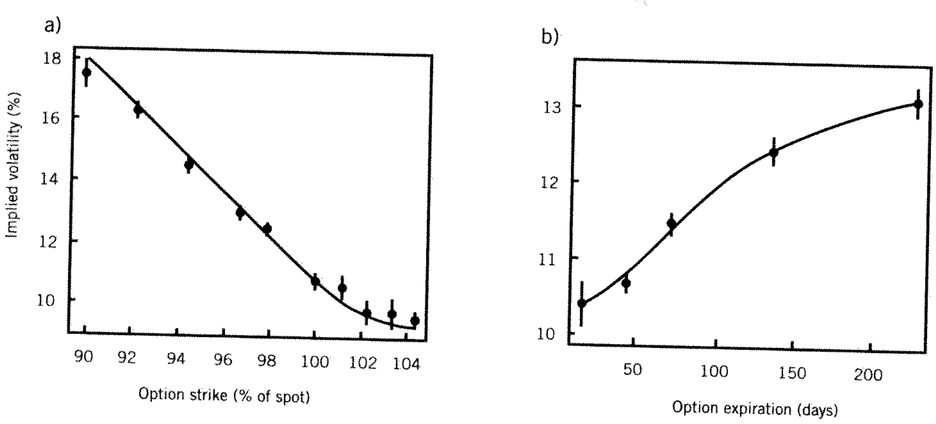
Market options prices are not exactly consistent with theoretical prices derived from the Black-Scholes formula. Nevertheless, the success of the framework has led traders to quote a call option's market price in terms of whatever constant local volatility σimp makes the Black-Scholes formula value equal to the market price. We call σimp the Black Scholes equivalent or implied volatility, to distinguish it from the theoretically constant local volatility σ assumed by the theory. In essence, σimp is a means of quoting prices.

The smile

How consistent are market option prices with the Black-Scholes formula? Figure 2(a) shows the decrease of σimp with the strike level of options on the S&P 500 index with a fixed expiration of 44 days, as observed on May 5, 1993. This asymmetry is commonly called the volatility "skew". Figure 2(b) shows the increase of σimp with the time to expiration of at-the-money options. This variation is generally called the volatility "term structure." In this article we will refer to them collectively as the volatility "smile."

In Figure 2(a) the data for strikes above (below) spot come from call (put) prices. In Figure 2(b) the average of at-the-money call and put implied volatilities is used. You can see that σimp falls as the strike level increases. Out-of-

2. Implied volatilities of S&P500 options
May 5, 1993



the-money puts trade at higher implied volatilities than out-of-the-money calls.

Though the exact shape and magnitude vary from day to day, the asymmetry persists and belies the Black-Scholes theory, which assumes constant local (and therefore, constant implied) volatility for all options. This persistence suggests a discrepancy between theory and the market. It may be convenient to continue quoting options prices in terms of Black-Scholes equivalent volatilities, but it is probably incorrect to calculate options prices using the Black-Scholes formula.

There have been various attempts to extend the Black-Scholes theory to account for the volatility smile. One approach incorporates a stochastic volatility factor;[1] another allows for discontinuous jumps in the stock price.[2]

These extensions cause several practical difficulties. First, since there are no securities with which to hedge the volatility or the jump risk directly, options valuation is in general no longer preference-free. Second, in these multi-factor models, options values depend on several additional parameters whose values must be estimated. This often makes confident option pricing difficult.

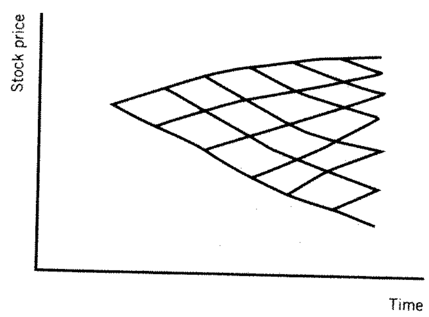
The implied tree

We want to develop an arbitrage-free model that fits the smile, is preference-free, avoids additional factors and can be used to value options from easily observable data. The most natural and minimal way to extend the Black-Scholes model is to replace equation (1) above by

$$\frac{dS}{S} = \mu(t)dt + \sigma(S,t)dZ \qquad (2)$$

where $\mu(t)$ is the risk-neutral drift depending only on time and $\sigma(S,t)$ is the local volatility function that is dependent on both stock price and time.

3. The implied tree



Other models of this type often involve a special parametric form for $\sigma(S,t)$. In contrast, our approach is to deduce $\sigma(S,t)$ numerically from the smile. We can completely determine the unknown function $\sigma(S,t)$ by requiring that options prices calculated from this model fit the smile.

In the binomial framework in which we work, the regular binomial tree of Figure 1 will be replaced by a distorted or implied tree (Fig. 3). Options prices for all strikes and expirations, obtained by interpolation from known options prices, will determine the position and the probability of reaching each node in the implied tree.

Constructing the tree

We use induction to build an implied tree with uniformly spaced levels, Δt apart. Assume you have already constructed the first n levels that match the implied volatilities of all options with all strikes out to that time period. Figure 4 shows the n*th* level of the tree at time t_n, with n

implied tree nodes and their already known stock prices s_i.

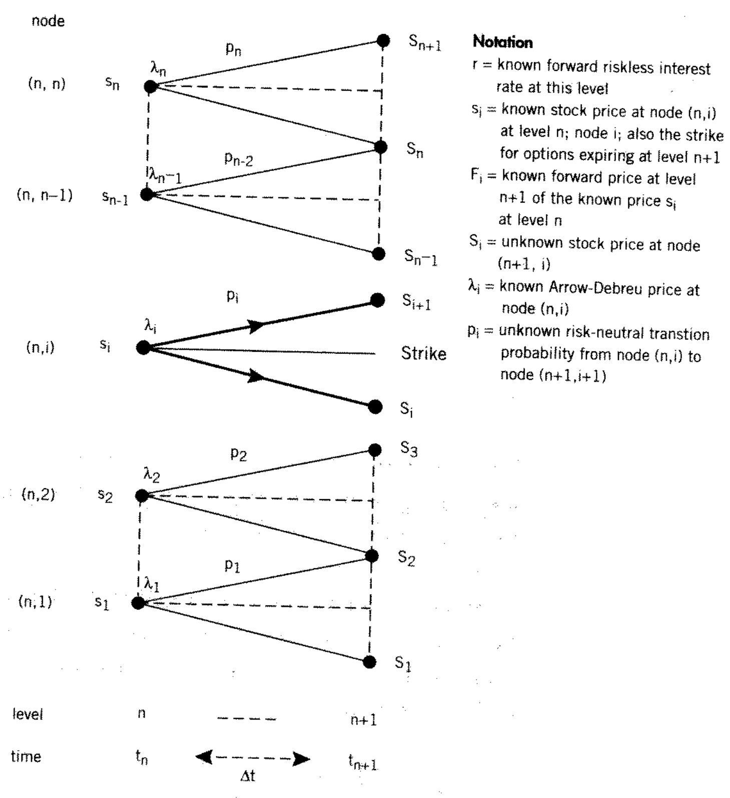
We call the continuously compounded forward riskless interest rate at the nth level r. In general this rate is time-dependent and can vary from level to level; for notational simplicity we avoid attaching an explicit level index to this and other variables used. We want to determine the nodes of the $(n+1)th$ level at time t_{n+1}. There are $n+1$ nodes to fix, with $n+1$ corresponding unknown stock prices S_i. Figure 4 shows the ith node at level n, denoted by **(n,i)** in bold. It has a known stock price si and evolves into an "up" node with price S_{i+1} and a "down" node with price S_i at level $n+1$, where the forward price corresponding to s_i is $F_i = e^{r\Delta t}s_i$. We call p_i the probability of making a transition into the up node. We call λ_i the Arrow-Debreu price at node (n,i); it is computed by forward induction as the sum over all paths, from the root of the tree to node (n,i), of the product of the risklessly-discounted transition probabilities at each node in each path leading to node (n,i). All λ_i at level n are known because earlier tree nodes and their transition probabilities have already been implied out to level n.

There are $2n+1$ parameters that define the transition from the nth to the $(n+1)th$ level of the tree, namely the $n+1$ stock prices S_i and the n transition probabilities p_i. We show how to determine them using the smile.

We imply the nodes at the $(n+1)th$ level by using the tree to calculate the theoretical values of $2n$ known quantities – the values of n forwards and n options, all expiring at time t_{n+1} – and requiring that these theoretical values match the interpolated market values. This provides $2n$ equations for these $2n+1$ parameters. We use the one remaining degree of freedom to make the centre of our tree coincide with the centre of the standard Cox-Ross-Rubinstein tree that has constant local volatility. If the number of nodes at a given level is odd, choose the central node's stock price to be equal to spot today; if the number is even, make the average of the natural logarithms of the two central nodes' stock prices equal to the logarithm of today's spot price. We now derive the $2n$ equations for the theoretical values of the forwards and the options.

The implied tree is risk-neutral. Consequently, the expected value, one period later, of the stock at any node (n,i) must be its known forward price. This leads to the equation

4. Constructing the (n+1)th level of the implied tree



$$F_i = p_i S_{i+1} + (1 - p_i) S_i \qquad (3)$$

where F_i is known. There are n of these forward equations, one for each i.

The second set of equations expresses the values of the n independent options,[3] one for each strike si equal to the known stock prices at the nth level, that expire at the $(n+1)th$ level. The strike level s_i splits the up and down nodes, S_{i+1} and S_i, at the next level, as shown in Figure 4. This ensures that only the up (down) node and all nodes above (below) it contribute to a call (put) struck at s_i. These n equations for options, derived below, together with equation (3) and our choice in centring the tree, will determine both the transition probabilities pi that lead to the $(n+1)th$ level and the stock prices S_i at the nodes at that level.

Let $C(s_i,t_{n+1})$ and $P(s_i,t_{n+1})$, respectively, be the known interpolated market values for a call and put struck today at s_i and expiring at t_{n+1}. We know the values of each of these calls and puts from interpolating the smile curve at time t_{n+1}. The theoretical binomial value of a call struck at K and expiring at t_{n+1} is given by the sum over all nodes j at the $(n+1)$th level of the discounted probability of reaching each node $(n+1,j)$ multiplied by the call payoff there, or

$$C(K,t_{n+1}) = e^{-r\Delta t}\sum_{j+1}^{n}\{\lambda_j p_j + \lambda_{j+1}(1-p_{j+1})\} \max(S_{j+1}-K,0) \quad (4)$$

When the strike K equals s_i, the contribution from the transition to the first in-the-money up node can be separated from the other contributions, which, using equation (3), can be rewritten in terms of the known Arrow-Debreu prices, the known stock prices s_i and the known forwards $F_i = e^{r\Delta t}s_i$, so that

$$e^{r\Delta t}C(s_i,t_{n+1}) = \lambda_i p_i(S_{i+1}-s_i) + \sum_{j=i+1}^{n}\lambda_j(F_j - s_i) \quad (5)$$

The first term depends upon the unknown pi and the up node with unknown price S_{i+1}. The second term is a sum over already known quantities.

Since we know both F_i and $C(s_i,t_{n+1})$ from the smile, we can simultaneously solve equations (3) and (5) for S_{i+1} and the transition probability p_i in terms of S_i:

$$S_{i+1} = \frac{S_i\left[e^{r\Delta t}C(s_i,t_{n+1}) - \Sigma\right] - \lambda_i s_i(F_i - S_i)}{\left[e^{r\Delta t}C(s_i,t_{n+1}) - \Sigma\right] - \lambda_i(F_i - S_i)} \quad (6)$$

$$p_i = \frac{F_i - S_i}{S_{i+1} - S_i} \quad (7)$$

where Σ denotes the summation term in equation 5.

We can use these equations to find iteratively the S_{i+1} and p_i for all nodes above the centre of the tree if we know S_i at one initial node. If the number of nodes at the $(n+1)th$ level is odd (that is, n is even), we can identify the initial S_i, for $i=n/2+1$, with the central node whose stock price we choose to be today's spot value, as in the Cox-Ross-Rubinstein tree. Then we can calculate the stock price S_{i+1} at the node above from equation (6), and then use equation (7) to find the p_i. We can now repeat this process, moving up one node at a time, until we reach the highest node at this level. In this way we imply the upper half of each level.

If the number of nodes at the $(n+1)th$ level is even (that is, n is odd), we start instead by identifying the initial S_i and S_{i+1}, for $i=(n+1)/2$, with the nodes just below and above the centre of the level. The logarithmic Cox-Ross-Rubinstein centring condition we chose is equivalent to choosing these two central stock prices to satisfy $S_i = S^2/S_{i+1}$, where $S = s_i$ is today's spot price corresponding to the Cox-Ross-Rubinstein-style central node at the previous level. Substituting this relation into equation (6) gives the formula for the upper of the two central nodes at level n+1, with n odd:

$$S_{i+1} = \frac{S\left[e^{r\Delta t}C(S,t_{n+1}) + \lambda_i S - \Sigma\right]}{\lambda_i F_i - e^{r\Delta t}C(S,t_{n+1}) + \Sigma} \quad (8)$$

$$\text{for } i = (n+1)/2$$

Once we have this initial node's stock price, we can continue to fix higher nodes as shown above.

In a similar way we can fix all the nodes below the central node at this level by using known put prices. The analogous formula that determines a lower node's stock price from a known upper one is

$$S_i = \frac{S_{i+1}\left[e^{r\Delta t}P(s_i,t_{n+1}) - \Sigma\right] + \lambda_i s_i(F_i - S_{i+1})}{\left[e^{r\Delta t}P(s_i,t_{n+1}) - \Sigma\right] + \lambda_i(F_i - S_{i+1})} \quad (9)$$

where here Σ denotes the sum

$$\sum_{j=1}^{i-1}\lambda_j(s_i - F_j)$$

over all nodes below the one with price s_i at which the put is struck. If you know the value of the stock price at the central node, you can use equations (9) and (7) to find, node-by-node, the values of the stock prices and transition probabilities at all the lower nodes.

By repeating this process at each level, we can use the smile to find the transition probabilities and node values for the entire tree. If we do this for small enough time-steps between successive levels of the tree, using interpolated call and put values from the smile curve, we obtain a good discrete approximation to the implied risk-neutral stock evolution process.

The transition probabilities p_i at any node in the implied tree must lie between zero and one. If $p_i > 1$, the stock price S_{i+1} at the up-node at the next level will fall below the forward price F_i in Figure 4. Similarly, if $p_i < 0$, the stock price S_i at the down-node at the next level will fall above the forward price F_i. Either of these conditions allows riskless arbitrage. Therefore, as we move through the tree node-by-node, we demand that each newly determined node's stock price must lie between the neighbouring forwards from the previous level, that is $F_i < S_{i+1} < F_{i+1}$.

If the stock price at a node violates the above inequality, we override the option price that produced it. Instead we choose a stock price that keeps the logarithmic spacing between this node and its adjacent node the same as that between corresponding nodes at the previous level. This procedure removes arbitrage violations (in this one-factor model) from input option prices, while keeping the implied local volatility function smooth.

How it works

We now illustrate the construction of a complete tree from the smile. To keep life simple, we build the tree for levels spaced one year apart. It can be done for more closely spaced levels on a computer.

We assume that the current value of the index is 100, its dividend yield is zero, and that the annually compounded riskless interest rate is 3% a year for all maturities. We assume that the annual implied volatility of an at-the-money European call is 10% for all expirations, and that implied volatility increases (decreases) linearly by 0.5 percentage points with every 10-point drop (rise) in the strike. This defines the smile.

Figure 5 shows the standard (not implied) Cox-Ross-Rubinstein binomial stock tree for a local volatility of 10% everywhere. This tree produces no smile and is the discrete binomial analogue of the continuous-time Black-Scholes equation. We use it to convert implied volatilities into quoted options prices. Its up and down moves are generated by factors $\exp(\pm\sigma/100)$. The transition probability at every node is 0.625.

Figure 6 displays the implied stock tree, the tree of transition probabilities and the tree of Arrow-Debreu prices that fits the smile. We illustrate how a few representative node parameters are fixed in our model.

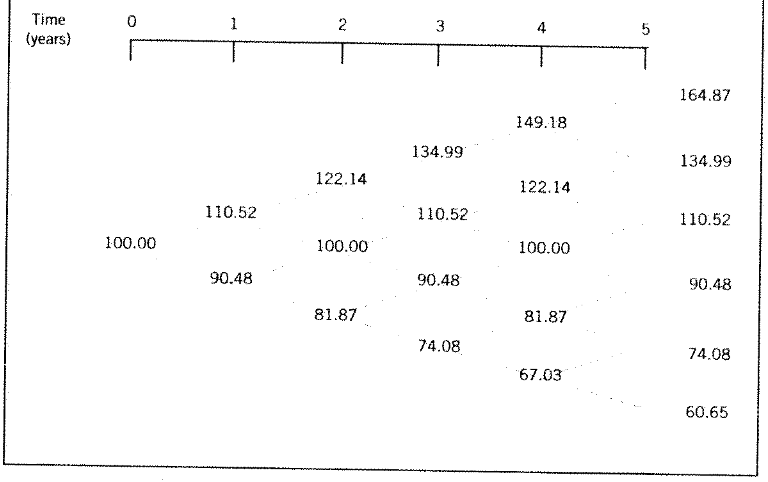
First, the assumed 3% interest rate means that the forward price one year later for any node is 1.03 times that node's stock price.

Today's stock price at the first node on the implied tree is 100, and the corresponding initial Arrow-Debreu price $\lambda_0 = 1.000$. Now let's find the node A stock price in level 2 of Figure 6. Using equation (8) for even levels, we set $S_{i+1}=S_A$, $S=100$, $e^{r\Delta t}$ and $\lambda_1=1.000$. Then

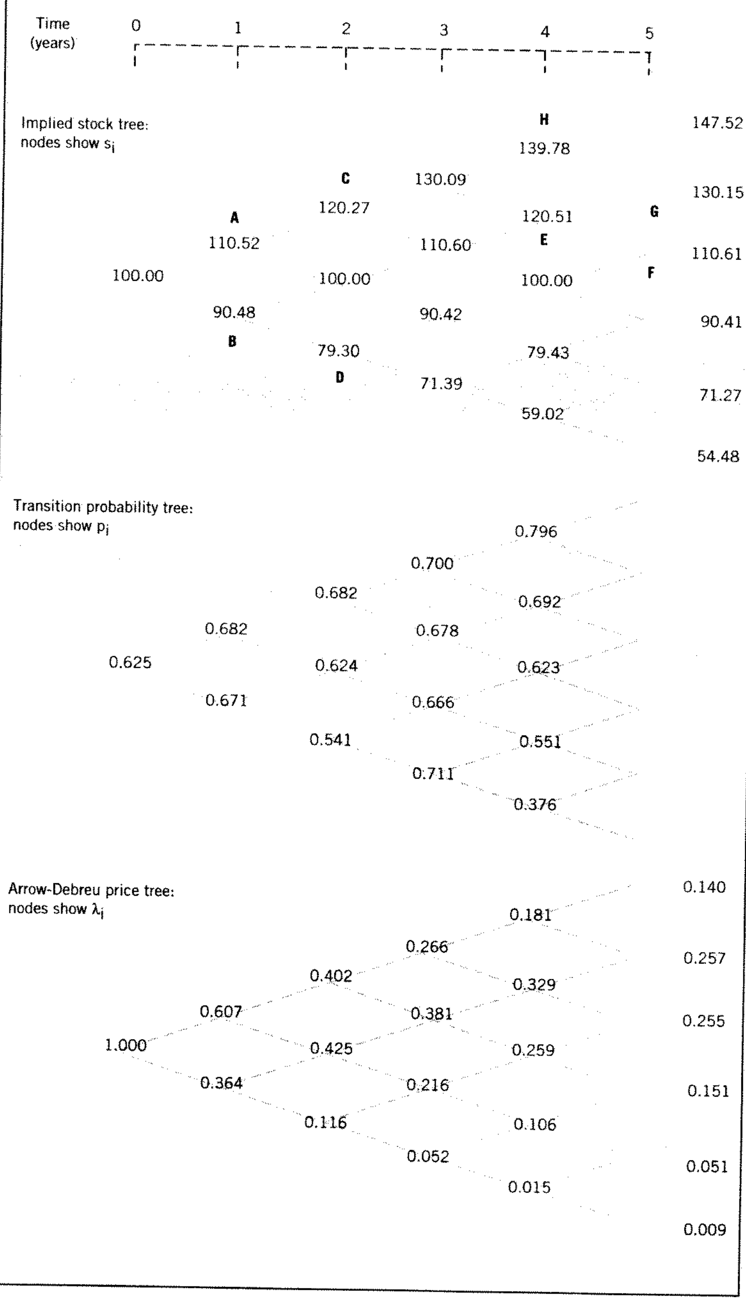
$$S_A = \frac{100[1.03 \times C(100,1) + 1.000 \times 100 - \Sigma]}{1.000 \times 103 - 1.03 \times C(100,1) + \Sigma}$$

where $C(100,1)$ is the value today of a one-year call with strike 100. Σ must be set to zero because there are no higher nodes than the one with strike above 100 at level 0. According to the smile, we must value the call $C(100,1)$ at an

5. Binomial stock tree with constant 10% stock volatility



6. Implied tree, probability tree and Arrow-Debreu tree



implied volatility of 10%. In the simplified binomial world we use here, $C(100,1) = 6.38$ when valued on the tree of Figure 5. Inserting these values into the above equation yields $S_A = 110.52$. The price SB corresponding to the lower node B in Figure 6 is given by our chosen centring condition $S_B = S^2/S_A = 90.48$. From equation (7), the transition probability at the node in year 0 is

$$p = \frac{(103 - 90.48)}{(110.52 - 90.48)} = 0.625$$

Using forward induction, the Arrow-Debreu price at node A is given by $\lambda_a = (\lambda_0 p)/1.03 = (1.00 \times 0.625)/1.03 = 0.607$, as shown on the bottom tree in Figure 6. In this way the smile has implied the second level of the tree.

Now let's look at the nodes in year two. We choose the central node to lie at 100. The next highest node, C, is determined by the one-year forward value $F_A = 113.84$ of the stock price $S_A = 110.52$ at node A, and by the two-year call $C(S_A,2)$ struck at S_A. Because there are no nodes with higher stock values than that of node A in year one, the S term is again 0 and equation (8) gives

$$S_C = \frac{100[1.03 \times C(S_A,2)] - 0.607 \times S_A \times (F_A - 100)}{1.03 \times C(S_A,2) - 0.607 \times (F_A - 100)}$$

The value of $C(S_A,2)$ at the implied volatility of 9.47% corresponding to a strike of 110.52 is 3.92 in our binomial world. Substituting these values into the above equation yields $S_C = 120.27$. Equation (7) for the transition probability gives

$$p_A = \frac{(113.84 - 100)}{(120.27 - 100)} = 0.682$$

We can similarly find the new Arrow-Debreu price λ_C. We can also show that the stock price at node D must be 79.30 to make the put price $P(S_B,2)$ have an implied volatility of 10.47% consistent with the smile.

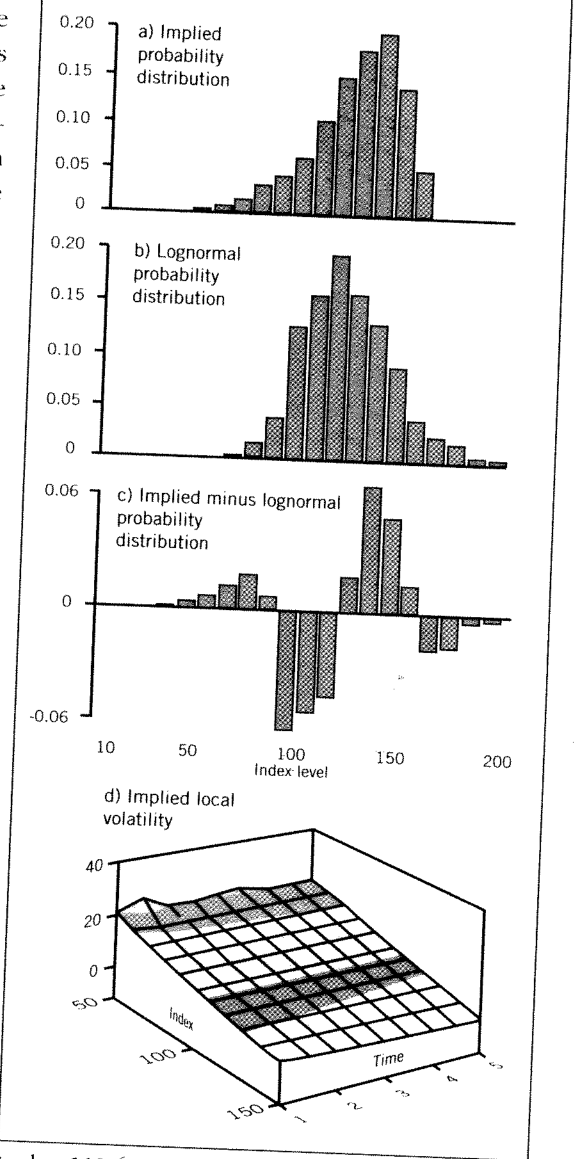
The implied local one-year volatility at node A in the tree is

$$\sigma_A = \sqrt{p_A(1 - p_A)} \log(120.27/100) = 8.60\%$$

Similarly, $\sigma_B = 10.90\%$. You can see that fitting the smile causes local volatility one year out to be greater at lower stock prices.

To leave nothing in doubt, we show how to find the value of one more stock price, that at node G in year five of Figure 6. Let's suppose we have already implied the tree out to year four, and also found the value of S_F at node F to be 110.61, as shown in Figure 6. The stock price S_G at node G is given by equation (8) as

7. The implied distributions



$$S_G = \frac{S_F[1.03 \times C(S_E,5) - \Sigma] - \lambda_E \times S_E \times (F_E - 110.61)}{[1.03 \times C(S_E,5) - \Sigma] - \lambda_E \times (F_E - 110.61)}$$

where $S_E = 120.51$, $F_E = 120.51 \times 1.03 = 124.13$ and $\lambda_E = 0.329$.

The smile's interpolated implied volatility at a strike of 120.51 is 8.86%, corresponding to a call value $C(120.51,5) = 6.24$. The value of the S term in the above equation is given by the contribution to this call from the node H above E in year four. From equation (5) and Figure 6 it is

$$\Sigma = \lambda_H(F_H - S_E)$$
$$= 0.181 \times (1.03 \times 139.78 - 120.51)$$
$$= 4.247$$

Substituting these values gives $S_G = 130.15$.

THE CONTINUOUS-TIME THEORY

In our one-factor model the risk-neutral stock price S diffuses with drift r(t) and local volatility σ(S,t) according to the equation

$$\frac{dS}{S} = r(t)dt + \sigma(S,t)dZ \tag{A}$$

The risk-neutral value C(S,K,T) of a standard European-style call with strike K and expiration T is given by

$$C(S,K,T) = D\int_K^\infty \phi(S',T;S)(S'-K)dS' \tag{B}$$

where D is the deterministic discount factor

$$\exp\left(-\int_0^T r(t')dt'\right)$$

The function φ (S′,T;S) describes the stock distribution at time T, and evolves according to the Fokker-Planck equation:

$$\frac{1}{2}\frac{\partial^2}{\partial S'^2}\left(\sigma^2 S'^2 \phi\right) - \frac{\partial^2}{\partial S'}(rS'\phi) = \frac{\partial\phi}{\partial T} \tag{C}$$

As Dupire has shown,[1] you can use equations (B) and (C) to relate the partial derivatives of the standard European-style call price C(S,K,T) to the local volatility σ(K,T) through the following "forward" equation, which will allow you to extract σ(K,t) from the smile

$$\frac{\partial C}{\partial T} = \frac{1}{2}K^2\sigma^2\frac{\partial^2}{\partial K^2}C - rK\frac{\partial C}{\partial K} \tag{D}$$

You can derive this equation as follows. First by differentiating equation (B) with respect to the strike K, you can show that

$$\frac{\partial C}{\partial K} = -D\int_K^\infty \phi(S',T;S)dS'$$

$$\frac{\partial^2 C}{\partial K^2} = D\phi(K,T;S) \tag{E}$$

Second, the time derivative of equation (B) is

$$\frac{\partial C}{\partial T} = -r(T)C + D\int_K^\infty\left[\frac{\partial}{\partial T}\phi(S',T;S)\right](S'-K)dS'$$

$$= -r(T)C + D\int_K^\infty\left[\frac{1}{2}\frac{\partial^2}{\partial S'^2}\left(\sigma^2 S'^2\phi\right) - \frac{\partial}{\partial S'}(rS'\phi)\right](S'-K)dS' \tag{F}$$

The second line of the above equation follows from equation (C).

Assuming that φ(S′,T;S) decreases to zero fast enough as S′ becomes large, you can integrate the right-hand side of equation (F) by parts twice and set the boundary terms at infinity to zero to obtain

$$\frac{\partial C}{\partial T} = -r(T)C + \frac{\sigma^2K^2}{2}D\phi + rD\int_K^\infty S'\phi(S',T;S)dS'$$

$$= -r(T)D\int_K^\infty \phi(S',T;S)(S'-K)dS' + \frac{\sigma^2K^2}{2}D\phi + rD\int_K^\infty \phi(S',T;S)dS'$$

$$= \frac{\sigma^2K^2}{2}D\phi + rD\int_K^\infty K\phi(S',T;S)dS'$$

$$= \frac{\sigma^2K^2}{2}\frac{\partial^2}{\partial K^2}C - r(T)K\frac{\partial C}{\partial K}$$

where the last line follows from equation (E).

[1] *Risk* January 1994, pp. 18–20, reprinted as Chapter 41 of the present volume.

Some distributions

Once you have an implied tree that fits the smile, you can look at distributions of future stock prices in the risk-neutral world. If you take the model seriously, these are the distributions the market is attributing to the stock through its quoted options prices.

The implied distributions in Figure 7 result from fitting an implied five-year tree with 500 levels to the following smile: for all expirations, at-the-money (strike=100) implied volatility is 10%, and increases by one percentage point for every 10% drop in the strike. We assume a continuously compounded interest rate of 3% a year, and no stock dividends.

Figure 7(a) shows the implied risk-neutral stock price distribution at five years, as computed from the implied tree. The mean stock price is 116.18; the standard deviation is 21.80%.

Figure 7(b) shows a lognormal distribution with the same mean and standard deviation. You can see that the implied tree has a distribution that is shifted towards low stock prices. Figure 7(c) shows the difference between the two distributions.

Figure 7(d) shows the local volatility $\sigma(S,t)$ in the implied tree at all times and stock price levels. To explain the smile you need local volatility to decrease sharply with increasing stock price and vary slightly in time.

In this example we have found the implied tree and its distributions resulting from a smile whose shape is independent of expiration time. We can do the same for more complex smiles, where volatility changes with time to expiration.

Conclusion

We have shown that you can use the volatility smile of liquid index options, as observed at any instant in the market, to construct an entire implied tree.[4] This tree will correctly value all standard calls and puts that define the smile. In the continuous time limit, the risk-neutral stochastic evolution of the stock price in our model has been completely determined by market prices for European-style standard options.

You can use this tree to value other derivatives whose prices are not readily available from the market – standard but illiquid European-style options, American-style options and exotic options – secure in the knowledge that the model is valuing all your hedging instruments consistently with the market. We believe the model may be especially useful for valuing barrier options, where the probability of striking the barrier is sensitive to the shape of the smile. You can also use the implied tree to create static hedge portfolios for exotic options,[5] and to generate Monte Carlo distributions for valuing path-dependent options.

Finally, it would be interesting to see to what extent the implied tree's local volatility function $\sigma(S,t)$ forecasts index volatility at future times and market levels.

1 *See, for instance, J. Hull and A. White, 1987, "The Pricing of Options on Assets with Stochastic Volatilities",* Journal of Finance *42, pp. 281–300.*

2 *See R. Merton, 1976, "Options Pricing when Underlying Stock Returns are Discontinuous",* Journal of Financial Economics *3, pp. 125–44.*

3 *There are only n independent options because puts and calls with the same strike are related through put-call parity, which holds in our model because the implied tree is constrained to value all forwards correctly.*

4 *We have become aware of recent works with similar aims by both Mark Rubinstein and Bruno Dupire (*Risk *January 1994, pp. 6, 18–20). Dupire's article is reprinted as Chapter 41 of the present volume.*

5 *Derman, Risk Exotic Options Conference, London, December 1993.*