

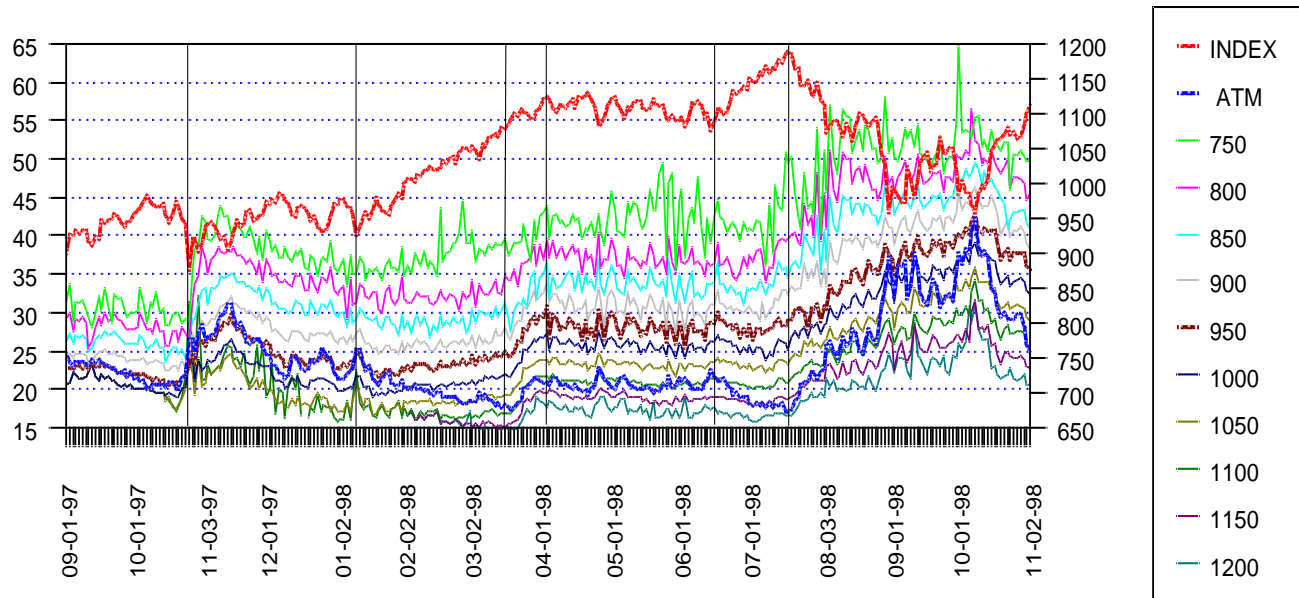
Quantitative Strategies Research Notes

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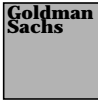
Regimes of Volatility

Some Observations on the Variation of S&P 500 Implied Volatilities

Three-Month Implied Volatilities of SPX Options



Emanuel Derman



QUANTITATIVE STRATEGIES RESEARCH NOTES

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SUMMARY

Since the 1987 stock market crash, the S&P 500 index options market has displayed a persistent implied volatility skew.

How should the skew vary as markets move? There are a variety of apocryphal rules and theoretical models, each leading to different predictions.

In this report I examine more than a year's worth of S&P 500 implied volatilities, qualitatively isolating several distinct periods in which different patterns of change seem to hold. For each period, I try to determine which rule or model the volatility market seems to be following, the possible reason why, and whether the change in volatility is appropriate.

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Editorial: Barbara Dunn

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INTRODUCTION

Since the 1987 stock market crash, global index options markets have been characterized by a persistent large implied volatility skew. How does this skew vary as index markets move? The correct answer to this question dictates the appropriate method for valuing and hedging all sorts of index options.

There are a variety of apocryphal rules that attempt to describe how implied volatilities vary. There are also theoretical models that, in attempting to account for the origin of the skew, lead to forecasts for skew variation. Each rule and model leads to different predictions.

In this report I examine more than a year's worth of S&P 500 implied volatilities, trying to understand the patterns in the data through the prism of models. I isolate several distinct periods in which different patterns of change seem to hold. For each period, I try to determine which rule or model the volatility market seems to be following, the possible reason why, and whether the change in volatility is appropriate.

My hope is that thinking about data in the context of models provides some additional insight into the information embedded in options prices.

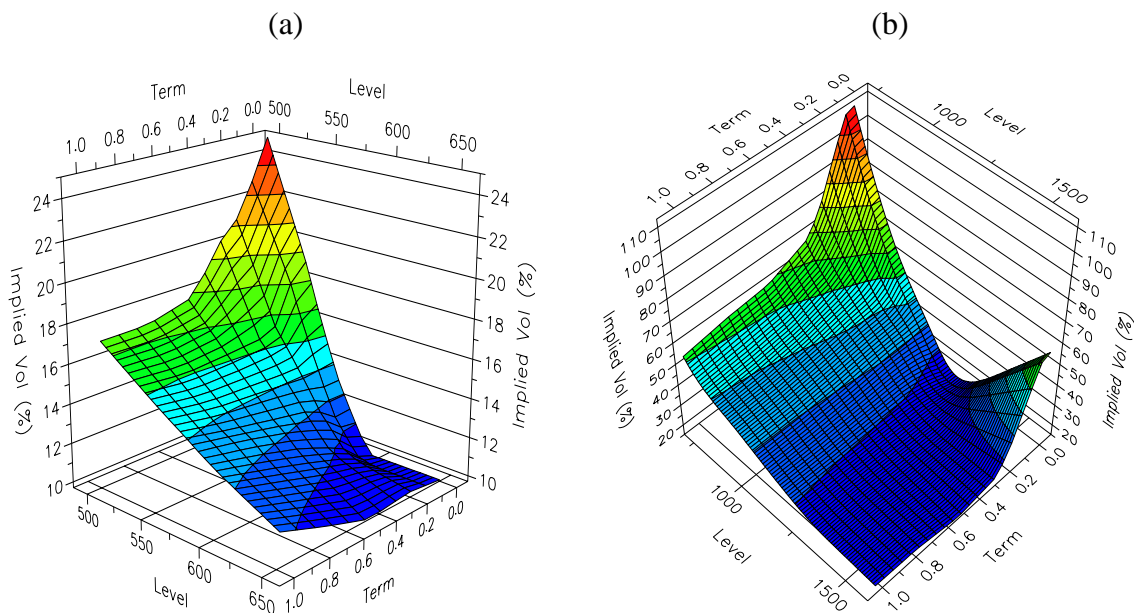
S&P 500
IMPLIED VOLATILITIES

Before the 1987 stock market crash, there was little volatility skew: the implied volatilities of equity index options of a given expiration were virtually independent of their strike level. Since then, global equity index markets have tended to display a “negative” volatility skew in which out-of-the-money puts typically command a premium of several volatility points over at-the-money or out-of-the-money calls. In the last year that premium has widened.

Levels Vary, But The
Skew is Always
Negative

Figure 1 shows the implied volatility surface¹ – the variation of Black-Scholes implied volatility with strike and expiration – constructed from S&P 500 options prices on two dates, September 27, 1995 and December 3, 1998.²

FIGURE 1. The mid-market S&P 500 implied volatility surface at the market close on (a) September 27, 1995 and (b) December 3, 1998.



1. The options prices used to compute these implied volatilities are mid-market closing prices obtained from Reuters' price feeds. Despite the difficulties of ascertaining whether closing options prices are quoted synchronously with closing index levels, we will use them to obtain the implied volatilities we analyze. The volatility patterns obtained do not seem to differ significantly from those obtained from traders' over-the-counter marks.
2. The wings of the surface for very low and high strikes are extrapolated.

The index level, the level of implied volatility and the implied volatility term structure in September 1995 all differed appreciably from their corresponding values in December 1998. Yet on both dates the skew was negative, the usual state of affairs.

The Skew Is Roughly Linear

As is evident from Figure 1, the skew varies approximately linearly as the strike moves away from its at-the-money value. Therefore, in much of this paper, I will parameterize the skew by the empirical formula

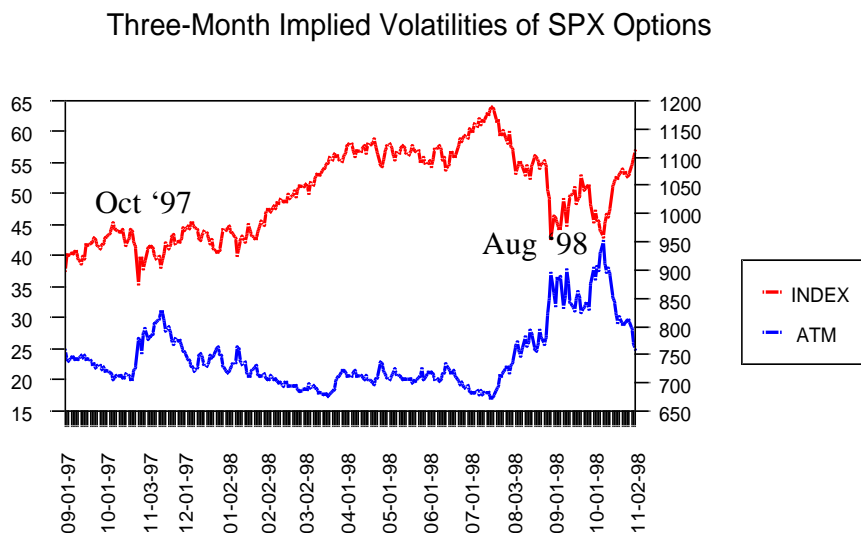
$$\Sigma(K, t) = \Sigma_{atm}(t) - b(t)(K - S_0) \tag{EQ 1}$$

where $\Sigma()$ denotes the Black-Scholes implied volatility for an option of strike K and expiration t when the index level is S_0 . The parameter $b(t)$ is the slope of the skew in annual percentage points of volatility per strike point, and is positive when the skew is negative. (Note that Equation 1 is presumed to describe only the strike-dependence for the current index level S_0 ; I have not yet said anything about the dependence of Σ on the general index level S .)

At-the-money Implied Volatilities Are Negatively Correlated With The Index

At-the-money options are usually the most liquid, and their implied volatility is the simplest measure of the prevailing volatility level. Figure 2 shows the variation of both the S&P 500 index and the rolling three-month at-the-money implied volatility of S&P 500 options from September 1997 through October 1998. Note the negative correlation between the index and its implied volatility, whose graph resembles the reflected image of the index, especially during the corrections of October 1997 and August 1998.

FIGURE 2. The S&P 500 index and its at-the-money rolling three-month



Fixed-Strike Implied
Volatilities Show A
Richer Structure

If you trade options, you don't own a rolling three-month at-the-money strike; instead, you are long or short options with definite expirations and strikes. The changes in their volatilities affect your profit and loss.

Figure 3 shows the behavior of the three-month³ implied volatility of S&P 500 options whose strikes range from 750 to 1200, during the period September 1997 through October 1998. Also shown is the three-month at-the-money implied volatility line that meanders from strike level to strike level as the index moves. Two features are noticeable.

The volatility skew is always negative. At any time, implied volatility increases monotonically as the strike level decreases.

The skew widened after the October 1997 market drop, and then expanded even more after the decline of August 1998. The double-headed arrows in Figure 3 provide a measure of the skew magnitude by indicating the spread in volatility between an 800 and a 1200 strike option. Before the index decline on October 27, this spread was about eight volatility points. It widened through the market drop, stabilizing at about 16 points by mid-December, and then increasing further in August 1998.

I have distinguished seven different regimes in Figure 3. The boundaries between each regime are obviously somewhat subjective. Table 1 shows the realized volatility in each regime.

Regime I: Sept. 1, 1997 ~ Oct. 24, 1997. The index increased as a preamble to the Oct. 27 correction. At-the-money volatility and fixed-strike volatilities declined slightly.

Regime II: Oct. 27, 1997 ~ Jan. 14, 1998. On Oct. 27 the S&P 500 index fell more than 7%. Subsequently, realized volatility increased and the skew widened. During this period, the implied volatility of every option varied approximately inversely with index level.

Regime III: Jan. 15, 1998 ~ Mar. 19, 1998. The index recovered and commenced a long, smooth, low-volatility ascent from about 950 to 1100. During this time, the implied volatility of each strike remained essentially stable, showing only small random variations about its equilibrium level. At-the-money implied volatility steadily rode down the unchanging skew curve as the strike level of at-the-money options increased.

3. The rolling three-month implied volatility for a particular strike has been obtained by interpolation from the closing, mid-market prices of options with expirations that straddle the three-month time to expiration.

FIGURE 3. The implied volatilities of S&P 500 options from September 1997 through October 1998.

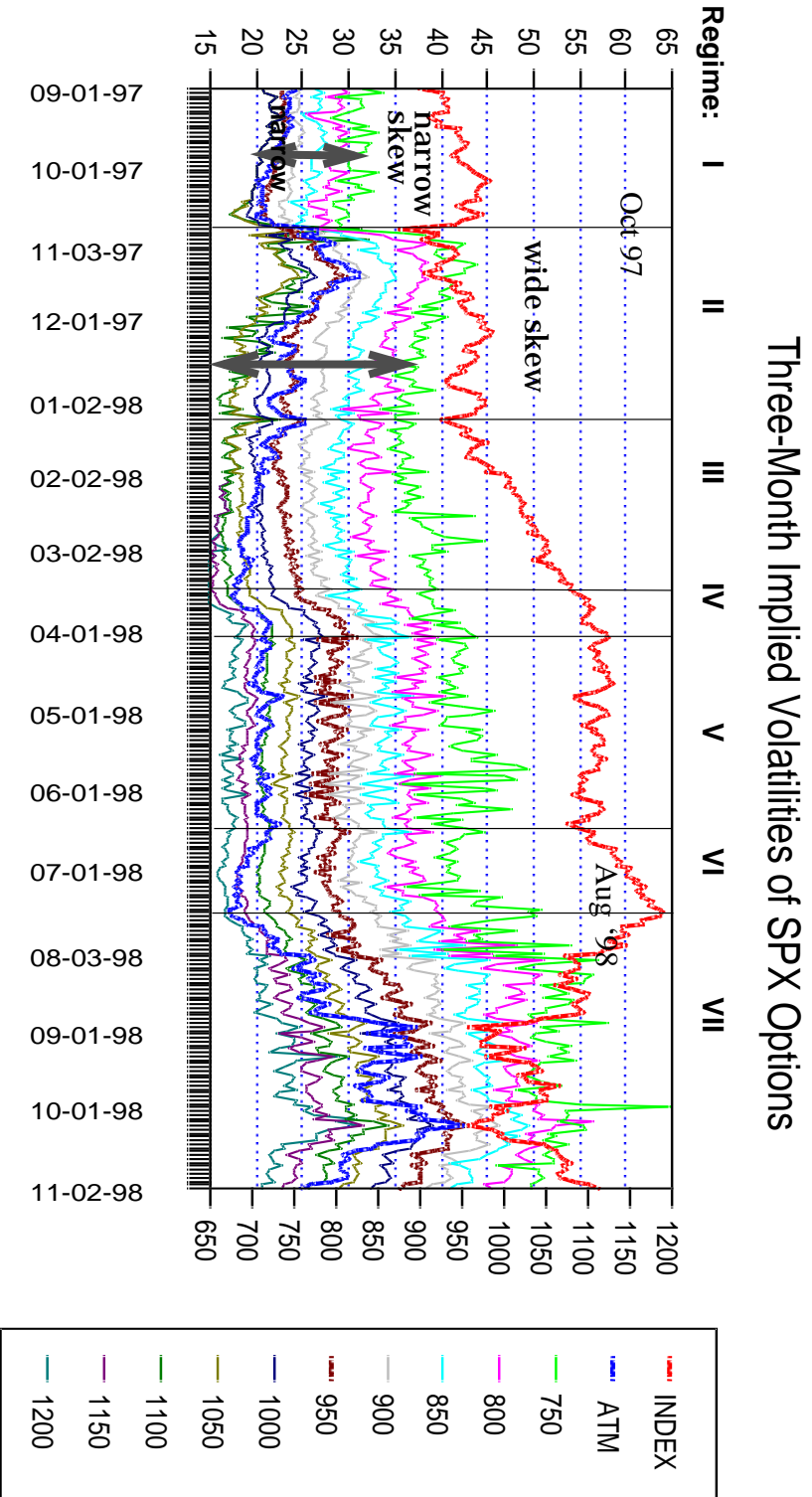


TABLE 1. The annualized realized volatility during each regime in Figure 3.

Regime	Index Vol. (%)
I	17
II	21
III	12
IV	9
V	13
VI	11
VII	30

Regime IV: Mar. 20, 1998 ~ Apr. 15, 1998. During this period the index continued its low-volatility rise, but, in contrast to Regime III, all implied volatilities suddenly increased by three to five volatility points.

Regime V: Apr. 16, 1998 ~ June 12, 1998. Both the index and each strike's implied volatilities remained within tight ranges.

Regime VI: June 15, 1998 ~ July 17, 1998. This is a period similar to Regime III. The index ascended from 1100 to close to 1200 with low volatility, while the implied volatilities of individual strikes remained approximately constant. At-the-money volatility again slid down the stable skew curve as the index rose.

Regime VII: July 20, 1998 ~ Nov. 2, 1998. The S&P 500 index began a period of precipitous declines and recoveries, characterized by record levels of realized and implied volatility. As in Regime II, the implied volatility of each individual option moved inversely to the index, rising as the index fell and then falling as the index rose.

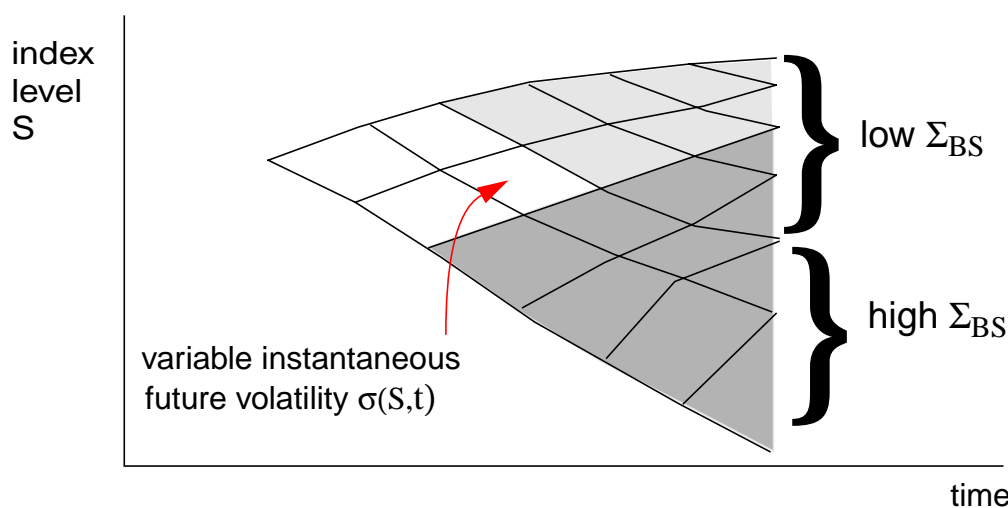
FUTURE TREES:
HOW WILL
VOLATILITIES EVOLVE?

The S&P 500 implied volatility skew at any instant is described by $\Sigma(K, t) = \Sigma_{atm}(t) - b(t)(K - S_0)$. This simple linear dependence on strike K tells us nothing about how implied volatilities will vary when the index level moves away from S_0 , the current index level; the $(K - S_0)$ term is intended to describe only the variation in K .

What *is* the S -dependence of $\Sigma(S, K, t)$, given its observed K -dependence? This dependence is important for obtaining both appropriate hedge ratios and current options values. In this note, I investigate the systematic connections between the changes in index level and index implied volatility in the framework of one-factor options pricing models. (Of course, stochastic changes in implied volatility can also occur; the at-the-money volatility $\Sigma_{atm}(t)$ and the skew slope $b(t)$ can change randomly, even within a particular regime. As long as these intra-regime changes are not too large, our analysis may still be valid.)

In a one-factor options pricing model, valuation is a statement of expectations about the future instantaneous volatility $\sigma(S, t)$ of the index, as illustrated in the tree of Figure 4. In this tree the future index volatilities are shown schematically, and are assumed to increase as the index declines. As a consequence, subtrees at higher index levels within the main tree have lower average instantaneous volatilities. These average instantaneous volatilities are a good proxy for the Black-Scholes implied volatility Σ_{BS} within the subtree⁴, so that in this tree Σ_{BS} also decreases as S increases.

FIGURE 4. A schematic illustration of the tree of future index evolution. The size of each binary fork in the tree is intended to represent the magnitude of the future instantaneous volatility $\sigma(S, t)$, which, in this example, increases as the index declines. In consequence, the Black-Scholes implied volatility Σ_{BS} displays a similar dependence on index level.



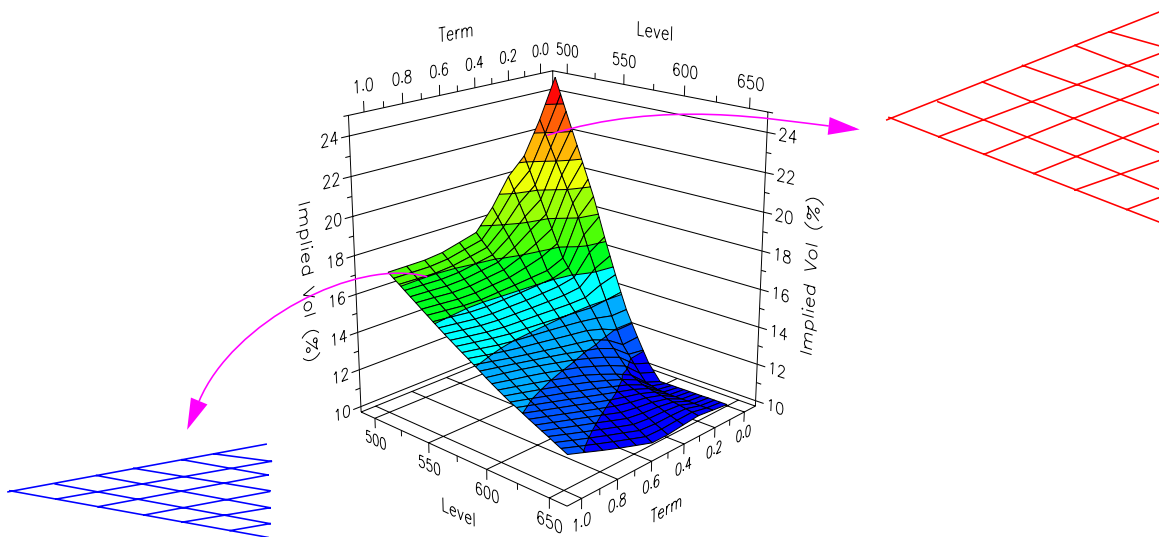
4. See the Appendix in [Derman, 1996].

The market's view of the future tree can be determined from current options prices via the implied tree model, as illustrated in [Derman, 1996], much as forward rates can be extracted from the market's current bond yields. Your personal view of future volatilities can be used to produce your own future tree. In either case, the chosen tree determines all options values and deltas. The implied tree delta for an option will generally differ from the Black-Scholes delta, even if both models agree on option value, because here, in contrast to the Black-Scholes model, implied volatility varies with index level.

The trees corresponding to the Black-Scholes world have future instantaneous volatilities that are constant, independent of time and index level. From the Black-Scholes point of view, as illustrated in Figure 5, options with high strikes imply a future tree with constant, large instantaneous volatility, whereas options with low strikes imply a future tree with constant, relatively smaller volatilities. This picture is inconsistent: how can the same index have two different expected future trees? Nevertheless, it is possible that such a view provides a good empirical description.

Whichever kind of tree you use to describe the current skew, you need to think about how it will change as the index moves.

FIGURE 5. The Black-Scholes trees corresponding to high and low volatilities in the skew.



RULES, MODELS AND THEORIES FOR FUTURE IMPLIED VOLATILITY

Given the current skew, how will it vary with index level in the future?

It is often easier to formulate models of change by describing what doesn't change. In physics or mathematics, such quantities are called invariants. In the world of options trading, it has become customary to refer to what doesn't change as "sticky." There are at least three different views on which aspects of the current skew are sticky as the index moves. The first two views are really heuristic rules rather than models; they are based on some sort of intuition or common sense that does not provide a consistent theoretical framework. The third view is truly a model of stickiness, in the sense that it provides an alternative, self-consistent (though not necessarily true) theory of options valuation. Obviously, none of these views are absolutely correct. Financial modeling is unlikely to provide a model of volatility forecasting that works over a long period. Our aim will be to see to what extent one or another of these relations between index level and implied volatility dominates the options market during a particular period.

The Sticky-Strike Rule

Given the current skew, some traders believe that, as the index moves, the volatility of an option with a particular strike remains unchanged – hence the "sticky-strike" appellation.

Mathematically, the sticky-strike rule is

$$\Sigma(S, K, t) = \Sigma_{atm}(t) - b(t)(K - S_0) \quad \text{Sticky-Strike Rule} \quad (\text{EQ 2})$$

This is equivalent to assuming that Equation 1 holds true for *any* index level S ; implied volatility simply has no dependence on index level. The value S_0 is present in the formula merely to provide a reference level for the current at-the-money volatility.

Intuitively, "sticky strike" is a poor man's attempt to preserve the Black-Scholes model. It allows each option an independent existence, and doesn't worry about whether the collective options market view of the index is consistent. It models the current skew by attributing to each option of a definite strike its own future Black-Scholes-style tree of constant instantaneous volatility. Then, as the index moves, each option keeps exactly the same constant future instantaneous volatility in its Black-Scholes valuation tree, by moving the previously current tree so that its root now sits at the current index level. The rules are illustrated graphically in Table 2.

In Equation 2 the implied volatility is independent of index level S . Therefore, the delta of the option equals the Black-Scholes delta. You can visualize this in Table 2: since the trees remain invariant as you move across a row, the change in the option value is affected only by the change in the moneyness of the option, just as in the Black-Scholes model.

TABLE 2. The evolution of trees in the sticky-strike model. The center column shows the Black-Scholes trees that match the current negative volatility skew for strikes of 90, 100 and 110, when the index level is 100. Each tree in a column has its root at the same index level. For a definite strike, irrespective of index level, all trees in a row have the same volatility structure, except that the root of the tree is relocated to the current index level. The heavy arrows illustrate the rule for relocating trees as the index level changes.

Index	90	100	110
Strike	←	Current Trees	→
90			
100			
110			

We can use Table 2 to see how at-the-money implied volatility changes with index level. The increasing at-the-money direction in the table is along the diagonal, moving from top left to bottom right. If you move your eye along this diagonal, you will see that the trees get progressively narrower, so that at-the-money volatility decreases as the index increases.

Table 3 summarizes the behavior of volatilities under the sticky-strike rule.

TABLE 3. Volatility behavior using the sticky-strike rule.

Quantity	Behavior
Fixed-strike volatility:	is independent of index level
At-the-money volatility:	decreases as index level increases
Exposure Δ :	$= \Delta_{BS}$

The Sticky-Delta Rule

The sticky-delta rule is a more subtle view of what quantity remains invariant as the index moves. It's easier to start by explaining the related concept of sticky moneyness.

Sticky moneyness means that an option's volatility depends only on its moneyness K/S ; the volatility's variation with both index level and strike stems from its dependence on the single variable moneyness. In mathematical terms,

$$\Sigma(S, K, t) = \Sigma_{atm}(t) - b(t)\left(\frac{K}{S} - 1\right)S_0 \quad \text{Sticky-Moneyness Rule} \quad (\text{EQ 3})$$

where S_0 is the initial index level in Equation 1 at which the skew is first observed.

In the Black-Scholes model, the exposure delta itself depends on K and S through the moneyness variable, so that "sticky moneyness" is equivalent to "sticky delta," with an at-the-money option corresponding approximately to $|\Delta_{BS}| = 0.5$. Options market participants think of the value of $(|\Delta_{BS}| - 0.5)$ as a measure of an option's out-of-the-moneyness.

For index levels S and strikes K close to S_0 , you can approximate Equation 3 by

$$\Sigma(S, K, t) = \Sigma_{atm}(t) - b(t)(K - S) \quad \text{Sticky-Delta Rule} \quad (\text{EQ 4})$$

The sticky-delta rule quantifies the intuition that the current level of at-the-money volatility – the volatility of the most liquid option – should remain unchanged as the index moves. Similarly, in this view, the option that is 10% out of the money after the index moves should have the same implied volatility as the 10% out-of-the-money option before the index move. Table 4 illustrates the rule graphically. You can see that, moving along a diagonal of increasing moneyness, the width

and structure of the tree remains invariant, so that at-the-money implied volatility does not change.

Equation 4 shows that the implied volatility for an option of strike K increases with index level S . Therefore, by the chain rule of calculus, the delta of the option is greater than the Black-Scholes delta at the same option price. You can visualize this in Table 4: since the trees increase in width as you move along a row, the change in the option value for a given strike is affected not only by a change in the option's moneyness, but also by an increase in the tree's volatility. Table 5 summarizes the behavior of volatilities and exposures under the sticky-delta rule.

TABLE 4. The evolution of trees in the sticky-delta model. The center column shows the Black-Scholes trees that match the current negative volatility skew for strikes of 90, 100 and 110, when the index level is 100. Each tree in a column has its root at the same index level. For a definite moneyness, irrespective of index level, all trees have the same structure, except that the root of the tree is relocated to the current index level. The heavy arrows indicate the rules for relocating trees as the index level changes.

Index	90	100	110
Strike	Current Trees		
90			
100			
110			

TABLE 5. Volatility behavior using the sticky-delta rule.

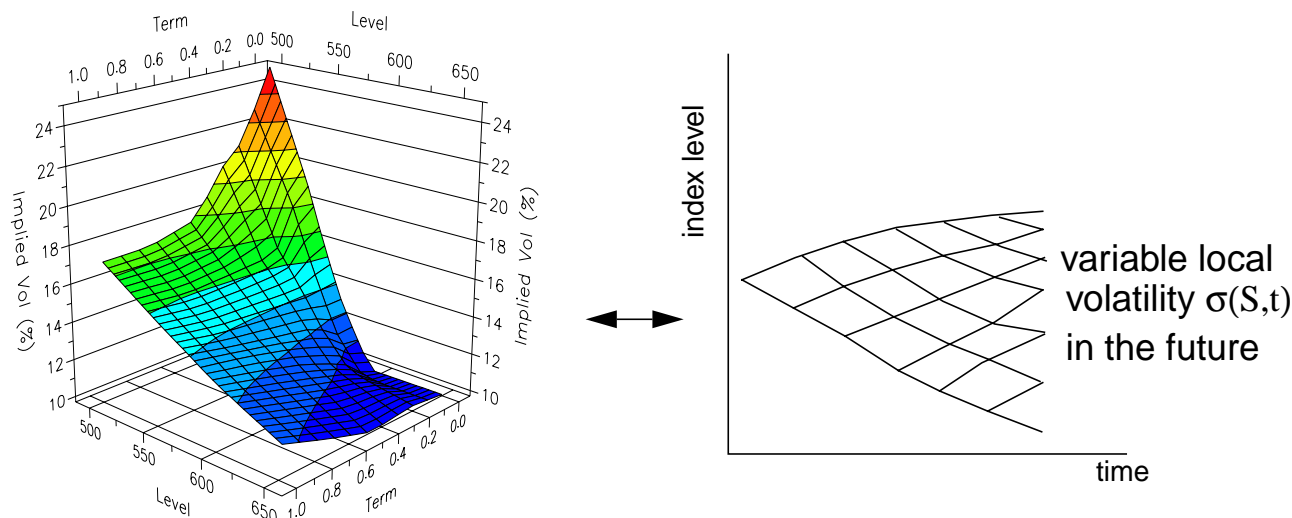
Quantity	Behavior
Fixed-strike volatility:	increases as index level increases
At-the-money volatility:	is independent of index level
Exposure Δ :	$> \Delta_{BS}$

The Sticky-Implied-Tree Model:
One Index, One Tree!

You can interpret all current index options prices as determining a single consistent unique tree – the implied tree⁵ – of future instantaneous index volatilities consistent with the current market and its expectations of future volatilities. This consistency contrasts with the two previous stickiness rules, where each option demands a different tree for the same index.

Figure 6 shows a schematic view of the implied tree, consistent with a particular implied volatility surface. These future instantaneous vola-

FIGURE 6. The implied tree corresponding to a given implied volatility surface.



5. For a summary and further references, see [Derman, 1996].

tilities, when chosen to match the current skew, are called *local volatilities*, and vary both with the future index level and the future time. They bear the same relation to current implied volatilities as forward rates bear to current bond yields.

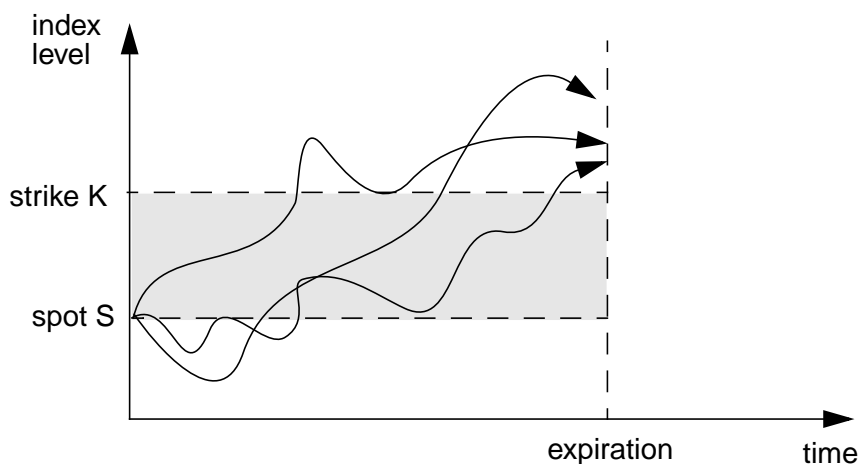
In the tree above, local volatility increases as the index level decreases. The model attributes the implied volatility skew to the market's expectation of higher realized instantaneous volatilities, as well as higher implied volatilities, in the event that the index moves down. You can also think of this market aversion to increased volatilities on a downward index move as representing an aversion to downward index jumps.

Once you have determined the future index tree implied by the current skew and current index level, you can isolate the future subtree at a lower index level to compute the option market's expectation of the future skew, were the index to collapse to that level. This is similar to rolling along the curve of forward rates in order to compute the bond market's expectation of future yields.

Extracting Local Volatilities From Implied Volatilities.

The implied tree model allows the detailed numerical extraction of future local and implied volatilities from current implied volatilities. As an example, consider the implied volatility $\Sigma(S,K,t)$ of a slightly out-of-the-money call option with strike K when the index is at S . Any index paths that contribute to the option value must pass through the region between S and K , shown shaded in Figure 7. The volatility of

FIGURE 7. Index evolution paths that finish in the money for a call option with strike K when the index is at S . The shaded region is the volatility domain whose local volatilities contribute most to the value of the call option.



these paths during most of their evolution is determined by the local volatility in the shaded region. Because of this, you can roughly think of the implied volatility for the option as the linear average of local volatilities between the current index level and the option's strike.

Table 6 displays a simple example with no term structure of volatility in which this approximation is used to extract future local and implied volatilities from the current skew. For strikes not far from the money, this approximation agrees surprisingly well with the results of more exact numerical methods.

TABLE 6. Extracting local volatilities from a sample of hypothetical Black-Scholes implied volatilities. The first two columns of the table show the current skew at an index level of 100, assuming no term structure. The next two columns show the resultant local volatilities, assuming that implied volatility for a given strike is computed as the linear average of the local volatilities between the current index level and the strike. The last two columns contain the skew at a new index level of 99. Note that the local volatilities in column 4 increase by two percentage points per one point change in index level. This is twice the rate at which implied volatilities in column 2 increase per strike point.

Index Level= 100		Index Level = 99			
Strike	Implied BS Vol. (%)	Index Level	Local Vol. (%)	Strike	Implied BS Vol. (%)
100	20%	100	20%	99	22%
99	21%	99	22%	98	23%
98	22%	98	24%	97	24%
97	23%	97	26%	96	

The first two columns in Table 6 show the current implied volatility skew when the index is at a level of 100. The skew is taken to be linear and negative, increasing at one volatility point per strike point. When the index is at 100, the 100-strike at-the-money volatility in column 2 is 20% per year. The local volatility at an index level of 100 in column 4 is therefore also 20%, because local volatility *is* effectively the short-term at-the-money implied volatility at that index level. The 99-strike

volatility in column 2 is 21%. This implies a value of 22% for at-the-money (local) volatility at an index level of 99: a local volatility of 20% at an index level of 100 and 22% at an index level of 99 results in an average (Black-Scholes implied volatility) value of 21% for the 99-strike option when the index level is 100.

Column 4 shows the local volatilities corresponding to this skew, computed using this averaging procedure. The averaging necessitates that local volatilities must increase twice as fast with index level as the implied volatilities increased with strike level. In the bond world, the analogous statement is that forward rates increase twice as fast with future time as bond yields increase with maturity.

Given the local volatilities in column 4, we can use them to reconstruct the implied volatilities at a different index level, say 99, as shown in column 6. The first entry, an implied volatility of 22% at a strike level of 99, is the local volatility in column 4 at an index level of 99. The second entry, 23% implied volatility for a strike of 98, is the average of the local volatility of 22% at an index level of 99 and 24% at a level of 98 in column 4.

Future Trees In The Sticky-Implied-Tree Model.

Table 7 illustrates the use of one consistent implied tree. As the index level within the tree rises, the local volatilities decline, monotonically and (roughly) linearly, in order to match the linear strike dependence of the negative skew in Equation 1. Therefore, an increase of either index level or strike leads to the same decrease in the average local volatility between index and strike within the tree. This average local volatility is a good measure of the Black-Scholes implied volatility. Consequently, in the sticky-implied-tree model, the linear dependence of implied volatilities on strike K induces a similar linear dependence on index level S , as described by

$$\Sigma(S, K, t) = \Sigma_{atm}(t) - b(t)(K + S) \quad \text{Sticky-Implied-Tree Model} \quad (\text{EQ 5})$$

Implied volatilities decrease as K or S increases. At-the-money implied volatility, for which K equals S , decreases twice as rapidly with S . Because volatility decreases as you move to higher index levels in the tree, an option's exposure delta in the model is smaller than the Black-Scholes delta of an option with the same volatility.

Table 8 summarizes the variation of implied volatility in the sticky-implied-tree model. The implied volatilities of all options are inversely correlated with the index, rising as the index falls and falling as it rises.

TABLE 7. Tree evolution in the sticky-implicit-tree model. The center entry shows the implied tree that matches the current negative skew. Each tree in a column has its root at the same index level. As the index moves, we simply slide along the tree to the node at the corresponding index level.

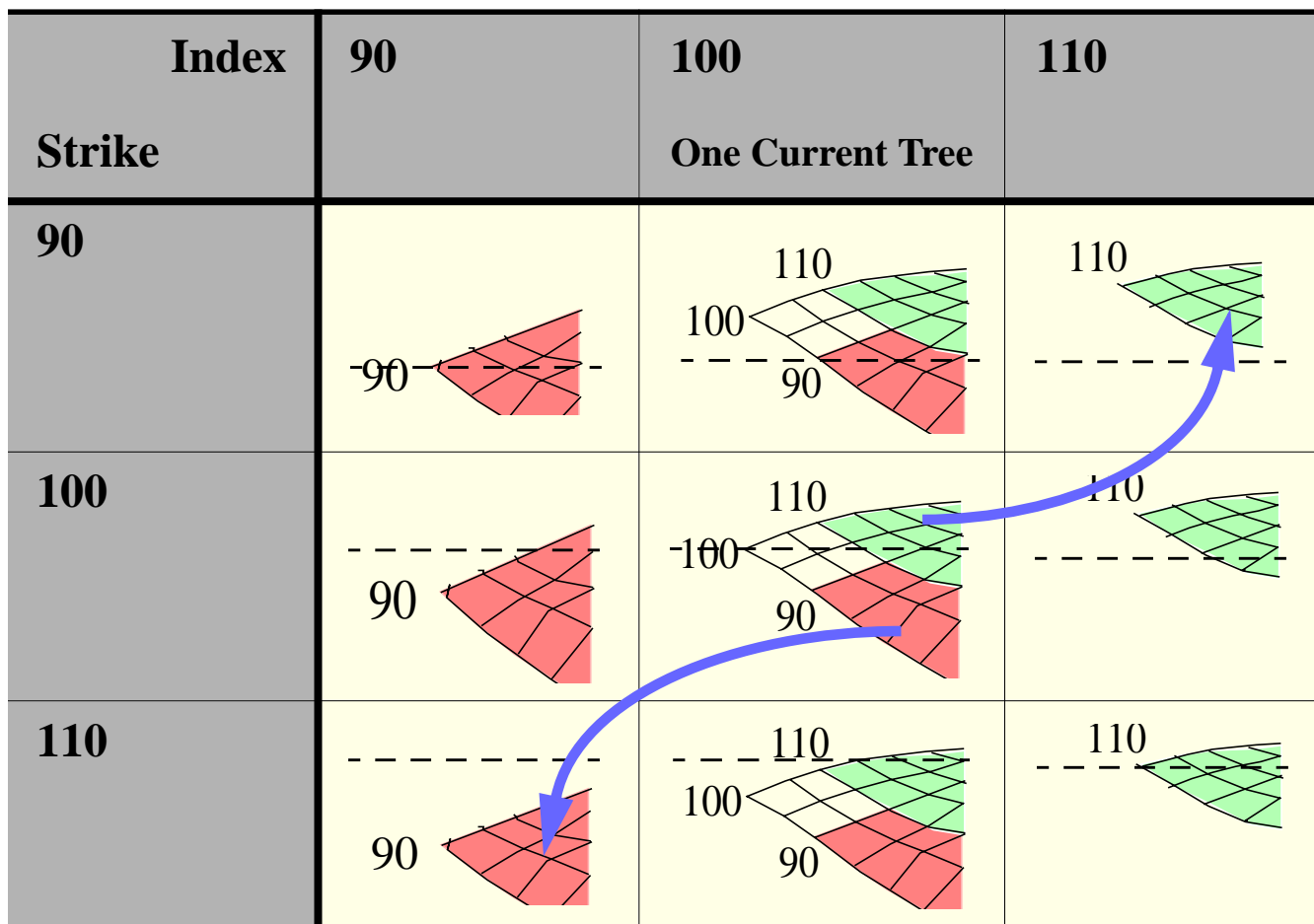


TABLE 8. Volatility behavior in the sticky-implicit-tree model.

Quantity	Behavior
Fixed-strike volatility:	decreases as index level increases
At-the-money volatility:	decreases twice as rapidly as index level increases
Exposure Δ :	$< \Delta_{BS}$

THE TRADING
PSYCHOLOGY BEHIND
THE MODELS

Table 9 summarizes the equations describing the three different models, and the behavior they dictate for implied volatilities and options exposures.

TABLE 9. The properties of the models.

Stickiness Model	Equation for $\Sigma(S, K, t)$	Behavior of		Delta
		Fixed-strike Option Volatility	At-the-money Option Volatility	
Strike	$\Sigma_{atm}(t) - b(t)(K - S_0)$	independent of index level	decreases as index level increases	$= \Delta_{BS}$
Delta	$\Sigma_{atm}(t) - b(t)(K - S)$	increases as index level increases	independent of index level	$> \Delta_{BS}$
Implied tree	$\Sigma_{atm}(t) - b(t)(K + S)$	decreases as index level increases	decreases twice as rapidly as index level increases	$< \Delta_{BS}$

Only the implied tree model provides a consistent (but not necessarily accurate) view of the world in which options prices are not arbitrageable. The other models are closer to heuristic rules than theories.

One can regard the implied volatility of an index option as reflecting the market's view of (at least) four future features, namely:

1. The behavior of the index: is it range-bound, trending or jumping?
2. The behavior of the realized volatility at which one will have to replicate an option: is it increasing, stable or decreasing?
3. The risk premium for hedging error⁶ and liquidity.
4. The impact of jumps (and fear of jumps) on future realized and implied volatilities.

Given a view on each of these aspects, there is a preferred model in Table 9 to use as an aid in modifying implied volatilities as the index moves. For each view below, I assume the current skew is negative.

6. In practice, the inevitably inaccurate estimation of future volatility and the discreteness of practical hedging pose significant risks.

**When To Use The
Sticky-Strike Rule**

Suppose you think that the index is in a regime where future moves are likely to be constrained to a trading range, without a significant change in current realized volatility.

Then, whatever the current skew, hedging costs and the risk premium will likely remain stable, and jumps are unlikely. Therefore, the simplest course is to preserve the current skew by leaving the implied volatility of every option unchanged.

**When To Use The
Sticky-Delta Rule**

Suppose you think that the index is in a regime where it is trending – that is, the index is undergoing some significant change in level, without a significant change in realized volatility.

Then, in the absence of a change in risk premium or an increased probability of jumps, the realized volatility will be the dominant input to the estimation of the implied volatility of (high-gamma) at-the-money options. As the index moves to new levels, it is sensible to re-mark the current at-the-money implied volatility to the value of the previous at-the-money volatility, because the realized volatility at which you need to hedge these high-gamma options remains unchanged. This means that 50-delta options will always have the same volatility.

The excess volatility for out-of-the-money strikes relative to at-the-money strikes reflects the risk premium for hedging errors and jumps, which we assume to be unchanged. Preserving this premium leads to the sticky-delta or sticky-moneyness strategy.

**When To Use The
Sticky-Implied-Tree
Model**

Suppose you think the index is in (or about to enter) a regime in which jumps are likely, especially downward jumps. You are then in a period of large potential index moves and increased realized volatility.

The implied tree at any instant represents the option market's view of exactly this likelihood of increased volatility on large downward index moves, as described in the section entitled **THE STICKY-IMPLIED- TREE MODEL: ONE INDEX, ONE TREE!** on page 13.

This is the right time to use the sticky-implied-tree model, adjusting individual volatilities up as the market declines and moving them back down as the market rises.

LOOKING BACK:
WHICH MODEL
REIGNED?

Fourteen months of data on S&P 500 option volatility during a turbulent period for index markets in Figure 3, and three different views of how volatility should change; what can we say about what happened?

Figure 8 displays the history, demarcated into regimes and annotated so as to allocate a dominant rule or model to each period.

Regime 1: Sticky strike. Prior to the October 27, 1997 collapse, the sticky-strike rule appears to be in effect: volatilities by strike are roughly constant, though some decline in the volatilities of individual strikes is noticeable towards the end of the period.

Regime 2: Sticky implied tree. A period of increased index volatility followed the October 27 downward jump in the index. This collapse in the market realized the fear of a downward move and a subsequent higher volatility expressed in the volatility skew. In this scenario, the sticky-implied-tree model is appropriate. Indeed, during this period, all option volatilities moved in the opposite direction to the index, and the appropriate hedge ratios for options were likely to be the implied tree hedges, lower than their corresponding Black-Scholes values.

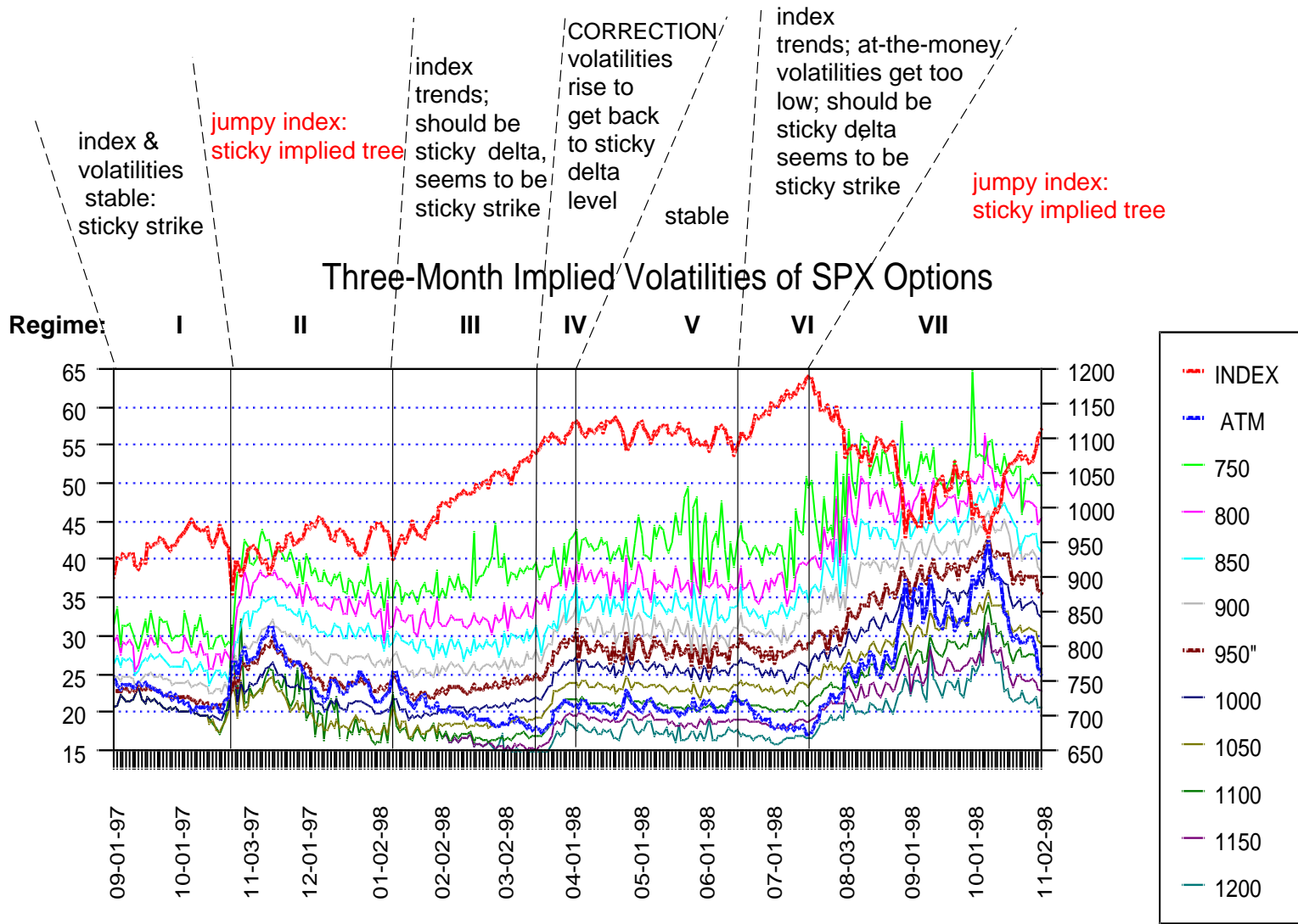
Regime 3: Sticky strike, but perhaps sticky delta would have been more appropriate. The index recovered from the turbulence of Regime 2, and now commenced a low-volatility, steady rise from about 950 to 1100. During this period, the sticky-strike rule reigned: the implied volatilities of individual strikes remain unchanged, except for small random oscillations. As a consequence, at-the-money volatility, denoted by the heavier blue line within the band of individual options volatilities, declined from about 25% to below 19%.

In this regime, the options market seemed to be intent on keeping volatility fixed, but interpreted this to mean keeping *each strike's volatility* fixed. In consequence, at-the-money volatility was steadily dragged down a stable skew curve as the index trended upward.

The decline in at-the-money volatility seems to have been irrational, or at best an oversight. At the end of the period, at-the-money, 1100-strike options were being issued at much lower volatilities than they were when the index was at 950. Was there some reason to think that realized volatility and/or the risk premium for at-the-money options was about to decline by six points?

If not, it would have been better to adopt the sticky-delta rule, as explained on page 19, and to have kept *at-the-money volatility* stable.

FIGURE 8. The regimes of three-month implied volatilities for S&P 500 options from September 1997 through October 1998.



Regime 4: A volatility correction. As the index approached 1100, the volatilities of all strikes climbed about five points in one week, even though the index continued to rise. By the end of the period, at-the-money volatility had returned to the approximate level it would have been at, had the sticky-delta rule been in effect for the past two regimes.

Regime 5: Stable index levels and volatilities. During this period the index remained in a trading range, and all volatilities were essentially unchanged. (The large observable fluctuations in 750-strike volatility were most likely a consequence of the illiquidity of the option, resulting in inaccurate prices at the market close.)

Regime 6: Sticky strike, but perhaps sticky delta would have been more appropriate. This seems to be a repetition of Regime 3. The index climbed from 1100 to close to 1200, but volatilities by strike remained unchanged, so that at-the-money volatility fell almost five points.

Again, there seems to have been no good reason to make markets in at-the-money options at such a relatively low level. The options market focused on keeping volatilities unchanged, but, in my view, mistakenly kept volatilities stable by strike, rather than keeping them stable by delta.

Regime 7: Sticky implied tree. In this final period, the index entered a period of high volatility and appreciable downward jumps. As explained in the section entitled WHEN TO USE THE STICKY-IMPLIED-TREE MODEL on page 19, it was now appropriate to employ the implied tree model, in which the volatility of each individual option varies inversely with index level and strike. During this period, you can see that the individual volatilities indeed moved up as the index declined, and then moved back down as the index recovered. Implied-tree hedge ratios are likely to have been more accurate here.

CONCLUSIONS

It helps to have theories, models or conjectures with which to view and describe data. The sticky-strike, sticky-delta and sticky-implied-tree models are intuitively useful ways of thinking about the patterns of variation in implied volatility that seem to correspond to modes of market behavior.

The word “volatility” is often used imprecisely. When market participants tell you, for example, that implied volatility increased as the market declined, it’s important to ask “Which volatility? At-the-money volatility or the volatility of a particular strike?” Usually, your portfolio contains particular strikes and expirations, and it’s their volatility that is relevant to your profit or loss.

At least in part, the negative skew reflects a fear of downward index moves and the increase in realized and implied volatilities that accompany it. Given the current skew, what is the appropriate way to modify implied volatilities as the index moves?

Looking back at S&P 500 volatilities since September 1997, one sees that the option market has subscribed to the sticky-strike rule when the index remained in a trading range (Regimes I and V).

In times of sudden downward index jumps (Regimes II and VII), the implied volatilities of all options tended to move inversely to the index, a feature of the sticky-implied-tree model. To me, this suggests that the current skew and the implied tree model are reasonable predictors of the future level of implied volatilities after a sudden downward index move.

During sustained smooth rallies in the index (Regimes III and VI), the options market seems to have decided to subscribe to the sticky-strike rule, as though it had whispered to itself: “Don’t change *individual* volatilities.” As the index ascended, this led to a steady decline in at-the-money implied volatility, sometimes culminating in a sudden upward volatility correction (Regime IV). Instead, most likely, the market should have insisted: “Don’t change *at-the-money* volatility!” The manner in which options market participants have adjusted their implied volatilities seems to have been most inappropriate when the index trended.

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