Introducing Hurst exponent in pair trading

Article *in* Physica A: Statistical Mechanics and its Applications · July 2017

DOI: 10.1016/j.physa.2017.06.032

3 authors:

José Pedro Ramos Requena
Universidad de Almería

Juan Evangelista Trinidad Segovia
Universidad de Almería

Miguel Angel Sánchez-Granero
Universidad de Almería

INTRODUCING HURST EXPONENT IN PAIR TRADING

J.P. Ramos-Requena[1], J.E. Trinidad-Segovia[2] *, M.A. Sánchez-Granero[3] †

[1−2]Departamento de Economía y Empresa, [3]Departamento de Matemáticas
Universidad de Almería, 04120, Almería.
[1]pepe_ramos@hotmail.com, [2]jetrini@ual.es, [3]misanche@ual.es

Abstract

In this paper we introduce a new methodology for pair trading. This new method is based on the calculation of the Hurst exponent of a pair. Our approach is inspired by the classical concepts of co-integration and mean reversion but joined under a unique strategy. We will show how Hurst approach presents better results than classical Distance Method and Correlation strategies in different scenarios. Results obtained prove that this new methodology is consistent and suitable by reducing the drawdown of trading over the classical ones getting as a result a better performance.

1 Introduction.

Following [1], pairs trading is a quantitative trading strategy which is aimed at exploiting price movements of assets which are related to each other. The main theme underlying pairs trading is the presence of an equilibrium relationship, which can be interpreted in many different ways. Today, traders are able to buy an overpriced security and thanks to the market tools simultaneously sell a similar underpriced security. The point is to track a pair of securities whose prices move together but if prices diverge, they buy the down stock and simultaneously sell the up stock. Traders profit if prices converge but lose money if prices diverge further.
Some researchers placed the origin of pair trading in the mid-1980s, when Nunzio Tartaglia, quantitative analyst at Investment Bank Morgan Stanley joined a group of mathematicians, physicists and computer scientists to develop quantitative arbitrage strategies using the most sophisticated technology available at

*J.E. Trinidad-Segovia is supported by grant DER2016-76053-R (MINECO/FEDER, UE)
†M.A.Sanchez-Granero is supported by grant MTM2015-64373-P (MINECO/FEDER, UE)

the time. One of their techniques was arbitrage, looking for securities that had the tendency to move together.
By the other hand Wilmontt [2] claims that Gerald Bamberger, a computer scientist and successful trader on Wall Street throughout his career, laid the foundation for pairs trading at Morgan Stanley earlier than Nunzio in 1982, but left in 1985. Finally, other authors claim that pairs trading existed before. For example, Alfred Winslow Jones, who created the first hedge fund ever in 1949, already applied the concept of pairs trading by going long on certain stocks and short on others [3, 4].
What make pair trading interesting for some researchers is that it is a proof against the Efficient Market Hypothesis (EMH) because under an efficient market pair trading strategies must not work in any case. Several authors such us [5, 6, 7, 8, 9] find that pair trading do not report any consistent profit during a significant period of time if transaction cost and commissions are considered, mainly because of market efficiency, but what it is clear is that in the last decades pair trading has reported important amount of money to financial industry around the world.
In this paper we present a new strategy for pair tradig selection based on the Hurst exponent of stock market returns. In section 2 we present the main estrategies used in pair selection. Section 3 introduces main aspects of Hurst exponent as well as the fundaments of its use in pair trading. Paper concludes with a comparative analysis of the results obtained using our methodology and the classical distance method and correlation.

2 Choosing a pair

The undelying idea of pair trading is quite simple, but the issue is clearly how to find an optimal pair. Literature has introduced several methodologies of which the more inportant are two: co-movement and the distance method.

2.1 Co-movement

Defined by [10] as the movement of assets that is shared by all assets at time t. Co-movement is based on correlation and cointegration.
Pearson correlation coefficient is a frequently used statistic to get an idea of how assets move mainly. The higher the correlation coefficient, the more the assets move in sync. In [11], the author analysed the relative performance of different correlation measures w.r.t. pairs trading by back testing three different types of measures over as many pairs as possible. Their most important conclusion is that different statistical correlation measures do show important differences in terms of risk and return.
The other issue about correlation-based pairs trading that should be pointed

out is the data frequency on which to use correlation measures. In fact, correlation is intrinsically a short-run measure because it is based on returns, which is a short memory process [12]. This last fact implies that the higher the trading frequency, the more likely a correlation-based pairs trading strategy will work and thus the more potential for profits.
Introduced by Engle and Granger [13], cointegration shows a different type of co-movement dynamic, since it refers not to co-movements in returns, but to co-movements in asset prices, exchange rates and yields. The most important point to understand here is that a cointegrated pair of stocks could very well be a faultless candidate pair for pairs trading. Engle and Granger [13] say that two series are cointegrated if a linear combination of the both is stationary and even though the combination is at times in disequilibrium. Cointegration has been shown in commodities or foreign currencies that are traded in multiple markets, and for stocks that are cross-listed, in future and spot rates.

2.2 The distance method

Introduced in [14], it is a straightforward methodology based on minimising the sum of squared differences between the normalised price series. There are thus two steps involved in calculating the historical relationship they use to match an assess candidate pairs. To start with, the prices are first normalised.
After normalising the prices, the so-called distance measure, is calculated as the sum of squared differences of the aforementioned normalised price series.
Thus, the pair selection method consists of calculating the distance for all possible pairs in the sample, the best pairs being those ranked from smallest to highest value of d (distance).
The advantages of this methodology are relatively clear: it is economic model free, and as such not subject to model mis-specifications and mis-estimations. It is easy to implement, robust to data snooping and results in statistically significant risk-adjusted excess returns. The main disadvantage is that the choice of Euclidean squared distance for identifying pairs is analytically suboptimal.
Recently [9, 15] replicated the distance methodology extending the sample period. They found a declining profitability in pairs trading, mainly due to an increasing share of nonconverging pairs. They also included trading costs proving that the original methodology becomes largely unprofitable. Do and Faff then used refined selection criteria to improve pairs identification. First restriction consisted in allowing for matching securities within the 48 Fama-French industries. Second, authors favoured pairs with a high number of zero-crossings in the formation period. This indicator is used as a proxy for mean-reversion strength to takes mean-reversion into account. The top portfolios resulted to be profitable enough, even after consideration of transaction costs.

3 Hurst exponent approach for pair selection

One of the well-known stylised facts of finance is that financial time series exhibit mean reversion patterns in different degrees and at different times. Recall that a cointegrated pair of securities is defined as having a long term stable or stationary relationship and that this does not necessarily imply mean reversion per se but that deviations from equilibrium can occur and be restored throughout time which, of course, implies some kind of mean reversion properties. However, to ask for a cointegrated pair of two independent securities is a very strong condition that is rarely fulfilled, specially if a long period of time is considered, since there can be events that affect one of the securities but not the other. Therefore, we look for a more flexible tool to find a pair with a high degree of co-movement.

Example of mean reversion techniques applied in pair trading can be found in [16], where a mean reverting Gaussian Markov chain model was used to analyse pairs trading, and in [17], that used a stochastic residual spread model for detecting mean reversion.

A final point to mention about mean reversion is that it stands in opposite direction of correlation. However the two procedures can clearly be reconciled by looking for pairs of securities which have an absolute value of the correlation coefficient of 0.80 or higher.

In this paper, Hurst exponent will be used as a measure of mean reversion: the lower the Hurst exponent, the greater degree of mean reversion than one can expect.

3.1 Hurst exponent in financial literature.

In the last decades, application of the long memory processes in social sciences has been extended from macroeconomics to finance. In this particular case as a valid alternative to test, in a relative simple way, Efficient Market Hypothesis, probably the most popular topic in finance (see [18, 19, 20, 21, 22, 23, 24, 25, 26]). The study of the long memory processes is realized through the Hurst exponent. This analysis was introduced by English hydrologist H.E. Hurst in 1951 [27] to deal with the problem of reservoir control near Nile River Dam. The most popular methodology to estimate Hurst exponent is the R/S analysis [28] and the DFA [29].

Considering that several authors [30, 31, 32, 33] has proved that accuracy of R/S analysis and DFA is not adequate when the length of the time series is too short, part of Hurst exponent literature has been focused in providing new accurate algorithms for a more efficient Hurst exponent estimation in financial time series. Alternative techniques are the Hudaks Semiparametric Method (GPH) [34], the Quasi Maximum Likelihood analysis (QML) [35], the Generalized Hurst Exponent (GHE) [36], the Periodogram Method [37], wavelets [38],

the Centered Moving Average (CMA) [39], the multifractal detrended Fluctuation analysis (MF-DFA) [40], non-linear tools such as the Lyapunov exponent [41, 42], geometric method-based procedures (GM) [43] and fractal dimension algorithms (FD)[44].
It is important to consider that some classical methods are valid to study long-memory only for fractional Brownian motions and others are also valid for Levy stable motions [45], while only some of them work for the more general self-similar processes. To conclude, [46] showed the importance of the underlying distributions in Hurst exponent estimation and the interpretation of the results. For our purpose, there are a few recent contributions [47, 48] were the connection between market efficiency and long memory is related. Recently, scaling patterns have been increasingly explored for financial markets, [49, 50, 51, 52, 53] constitute a sample of quite interesting contributions. These contributions show that market agents may be essentially distinguished by the frequency at which they operate in markets linking the so-called Fractal Market Hypothesis (FMH) and the EMH.
Firstly, FMH emphasizes the impact of information and expectations on the investor behavior [24, 54]. In classical finance theory, information is treated as a generic item, so EMH implies that all types of information impact investors (also generic) in a similar way. In addition to that, FMH states that information is valued according to each investor's horizon. Traders focus only on short terms and investors are mainly interested in long term investments. This relationship between both theories could be proved by the fact that Capital Asset Pricing Model (CAPM) seems to work fine with stable markets, except during panics or crisis when correlations increase [55].
FMH is based on liquidity, which throws smooth pricing market processes, making it more stable. Therefore, the existence of investors having different horizons leads to a stable market evolution, though market may become unstable when one horizon becomes dominant since liquidity ceases. In this way, FMH predicts that critical events are connected to dominating investment horizons.
This link between market memory and market equilibrium suggests that Hurst exponent could be a good indicator to detect stocks divergence and make pair trading strategies profitable.

3.2 GHE methodology for Hurst exponent calculation

Introduced in [36], GHE is one of the most popular methods for Hurst exponent calculation. It is a generalization of the classical approach provided by [27] and it is related with the scaling behavior of some statistical properties of a time series. It is considered a powerful tool to detect multifractality by means of the scaling of qth-order moments of the distribution of the increments. Such a scaling property is determined by an exponent H_{GHE} which is usually connected with the long-term statistical dependence of the time series.
In particular, it has been verified that these statistical properties of time series scale with both the period of observation (T) and the resolution of the time

window. To do this, the following statistic $K_q(\tau)$ is considered

$$K_q(\tau) = \frac{< |X(t+\tau) - X(t)|^q >}{< |X(t)|^q >}$$

where τ can vary between 1 and τ_{max}, τ_{max} is usually chosen as a quarter of the length of the series, and $< \cdot >$ denotes the sample average over the time window. Hence, the GHE is defined from the scaling behavior of the statistic $K_q(\tau)$ given by the power-law:

$$K_q(\tau) \propto \tau^{qH(q)}. \tag{1}$$

The GHE is calculated as an average of a list of values from the expression contained in (1) for different values of τ [56, 57, 58]. However, the scaling of a time series can also be characterized through the next alternative statistic [48]:

$$K_q(\tau) = \sum_{t=1}^{T-\tau} \frac{|X(t+\tau) - X(t)|^q}{T - \tau + 1} \tag{2}$$

for time series $X(t)$ of length T, which also scales as provided in (1). Note that all the information about scaling properties of a time series is contained in the scaling exponent $H(q)$ which makes the analysis based on GHE quite simple.
In particular, note that for $q = 2$, $K_q(\tau)$ is proportional to the autocorrelation function of the increments, $C(t,\tau) =< X(t+\tau)X(t) >$, and it is related to the power spectrum, which is important from the point of view of long-range dependence detection. Thus, it is possible to estimate the Hurst exponent $H(2)$ from (1) for $q = 2$, which is similar to estimate the parameter $H_{R/S}$ of R/S Analysis, and H_{DFA} of DFA, respectively. In addition, for $q = 1$, $H(1)$ determines the scaling properties of the absolute deviations of the time series which is close to the original Hurst exponent.
In this paper, we are not interested in the multifractal aspect of GHE, so we will use GHE with $q = 1$. Note that GHE is easy to calculate and it is accurate with financial time series (see [44]). In particular, GHE can be used with short series, while other popular methods to calculate the Hurst exponent fail to work fine with short series ([44], [58]) and GHE with $q = 1$ works with a wider range of self similar processes than GHE with $q = 2$ ([58]).

3.3 Using Hurst exponent in pair trading

The interpretation of H is simple. When H is less than 0.5, the process is anti persistent, when H is greater than 0.5, it is persistent and when H is equal to 0.5 it is diffusive. As we pointed out in previous sections, in pair trading, researchers look for correlated (or cointegrated) stocks, since then the pair will have reversion to the mean properties, so it seems natural to look for pairs with low Hurst exponent in order to apply reversion to the mean strategies.

In this paper, given stocks A and B, the series of the pair AB will be log-price(A) $-$ $b*$log-price(B), where log-price is the logarithm of the price of the stock and b is a constant used to normalize the log-prices of A and B. There are some alternatives to choose b, for example, if the stocks are cointegrated, we can estimate b from the cointegration model. However, in this paper we will use a simpler method to calculate b, since we will calculate it as $b =$ std(log-rent(A))/std(log-rent(B)), where log-rent is the logarithmic return of the stock and std is the standard deviation. The reason of this choice is because when we buy the pair, we will short sell b shares of B for each share of A that we buy. Therefore, by definition of b, we will have that our position in stock A has the same volatility that our position in stock B, so we are normalizing both stocks. Furthermore, this estimation of b is faster to calculate than other, more complex, estimations.

If we want to invest an amount T in pair AB, if x is our inversion in stock A, then our inversion in stock B will be bx and hence $x + bx = T$, so $x = \frac{T}{1+b}$ is the total amount we invest in A and bx the amount we invest in B. Therefore, when we buy the pair, we buy $\frac{T}{1+b}$ of stock A and short sell $bx = \frac{bT}{1+b}$ of stock B. When we sell the pair, we short sell $\frac{T}{1+b}$ of stock A and buy $bx = \frac{bT}{1+b}$ of stock B.

The selection of the pairs is as follow: for each possible pair, we calculate the Hurst exponent of the series of the pair (as described previously) and we choose the pairs with the lowest Hurst exponent.

Once we have selected a pair, the estrategy of the inversion in the pair is as follow: if the series of the pair is greater than the mean of the series in the previous 3 months plus one time the standard deviation of the diference between the series and its mean, then we sell the pair as described previously. The position is closed when the series is less than the mean (that is, the stock have reverted to the mean) or greater than the mean plus 2 times the standard deviation (since we will understand that in this case the co-movement of the stocks has been broken, at least temporarily).

So, if s is the series of the pair (as described previously), m is the rolling mean of s with a window of three months and σ the standard deviation of $m - s$, with a window of three months, then:

- when $s > m + \sigma$: sell the pair. Close the position when $s < m$ or $s > m + 2\sigma$.
- when $s < m - \sigma$: buy the pair. Close the position when $s > m$ or $s < m - 2\sigma$.

In order to test this approach, we first proceed to the selection of the pairs. The selection of the pairs is made each six months. To select the pairs, we calculate the Hurst exponent of each possible pair and choose the best N pairs (the pairs with the lowest Hurst exponent), where N is a parameter that we can change. We will use these N pairs for the following six months until we make the selection again. The Hurst exponent is calculated with a window of 1 year of daily data for each pair.

Once we have selected the pairs, we assign $1/N$ of the total budget to each pair and wait for the corresponding signals to buy or sell each of the pairs. So, maybe we are invested at 100% of the budget on a given date or we are not invested in any of the pairs on another date. In the six month period it is possible to be invested in a given pair for more than one time (for example, buy the pair, close the position, sell the pair, close the position, etc.).

4 Experimental results

For testing the results of our strategy we have used data corresponding to stocks of the Dow Jones Index during the period January 1, 2000 through December 31, 2015. Note that the selected period contains bull and bear markets, so it is quite representative. Strategies corresponding to the two basic models have been simulated to be compared with the Hurst exponent one. We have considered 7 scenarios, dependending on the amount of pairs included in the portfolio.

As we can see in Table 1, Hurst approach performance is better than others strategies in all cases except when less than 10 pairs are considered. It is interesting to see how, when the number of pairs is increased, Hurst approach increases its advantage over the others because it is able to reduce the number of losing operations while increasing the winning ones. By the other hand, profit average in winning operations is slightly higher and profit average in losing operations is lower.

Optimun performance seems to be obtained for 20 pais with an average return of 0.06714% and a standard deviation of 0.15754 per operation.
Figures 1 and 2 show the consistence of Hurst approach, as it get a better performance and the equity curve is more robust than the other two approaches.

As Figures 1 and 2 reflect, our model achieves a superior yield over the classical models and the Dow Jones benchmark. It can be observed that, during the considered period, at all times the performance obtained by our model is superior to the classical models and the index. Faced with the financial turmoil that may affect markets, our model is favored. As we can see from 2008 until the end of 2009 the benchmark falls significantly, while the performance of our model increases the profit to reach the maximum levels of the period studied. During the last year of study, we can see that the performance of the index is approaching or at some point can surpass the performance of the model. Anyway, it is obvious that the risk of the proposed strategy is much less than the index risk, so, even with similar profits on the considered period, the drawdowns of the index are much greater than the drawdowns of the model.

Method	N	Oper	%WO	%LO	%PAOW	%PAOL	%Profit
Distance	2	1287	43.90	55.60	1.426	-0.932	73.00
Correlation	2	1269	42.20	57.40	0.913	-0.699	-8.00
Hurst	2	1216	42.60	56.80	1.382	-0.946	34.50
Distance	5	3194	43.60	56.00	1.436	-1.000	48.50
Correlation	5	3175	43.00	56.50	1.107	-0.799	20.20
Hurst	5	3053	43.60	55.90	1.380	-0.963	43.90
Distance	10	6397	42.80	56.70	1.385	-0.986	28.20
Correlation	10	6414	42.00	57.30	1.126	-0.833	2.50
Hurst	10	6211	44.20	55.30	1.334	-0.971	39.00
Distance	15	9625	43.30	56.10	1.368	-0.997	28.60
Correlation	15	9449	42.40	56.90	1.196	-0.853	19.90
Hurst	15	9490	44.00	55.50	1.335	-0.963	40.20
Distance	20	12901	43.20	56.30	1.347	-0.985	24.90
Correlation	20	12534	42.2	57.20	1.203	-0.859	16.20
Hurst	20	12628	44.10	55.40	1.357	-0.959	49.40
Distance	25	16058	42.90	56.60	1.356	-0.984	23.40
Correlation	25	15661	42.40	57.00	1.205	-0.865	17.00
Hurst	25	15750	44.00	55.60	1.363	-0.957	49.40
Distance	30	19305	43.00	56.50	1.345	-0.981	23.30
Correlation	30	18998	42.4	57.10	1.202	-0.873	13.00
Hurst	30	18982	43.7	55.80	1.363	-0.957	46.70

Table 1: Comparative results of Hurst approach; (where N is the number of pairs; WO is percentage of Winning Operations; LO is percentage of Losing Operations; PAOW is the Profit Average of Winning Operations; and PAOL is the Profit Average of Losing Operations)

5 Conclusion

In this paper we propose a new approach to pair trading by using the Hurst exponent as a new method for the selection of a pair. This new approach is based on the idea of mean reversion and correlation because by looking for pairs with low Hurst exponent we look for pairs which move in sync. To obtain the Hurst exponent of a pair we propose the GHE approach, which is a well-known methodology consistent for small samples.
With the introduction of Hurst exponent as a pair selection method, we can see how we get better results than with the classical methods. Our model performs better when the portfolio has ten or more pairs and the number of operations increases. It is the only methodology which is able to maintain the portfolio return when the number of pairs is increased and that shows consistent results when different numbers of pairs are used.

We can conclude by noting that the model that we introduce in this paper is quite market neutral, as expected of a pair trading strategy. It performs

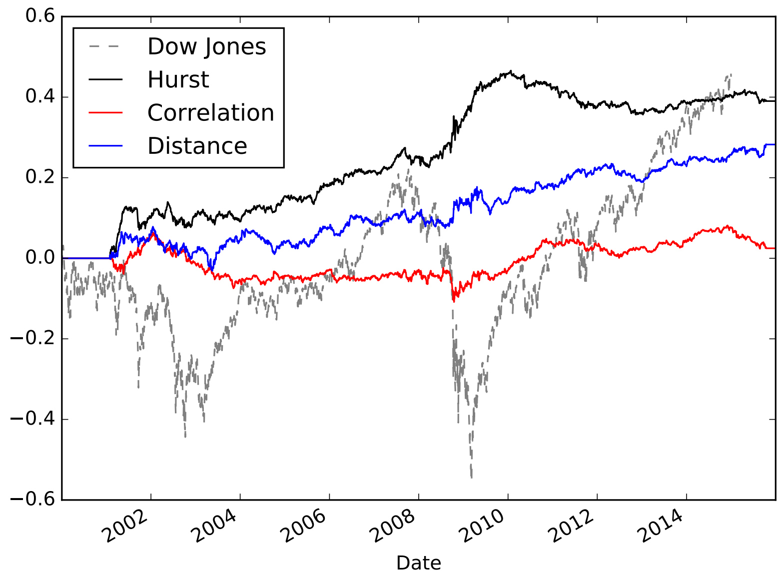

Figure 1: Log of the equity line of the 3 methods and the Dow Jones index, for a portfolio with 10 pairs.

even better during market crashes, as it seems to be able to capture markets inefficiencies on such periods. On the other hand, Hurst exponent approach seems to have a low correlation with the distance or the correlation methods, so it can be used as an alternative, or as a complement, to other pairs trading strategies.

References

[1] G. Vidyamurthy, *Pairs trading: quantitative methods and analysis*, John Wiley and Sons: New Jersey, 2004.

[2] P. Wilmontt, *The best of Wilmott: Incorporating the quantitative finance review (volume 1)*, John Wiley and Sons: New Jersey, 2004.

[3] H. Lindgren, *Long-short story short*, New Yoirk Magazine, 2007.

[4] S. Mallaby, *Learning to love hedge funds*, Wall Street Journal, 2010.

[5] M. Mitchell and T. Pulvino, *Characteristics of risk and return in risk arbitrage*, Journal of Finance 56, 2001, 2135-2175.

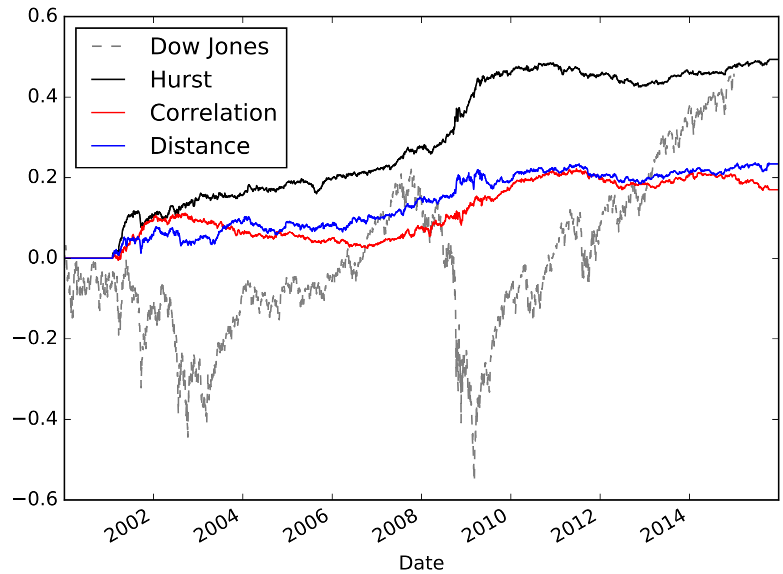

Figure 2: Log of the equity line of the 3 methods and the Dow Jones index, for a portfolio with 25 pairs.

[6] R. Korajczyk and R. Sadka, *Are momentum profits robust to trading costs?*, Journal of Finance 59, 2004, 1039-1082.

[7] B. Grundy and J. Martin, *Understanding the nature of the risks and the sources of the rewards to momentum investing*, Review of Financial Studies 14, 2001, 29-78.

[8] W.H. Chan, R. Jha and M. Kalimipalli, *The economic value of using realized volatility in forecasting future implied volatility*, Journal of Financial Research 32, 2009, 261-287.

[9] B. Do and R. Faff, *Are pairs trading profits robust to trading costs?*, Journal of Financial Research 35, 2012, 261-287.

[10] D. Baur, *What is co-movement?*, Technical Report, European Commission, Joint Research Center, Ispra (VA), 2003, Italy. IPSC-Technological and Economic Risk Management.

[11] J. Wang, C. Rostoker and A. Wagner, *A high performance pair trading application.*, IEEE International Symposium on Parallel and Distributed Processing Symposium, 2009, 1-8.

[12] C. Alexander, I. Giblin, and W. Weddington, *Cointegration and asset allocation: A new active hedge fund strategy*, ISMA Centre Discussion Papers in Finance, 2001, 2001-2003.

[13] R.F. Engle and C.W.J. Granger, *Co-integration and error correction: representation, estimation, and testing*, Econometrica, 5 (2), 1987, 251-276.

[14] R. Gatev, W.N. Goetzmann, and K.G. Rouwenhorst, *Pairs trading: Performance of a relative-value arbitrage rule*, Review of Financial Studies 19 (3), 2006, 797-827

[15] B. Do and R. Faff, *Does simple pairs trading still work?*, Financial Analysts Journal, 66(4), 2010, 83-95.

[16] R.G. Elliott, J. van der Hoek, and W.P. Macolms, *Pair trading*, Quantitative Finance 5 (3), 2005, 271-276.

[17] B. Do, R. Faff and K. Hamza, *A new approach to modeling and estimation for pairs trading*, 2006, Obtained online from http://citeseer.ist.psu.edu

[18] F.X. Diebold and G.D. Rudebusch, *Long memory and persistence in an aggregate output*, Journal of Monetary Economics 24, 1989, 189-209.

[19] R.T. Baillie, C. Chung and M.A. Tieslau, *Analyzing inflation by fractional integrated ARFIMA-GARCH model*, Journal of Applied Econometrics 11, 1995, 23-40.

[20] U. Hassler, *(Mis)specification of long memory in seasonal time series*, Journal of Time Series Analysis 16 (1994) 19-30.

[21] U. Hassler and J. Wolters, *Long memory in inflation rates: International evidence*, Journal of Business and Economic Statistics 13, 1995, 37-45.

[22] G.S. Shea, *Uncertainty and implied variance bounds in long memory models of the interest rate term structure*, Empirical Economics 16, 1991, 287-312.

[23] D. Backus and E. Zin, *Long memory inflation uncertainty: Evidence from the term structure of interest rates*, Journal of Money, Credit and Banking 25 (3), 1993, 681-700.

[24] E. Peters, *Chaos and Order in the Capital Markets: A New View of Cycles, Prices, and Market Volatility*, second ed., John Wiley and Sons, Inc., 1996.

[25] D. Conniffe and J.E. Spencer, *Approximating the distribution of the R/S statistic*, The Economic and Social Review 31 (3), 2000, 237-248.

[26] M. Couillard and M. Davison, *A comment on measuring the Hurst exponent of financial time series*, Physica A 348, 2005, 404-418.

[27] H. Hurst, *Long term storage capacity of reservoirs*, Transactions of the American Society of Civil Engineers 6, 1951, 770-799.

[28] B. Mandelbrot and J.R. Wallis, *Robustness of the rescaled range R/S in the measurement of noncyclic long-run statistical dependence*, Water Resources Research 5, 1969, 967-988.

[29] C.K. Peng, S. V. Buldyrev, S. Havlin, M. Simons, H. E. Stanley, and A. L. Goldberger, *Mosaic organization of DNA nucleotides*, Phys. Rev. E 49, 1994, 1685-1689

[30] A.W. Lo, *Long-term memory in stock market prices*, Econometrica 59 (5), 1991, 1279-1313.

[31] M.A. Sanchez-Granero, J.E. Trinidad Segovia and J.E. Garcia Perez, *Some comments on Hurst exponent and the long memory processes on capital markets*, Physica A 387, 2008, 5543-5551.

[32] R. Weron, *Estimating long-range dependence: finite sample properties and confidence intervals*, Physica A 312 (1-2), 2002, 285-299.

[33] W. Willinger, M.S. Taqqu and V. Teverovsky, *Stock market prices and long-range dependence*, Finance Stoch. 3 (1), 1999, 1-13.

[34] J. Geweke and S. Porter-Hudak, *The estimation and application of long memory time series models*, J. Time Ser. Anal. 4 (4), 1983, 221-238.

[35] J. Haslett and A.E. Raftery, *Space time modelling with long memory dependence: assessing Irelands wind power resource*, Appl. Stat. 38 (1), 1989, 1-50.

[36] A.L. Barabasi and T. Vicsek, *Multifractality of self affine fractals*, Phys. Rev. A 44 (4), 1991, 2730-2733.

[37] M.S. Taqqu and V. Teverovsky, *Estimators for long range dependence: an empirical study*, Fractals 3 (4), 1995, 785-798.

[38] D. Veitch and P. Abry, *A wavelet-based joint estimator of the parameters of long-range dependence*, IEEE Trans. Inf. Theory 45 (3), 1999, 878-897.

[39] E. Alessio, A. Carbone, G. Castelli and V. Frappietr, *Second-order moving average and scaling of stochastic time series*, Eur. Phys. J. B 27, 2002, 197-200.

[40] J.W. Kantelhardt, S.A. Zschiegner, E. Koscielny-Bunde, S. Havlin, A. Bunde and H.E. Stanley, *Multifractal detrended fluctuation analysis of nonstationary time series*, Physica A 316 (1-4), 2002, 87-114.

[41] A. Bensaida, *Noisy chaos in intraday financial data: evidence from the American index*, Appl. Math. Comput. 226, 2014, 258-265.

[42] A. Das and P. Das, *Does composite index of NYSE represents chaos in the long time scale?*, Appl. Math. Comput. 174 (1), 2006, 483-489.

[43] M.A. Sanchez-Granero, M. Fernandez-Martinez and J.E. Trinidad Segovia, *Introducing fractal dimension algorithms to calculate the Hurst exponent of financial time series*, Eur. Phys. J. B 85, 2012, 86.

[44] M. Fernandez-Martinez, M.A. Sanchez-Granero and J.E. Trinidad Segovia, *Measuring the self- similarity exponent in Levy stable processes of financial time series*, Physica A 392, 2013, 5330-5345.

[45] M.A. Sanchez-Granero, J.E. Trinidad Segovia, J. Garcia and M. Fernandez-Martinez, *The effect of the underlying distribution in Hurst exponent estimation*, Plos ONE 10 (5), 2015, e0127824

[46] T. Di Matteo, T. Aste and M.M. Dacorogna, *Scaling behaviors in differently developed markets*, Physica A 324 (1-2), 2003, 183-188.

[47] T. Di Matteo, *Multiscaling in finance*, Quant. Financ. 7, 2007, 21-36.

[48] T. Di Matteo, T. Aste and M.M. Dacorogna, *Long term memories of developed and emerging markets: Using the scaling analysis to characterize their stage of development*, J Bank Financ. 29, 2005, 827-851.

[49] R. Gençay, M.M. Dacorogna, U.A. Muller, O. Pictet and R. Olsen, *An Introduction to High-Frequency Finance*, Academic Press: San Diego, 2001.

[50] R.N. Mantegna and H.E. Stanley, *Scaling behaviour in the dynamics of an economic index*, Nature 376 , 1995, 46-49.

[51] C.J.G. Evertsz, *Fractal geometry of financial time series*, Fractals-Complex Geom Patterns Scaling Nat Soc. 3, 1995, 609-616.

[52] S. Ghashghaie, W. Breymann, J. Peinke, P. Talkner and Y. Dodge, *Turbulent cascades in foreign exchange markets*, Nature, 381, 1996, 767-770.

[53] H.E. Stanley and R.N. Mantegna, *An introduction to econophysics*, Cambridge University Press: Cambridge, 2000.

[54] E.E. Peters, *Applying Chaos Theory to Investment and Economics*, New York: Wiley, 1994.

[55] A. Weron and R. Weron, *Fractal market hypothesis and two power-laws*, Chaos Solitons Fractals, 11, 2000, 289-296.

[56] T. Aste, *Generalized Hurst exponent of a stochastic variable.* http://www.mathworks.com/matlabcentral/fileexchange/30076.

[57] T. Di Matteo, T. Aste and M.M. Dacorogna, *Long-term memories of developed and emerging markets: using the scaling analysis to characterize their stage of development*, J. Bank. Finance 29 (4), 2005, 827-851.

[58] J. Barunik and L. Kristoufek, *On Hurst exponent estimation under heavy-tailed distributions*, Physica A 389 (18), 2010, 3844-3855.