FX Derivatives: Stochastic-Local-Volatility Model

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Agenda

1. Review Vanna Volga
   - Early Days and Drawbacks
   - Versions of Vanna-Volga
   - Design and Consistency Issues
2. Stochastic-Local-Volatility
   - LV and SV vanilla smile fit
   - SLV Step by Step
   - SLV Pricing / Validation
3. Mixture Local Volatility Model
   - MLV Main Features
   - Vol Process Comparison MLV vs. SLV
   - Granular Model Marking
4. Summary
   - Product/Model Matrix
   - Key Take-Aways

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1. Pricing in Vanna Volga (VV) model is popular in FX as it still allows to compute prices nearly as fast as in the BS model, using analytical formulas.
2. For a long time it matched market prices for barrier options and touch contracts quite good.
3. Market fit at least better than the alternatively available Local Volatility (LV) or Stochastic Volatility (SV) models.

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5. Volatility too flat in the wings

Wystup/Traders' Rule of Thumb 2003 (VV)

[Wystup, 2003], [Wystup, 2006]: compute the cost of the *overhedge* of risk reversals (RR) and butterflies (BF) to hedge vanna and volga of an option EXO.

$$\text{VV-value} = \text{TV} + p[\text{cost of vanna} + \text{cost of volga}] \tag{1}$$

with

$$\text{cost of vanna} = \frac{vannaEXO}{vannaRR} \times \text{OH RR} \tag{2}$$

$$\text{cost of volga} = \frac{volgaEXO}{volgaBF} \times \text{OH BF} \tag{3}$$

$$p = \text{no-touch probability or modifications} \tag{4}$$

$$\text{OH} = \text{overhedge} = \text{market price} - \text{TV} \tag{5}$$

Castagna/Mercurio 2007 (VV2)

[Castagna and Mercurio, 2007], [Castagna and Mercurio, 2006]: portfolio of three calls hedging an option risk up to second order (in particular the vanna and volga of an option

$$c(K,\sigma_K) = c(K,\sigma_{BS}) + \sum_{i=1}^{3} x_i(K)[c(K_i,\sigma_i) - c(K_i,\sigma_{BS})] \tag{6}$$

with

$$\begin{aligned} x_1(K) &= \frac{\frac{\partial c(K,\sigma_{BS})}{\partial\sigma}}{\frac{\partial c(K_1,\sigma_{BS})}{\partial\sigma}} \frac{\ln\frac{K_2}{K}\ln\frac{K_3}{K}}{\ln\frac{K_2}{K_1}\ln\frac{K_3}{K_1}} \\ x_2(K) &= \frac{\frac{\partial c(K,\sigma_{BS})}{\partial\sigma}}{\frac{\partial c(K_2,\sigma_{BS})}{\partial\sigma}} \frac{\ln\frac{K}{K_1}\ln\frac{K_3}{K}}{\ln\frac{K_2}{K_1}\ln\frac{K_3}{K_2}} \\ x_3(K) &= \frac{\frac{\partial c(K,\sigma_{BS})}{\partial\sigma}}{\frac{\partial c(K_3,\sigma_{BS})}{\partial\sigma}} \frac{\ln\frac{K}{K_1}\ln\frac{K}{K_2}}{\ln\frac{K_3}{K_1}\ln\frac{K_3}{K_2}} \end{aligned} \tag{7}$$

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- Hedge with BF and RR vs Hedge with 3 Vanillas, and if vanillas then which ones?

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- Not clear how to translate to Double-Touch products
- Which volatility to use for the touch probability: ATM, average of ATM and barrier vol, derived from equilibrium condition NTvv=NTbs+NTvv*...[Bossens et al., 2010]

Vanna-Volga Smile Fit

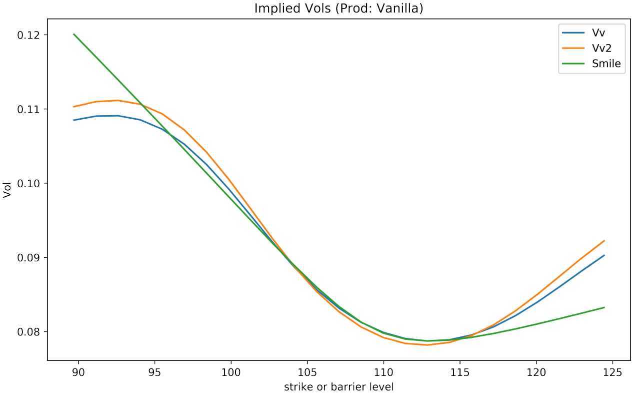
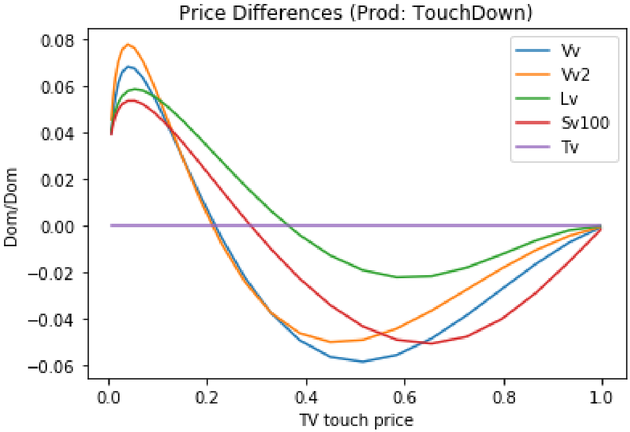


Figure: Smile 9-M-USD-JPY Horizon 23 Jan 2018 Spot 110.31

Strikes: 96.9873 103.1424 108.686 113.775 118.8013

Comparison: Heston-Local-VV USD-JPY OT Down Mustache



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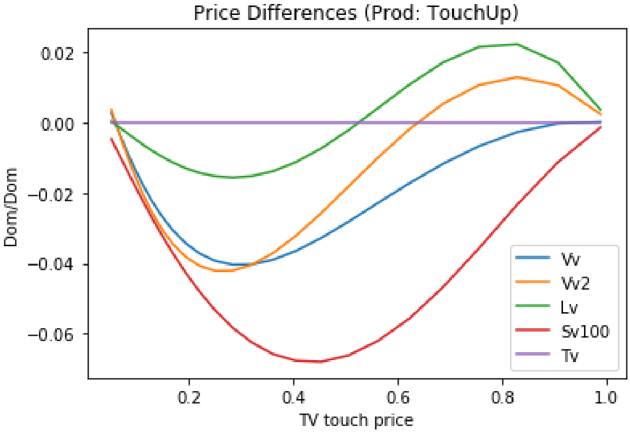

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Vanna-Volga Consistency Issues

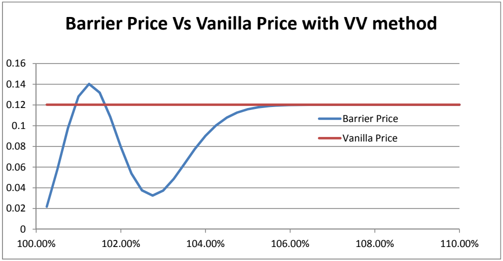

Figure: Convergence of a RKO EUR call CHF put to vanilla, strike 1.0809, 60 days, Market data of April 11 2012: Spot ref 1.20105, 2M EUR rate 0.055%, 2M-Forward -5.65, 10D BF 4.10, 25D BF 1.4755, ATM 3.00, 25D RR -0.7010, 10D RR -1.70.

Vanna-Volga as in [Wystup, 2010] causes arbitrage in extreme markets.

Vanna-Volga Consistency Issues

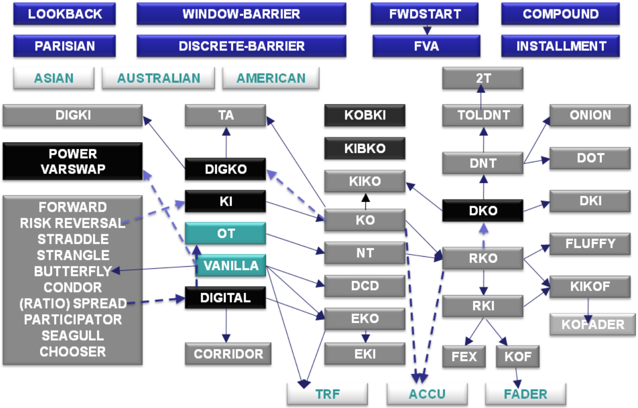

Figure: Exotic Options Pedigree

Vanna-Volga and the Greeks

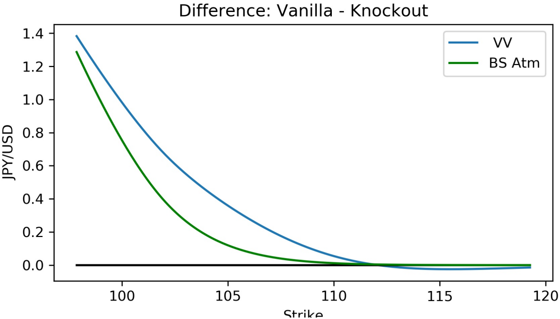

Figure: Difference of vanna-volga based KO call option value and its corresponding vanilla option value, strike on the x-axis

Down-and-out call in USD-JPY barrier 102, spot 109.24

Vanna-Volga and the Greeks

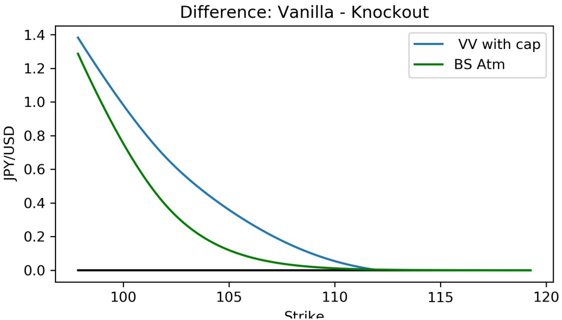

Figure: Difference of vanna-volga based KO call option value and its corresponding vanilla option value, floored at zero

Down-and-out call in USD-JPY barrier 102, spot 109.24

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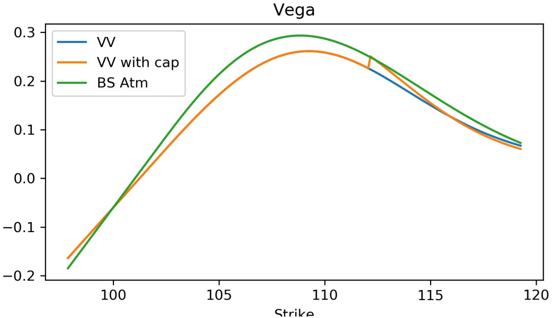


Figure: Vega on the strike space of a regular knock-out call, comparing vanna-volga approach with and without consistency rule (cap)

Implementing consistency rule is easy, but we now lose smoothness of the value function. Effect: jumps and spikes in the Greeks, especially when we compute derivatives by finite differences, i.e. bumping market data.

Vanna-Volga and the Greeks

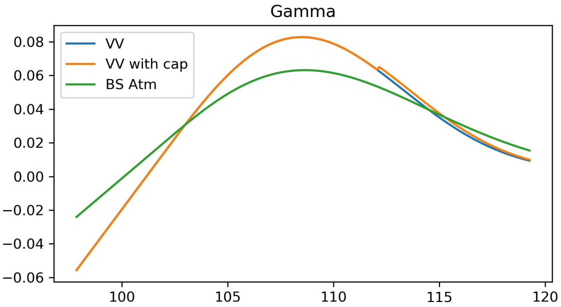


Figure: Gamma on the strike space of a regular knock-out call, comparing vanna-volga approach with and without consistency rule (cap)

kinks and jumps unpleasant, but not dramatic. The problem is that the kinks occur at parameter levels that are not easy to predict - in contrast to non-smooth behavior at a barrier level, which is known in advance and allows us to compute one-sided finite differences or shift the barrier.

Vanna-Volga and the Greeks

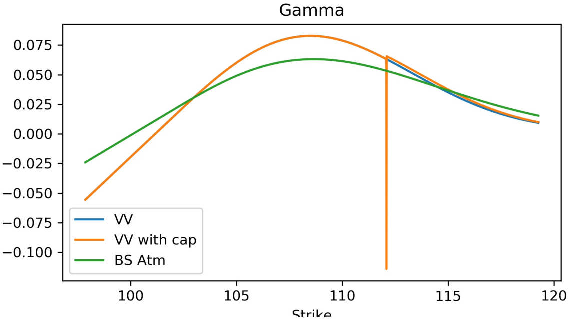
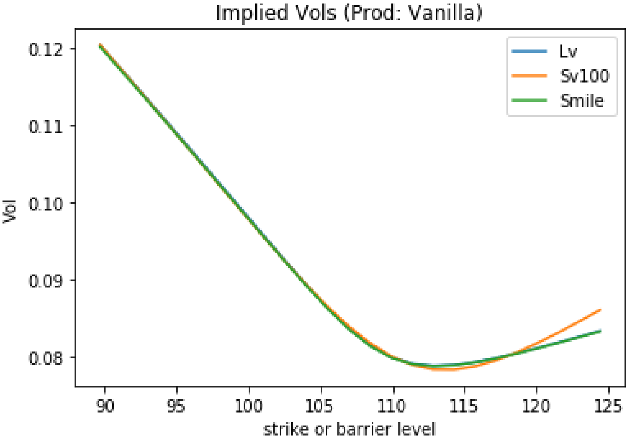


Figure: Exploding gamma on a different strike grid of a regular knock-out call, caused by a vanna-volga approach with consistency rule

LV and SV vanilla smile fit



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- Those computed in the Stochastic Model (SV) do not fit perfectly.
- The SV fits here only appear good, as it has been calibrated exactly to the smile at this expiry. It is difficult to find SV parameter that fit the whole surface.
- Nevertheless: if one needs to choose between LV or SV, in FX one would choose the SV model, as its dynamic better covers how in FX one thinks how a spot movement affects the volatility smile.

SLV Model Explained

- Start with the dynamic of an SV model
- Fit the vanilla options as well as a LV model
- Add some *mixing factor* or *cursor* to vary for prices of exotics between a LV ans SV model.

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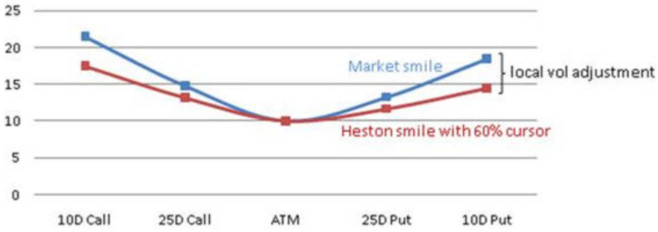

Figure: The Principle of Calibration.

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- Solve numerically the Forward PDE for the density in the (reduced) SV model and in each time step calibrate a leverage function to fit the marginal density to that implied by the LV model.
- Use the calibrated leverage function and the parameter of the SV model to price options either by solving numerically backward PDEs or by simulation with the Monte Carlo method.

Volatility Surface

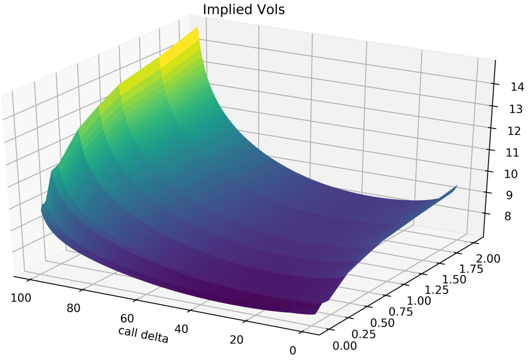


Figure: Volatility Surface

It all starts with a good volatility surface.

Local Volatility Surface

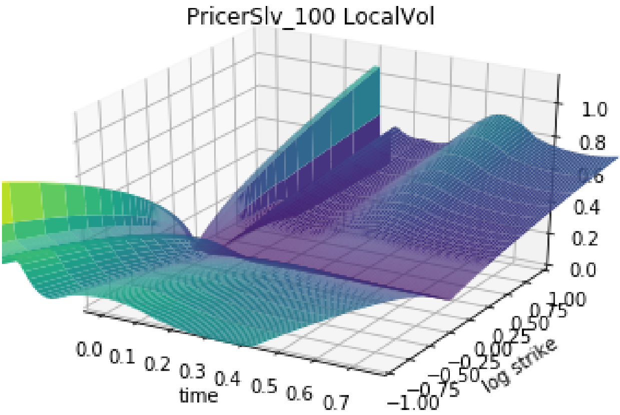
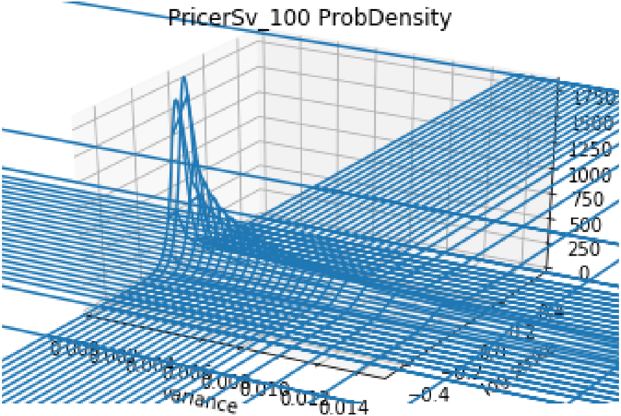


Figure: Local Volatility Surface

Probability Density



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Leverage Function

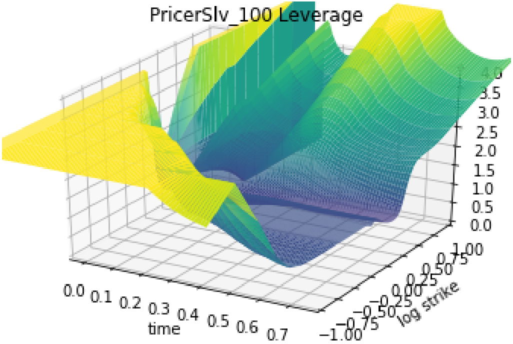

Figure: Leverage Function: how much local vol correction is required

$$\lambda(t, S) = \frac{\sigma_{loc}(t, S)}{\sqrt{\boldsymbol{E}[\sigma_t^2 | S_t = S]}} \tag{8}$$

Vanilla Smile Fit Revisited

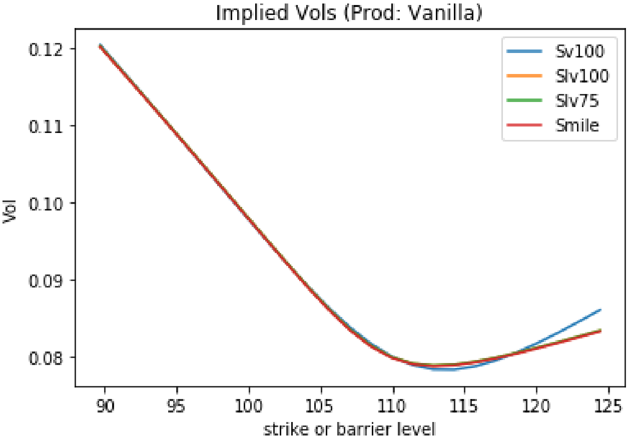

Figure: Vanilla Smile Fit with SLV 9-M-USD-JPY Horizon 23 Jan 2018 Spot 110.31

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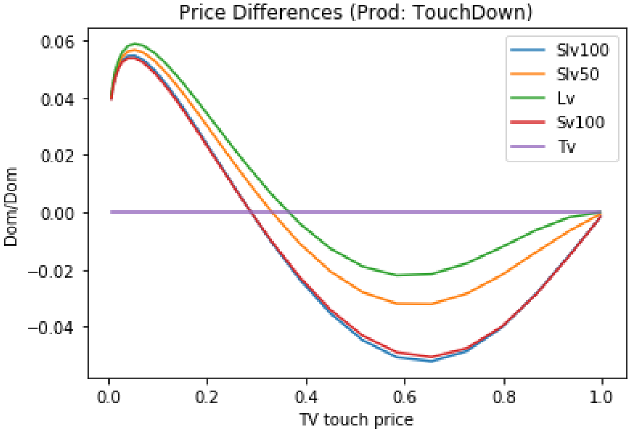

Figure: SLV Model Comparison 9-M-USD-JPY OTD: Horizon 23 Jan 2018 Spot 110.31

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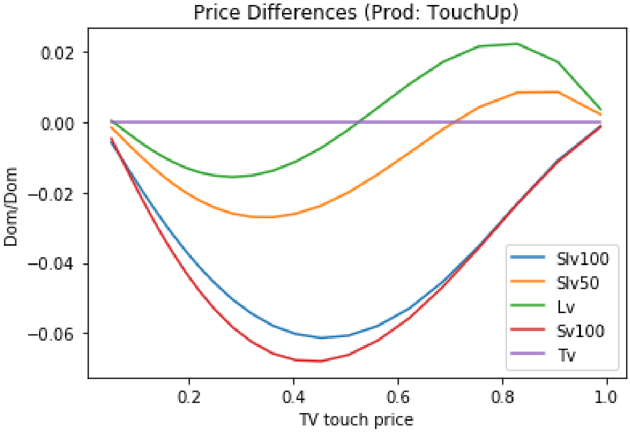

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Monte Carlos and PDE Pricing in SLV Compared

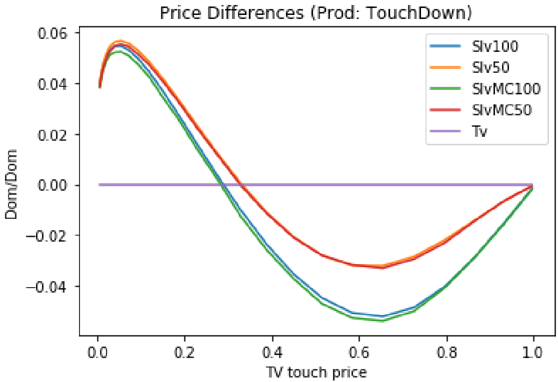

Figure: SLV Pricing Monte Carlo and PDE compared 9-M-USD-JPY OTD: Horizon 23 Jan 2018 Spot 110.31

... should be part of model validation.

MLV main features

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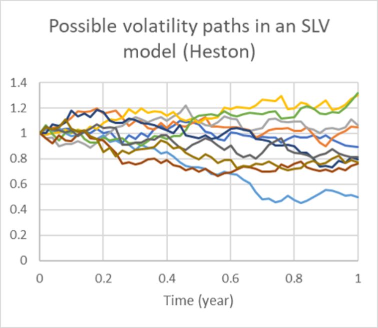
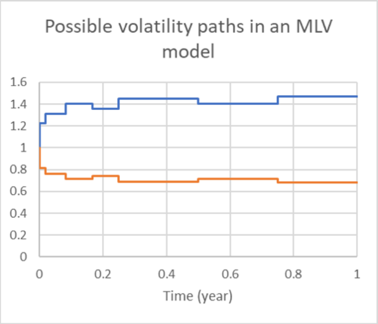
  - Skew generated exclusively from Local Vol effects
- An order of magnitude faster than SLV for calibration and pricing
- Granular calibration to term-structure of DNTs
- Arguably the market standard for pricing a large range of FX 1st generation exotics

Vol process comparison, MLV vs SLV



SLV: Volatility driven by a diffusive process (CIR). Continuous distribution.



MLV: Volatility IS stochastic, but randomness only in t=0. Discrete distribution.

Granular model marking

	ATM	RR25	BF25	RR10	BF10	MIX
ON	10.0%	-0.50%	0.30%	-0.95%	1.08%	30.00%
1W	9.0%	-0.50%	0.35%	-0.95%	1.26%	35.00%
2W	8.0%	-0.70%	0.40%	-1.33%	1.44%	40.00%
3W	8.5%	-0.70%	0.40%	-1.33%	1.44%	40.00%
1M	8.7%	-0.70%	0.40%	-1.33%	1.44%	45.00%
2M	9.0%	-0.80%	0.40%	-1.52%	1.44%	45.00%
3M	9.2%	-0.80%	0.40%	-1.52%	1.44%	50.00%
6M	9.5%	-0.80%	0.40%	-1.52%	1.44%	50.00%
9M	10.0%	-0.80%	0.40%	-1.52%	1.44%	55.00%
1Y	11.0%	-0.80%	0.40%	-1.52%	1.44%	55.00%
18M	11.5%	-0.80%	0.40%	-1.52%	1.44%	55.00%
2Y	12.0%	-0.80%	0.40%	-1.52%	1.44%	55.00%

45% of BF25 generated by Local-Vol

55% of BF25 generated by the mixture (pseudo stoch-vol)

Trader mark MIX empirically, to match a set of symmetric DNTs

A statistical estimate of MIX can also be obtained by looking at historical correlation between Spot and RR25

Figure: MLV: calibrate a per-tenor mixing factor allows to accurately and consistently price a term-structure of exotic instruments with a single model.

Product/Model Matrix

		Payoffs				
		vanillas & European pauoffs	TRFs, tarns (target but no barrier)	1st gen exotics (barriers, discrete barriers)	2nd gen exoticsForward-starting barriers, forward strikes, FVAs, cliquets	Option on realized Vol / variance
Models	BS + Vol interpolation	✓	×	×	×	×
	Vanna - Volga	✓×	×	✓×	×	×
	Dupire	✓	✓	×	×	×
	Local-Vol Mixture	✓	✓	✓	×	×
	Local-Vol Mixture with transition	✓	✓	✓	✓	✓×
	SLV	✓	✓	✓	✓	✓
	Heston	✓×	✓×	✓×	✓×	✓×

Too simple model, mispricing

Too complex model, overkill

Right complexity

Figure: Which model to use for which product

Key Take-Aways

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2. SLV is a common trend in FX 1st generation exotics flow business. Calibration of SLV models is the critical challenge.
3. Mixture local volatility (MLV) models act as compromise between precision and speed.

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