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Fast and unbiased estimator of the time-dependent Hurst exponent

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We combine two existing estimators of the local Hurst exponent to improve both the goodness of fit and the computational speed of the algorithm. An application with simulated time series is implemented, and a Monte Carlo simulation is performed to provide evidence of the improvement. *Published by AIP Publishing.* https://doi.org/10.1063/1.5025318

The estimation of the Hurst exponent of a time series is a recurring problem of great interest in many fields: finance, biology, hydrology, ecology, and signal processing, to quote a few. Since most estimators are asymptotic, large data samples are needed to obtain reliable estimates, and when the exponent changes through time, the estimates fail to capture the dynamics timely. In order to overcome this limit, we combine two techniques built using the quadratic variation estimators: the former is unbiased but displays a large variance; the latter, with a low variance, exhibits a bias which can be corrected at the cost of a computationally intensive procedure that slows down the estimation. Our simple and effective idea is to shift the biased estimates of the average difference between the unbiased and biased sequences. This removes the bias and improves the speed of the algorithm since it reduces the computations that must be carried out. Monte Carlo simulations for different sequences of the Hurst functional parameter show that (a) the estimates improve in almost all the cases considered and (b) our approach is computationally very efficient (the estimation time reduces drastically).

I. INTRODUCTION

It is well-known that the fractional Brownian motion $B_H(t)$ provides a powerful model for many natural as well as artificial phenomena in economics, medicine, and geoscience, to list a few. The fractional (or Hurst or Hölder) index H quantifies the smoothness of the sample paths of $B_H(t)$, and the problem of identifying this index is being widely investigated for years (see, e.g., Refs. 1–6). However, $B_H(t)$ has strong stationary increments, and this is too restrictive for many applications. To account for the complex dynamics exhibiting time-changing levels of magnitude, two generalizations of $B_H(t)$ were proposed independently by Péltier and Lévy Véhel[7] and Benassi *et al.*;[3] both these are called multifractional Brownian motions and denoted in the following by $B_{H(t)}(t)$. Multifractional processes have been widely used in turbulence analysis,[8–11] texture modeling, classification and segmentation,[12–14] medical image analysis (see Refs. 15–17 for a review of the literature), and finance modeling (see Refs. 18–20 or 21 for a review).

The pliancy that multifractional Brownian motion (mBm) exhibits in describing many complex dynamics has a counterpart in the difficult estimation of its functional parameter $H(t)$, a target that has attracted many contributions in the last twenty years. In the same seminal paper introducing the mBm, Péltier and Lévy Véhel discuss an estimator based on the average variation of the sampled process. Using the method defined by Benassi and Istas[22] for filtered white noise, Istas and Lang[2] build a convergent estimator of the local Hölder index and prove a central limit theorem. Benassi *et al.*[23] introduce a semi-parametric estimator for a piece-wise constant Hurst coefficient of a step fractional Brownian motion (SFBM), with the aim of detecting abrupt changes in the Hurst index for a Gaussian process with almost surely continuous paths. Coeurjolly[24] extends this result by introducing a local estimator of the second-order moment of a unique discretized filtered path and provides limit theorems for this class of functional estimators. His contribution allows us to manage Hölderian functions of arbitrary positive order. In the context of financial applications, Bianchi[25] and Bianchi *et al.*[26] extend to the mBm a class of estimators introduced for fBm by Péltier and Lévy Véhel[1] and study its Gaussian limiting distribution with a $\sqrt{\nu}\log n$ rate of convergence (here, ν and n denote the length of the estimation window and the number of sampling points, respectively). To identify the functional parameter of an even more general mBm (the generalized mBm), Ayache and Lévy Véhel[27–29] use the generalized quadratic variation and derived a central limit theorem for their estimator. More recently, Loutridis[30] proposed an algorithm based on the scaled window variance method for estimating both global and local scaling exponents and reported its simplicity and computational efficiency with respect to other techniques. Acting on convex combinations of sample quantiles of discrete variations, Coeurjolly[6] defines a class of consistent estimators and derives their almost sure convergence and their asymptotic normality. Finally, Garcin[31] uses the variational calculus to build a non-parametric smooth estimate for the estimator in Refs. 3 and 22 and claims his approach to be more accurate and easier than other existing non-parametric estimation techniques.

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The literature recalled above testifies about the interest towards the estimation of the functional Hurst exponent; nonetheless, many estimators suffer from several drawbacks, such as, e.g., their slow rate of convergence, their complexity, or even the arbitrary choice of some running parameters. As a consequence, some estimators can hardly be implemented. In this paper, we suggest to minimize the quadratic mean error of the sequence of differences between two estimates that already have proved to be fitting and robust, as well as rather simple to be implemented. We show that the resulting estimator both overperforms the previous estimates and increases the computational speed.

This paper is organized as follows: in Sec. II, the quadratic variation estimator is discussed along with a correction procedure used to reduce the bias of the estimator defined in Ref. 26. In this section, the procedure to improve the estimation is also introduced. Section III illustrates the Monte Carlo simulation run to study the effectiveness of the correction procedure and discusses the results. Section IV concludes this paper.

II. ESTIMATION OF THE HURST EXPONENT

Let us consider the multifractional Brownian motion (mBm) of functional Hurst parameter $H(t)$. It is well known that its non-anticipative moving average representation reads as

$$B_{H(t)}(t) = KV_{H(t)} \int_{\mathbb{R}} \left((t-s)_+^{H(t)-\frac{1}{2}} - (-s)_+^{H(t)-\frac{1}{2}} \right) dB(s), \quad (1)$$

where $V_{H(t)} = \frac{[\Gamma(2H(t)+1)\sin(\pi H(t))]^{1/2}}{\Gamma(H(t)+\frac{1}{2})}$ is a normalizing factor, K is a positive scaling parameter, and B is the Brownian motion. Notice that when $H(t) = H$, the fBm is recovered as the special case of the mBm.

The estimators of $H(t)$ described in Sec. II A are mostly based on the following relevant property of the mBm. For each $u \in \mathbb{R}^+$

$$\lim_{\epsilon \to 0^+} \frac{B_{H(t_0+\epsilon u)}(t_0+\epsilon u) - B_{H(t_0)}(t_0)}{\epsilon^{2H(t_0)}} \stackrel{d}{=} K\left(B_{H(t_0)}(u)\right), \quad (2)$$

where $\stackrel{d}{=}$ denotes the equality in distribution. Roughly speaking, the property states that at any point t_0, there exists an fBm of parameter $H(t_0)$ tangent to the mBm.[3] Therefore, recalling that the fBm is a Gaussian process, the infinitesimal increment of the mBm at any time t_0, normalized by $\epsilon^{2H(t_0)}$, is normally distributed with mean 0 and variance $K^2|u|^{2H(t_0)}$.

A. Quadratic variation estimator

The idea of estimating the functional parameter of the mBm by extending the methods available for the fBm is not new, but only a few estimators display a good rate of convergence, ensuring reliable estimates when H is allowed to change (even abruptly) through time. Of the many estimators that fit, more or less accurately, the fBm case[32–34] we will focus on the variation statistics,[2,4,23,24,35] which have been found to be more accurate than alternative techniques. For example, Storer *et al.*[34] conclude that the second order moment estimator is superior to Whittle's method, which in its turn outperforms—together with Haslett-Raftery—the estimators such as aggregated variance, boxed periodogram, difference variance, Geweke-Porter-Hudak estimator, Higuchi, Peng, periodogram, rescaled range, and wavelet.[33] For the above reason, in the following, we will analyze only the second order moment statistics.

With regard to the discrete version $\boldsymbol{X} = (X(i/n))_{i=1,..,n}$ of the mBm $\{B_{H(t)}(t), t \in [0,1]\}$, Coeurjolly[24] introduces a local version of the k-th order variation statistics that fits the case of a Hölderian function $H : t \in [0,1] \to H(t)$ of order $0 < \alpha \leq 1$ such that $\sup_t H(t) < \min(1, \alpha)$. Since the variance of the estimator is minimal for $k = 2$, in the following, the discussion will be referred only to the case $k = 2$. Given the two integers ℓ and p, a filter $\boldsymbol{a} := (a_0, \ldots, a_\ell)$ of length $\ell + 1$ and order $p \geq 1$ is built with the following properties:

$$\sum_{q=0}^{\ell} a_q q^r = 0, \text{ for } r = 0, \ldots, p-1 \quad \text{and} \quad \sum_{q=0}^{\ell} a_q q^p \neq 0.$$

The filter is a discrete differencing operator; for example, $\boldsymbol{a} = (1, -1)$ returns the discrete differences of order 1 of $\boldsymbol{X}$; $\boldsymbol{a} = (1, -2, 1)$ returns the second order differences, and so on. The filter also acts to make the sequence locally stationary and to weaken the dependence between the observations of $\boldsymbol{X}$ and defines the new time series

$$V^{\boldsymbol{a}}\left(\frac{j}{n}\right) = \sum_{q=0}^{\ell} a_q X\left(\frac{j-q}{n}\right) \text{ for } j = \ell+1, \ldots, n-1.$$

Given the neighborhood of t, $\mathcal{V}_{n,\varepsilon_n}(t) := \{j = \ell+1, \ldots, n : |j/n - t| \leq \varepsilon_n\}$ for some $\varepsilon_n > 0$ such that $n\varepsilon_n \to 0$ as $n \to \infty$ and denoted by $\nu := \nu_n(t)$, the number of observations in $\mathcal{V}_{n,\varepsilon_n}(t)$, and the quadratic variation statistics associated with the filter $\boldsymbol{a}$ is defined as

$$V_{n,\varepsilon_n}(t, \boldsymbol{a}) = \frac{1}{\nu} \sum_{j \in \mathcal{V}_{n,\varepsilon_n}(t)} \left\{ \frac{V^{\boldsymbol{a}}(j/n)^2}{\mathbb{E}(V^{\boldsymbol{a}}(j/n)^2)} - 1 \right\}. \quad (3)$$

Under some technical conditions on the form of the neighborhood $\mathcal{V}_{n,\varepsilon_n}(t)$ aiming at ensuring that it contains asymptotically an infinite number of points and to be of length asymptotically zero, it can be proved that $V_{n,\varepsilon_n}(t, \boldsymbol{a}) \to 0$ as $n \to \infty$. The asymptotic behavior of $V_{n,\varepsilon_n}(t, \boldsymbol{a})$ allows us to define estimators of $H(t)$ by acting in two directions:

(a) As the filter $\boldsymbol{a}$ is dilated m times, exploiting the local $H(t)$-self-similarity of $\boldsymbol{X}$, the linear regression of $\log \mathbb{E}(\frac{1}{\nu} \sum_{j \in \mathcal{V}_{n,\varepsilon_n}(t)} V^{\boldsymbol{a}^m}(j/n)^2)$ versus $\log m$ for $m = 1, \ldots, M$ defines the class of unbiased estimators $\hat{H}_{n,\varepsilon_n}(t, \boldsymbol{a}, M)$ independent of K whose variance is $\mathcal{O}((n\varepsilon_n)^{-1})$. In particular, Coeurjolly[24] proves that a filter of order $p \geq 2$ ensures asymptotic normality for any value of $H(t)$, whereas if $\boldsymbol{a} = (1, -1)$, the convergence holds if and only if $0 < \sup_t H(t) < \frac{3}{4}$. In this case, by taking $\varepsilon_n = \kappa n^{-\alpha} \ln(n)^\beta$ with $\kappa > 0$, $0 < \alpha < 1$, and $\beta \in \mathbb{R}$, it follows that $Var(\hat{H}_{n,\varepsilon_n}(t, \boldsymbol{a}, M)) = \mathcal{O}(\frac{1}{\kappa n^{1-\alpha} \ln(n)^\beta})$;

(b) By setting $p=1$ and $a=(1,-1)$, for $j=t-\nu,\dots,t-1$ and $t=\nu+1,\dots,n$, Eq. (3) becomes

$$V_{n,\varepsilon_n}(t,\boldsymbol{a})=\frac{\frac{1}{\nu}\sum_j|X_{j+1}-X_j|^2}{K^2(n-1)^{-2H(t)}}-1. \tag{4}$$

From this Refs. 25, 26, and 19, define the estimator

$$\hat{H}_{\nu,q,n,K}(t)=\frac{\ln\left(\frac{\sum_j|X_{j+q}-X_j|^2}{\nu-q+1}\right)}{2\ln\left(\frac{q}{n-1}\right)}-\frac{\ln K}{\ln\left(\frac{q}{n-1}\right)} \tag{5}$$

for $j=t-\nu,\dots,t-q$ and $q=1,\dots,\nu$.

When actual data are considered, K is generally unknown and—as it is evident from (5)—a misleading value reflects in estimates which are shifted with respect to the true values; the shift can be significant even for large n, with the logarithm being slowly varying at infinity. In Ref. 26, a prefiltering procedure (named **P1** hereafter) is proposed to estimate K: once an estimate has been obtained using an arbitrary K^*, because of the local normality of $B_{H(t)}(t)$ stated by (2), as $\rho\to0^+$, the conditional subsets

$$S_q(H^*,\rho)=\left\{X_{j+q}-X_j:\hat{H}_{\nu,q,n,K^*}(t)\in(H^*-\rho,H^*+\rho)\right\} \tag{6}$$

are normally distributed with mean zero and variance equal to $K^2(\frac{q}{n-1})^{2H}$, with

$$H=H^*+\frac{\ln(K^*/K)}{\ln(q/(n-1))}, \tag{7}$$

where the error term $\frac{\ln(K^*/K)}{\ln(q/(n-1))}$ accounts for the shifting bias. Recalling that for the Gaussian random variable Y, one has $\mathbb{E}(|Y|^k)=\frac{2^{k/2}\Gamma\left(\frac{k+1}{2}\right)}{\Gamma(1/2)}\sigma^k$, for each fixed H^*, it follows:

$$\ln\left(\frac{\sum_j|X_{j+q}-X_j|^k}{\nu-q+1}\right)=\ln\left[\frac{2^{k/2}\Gamma\left(\frac{k+1}{2}\right)\hat{K}^k}{\Gamma\left(\frac{1}{2}\right)}\right]+k\hat{H}\ln\left(\frac{q}{n-1}\right).$$

Therefore, K can be estimated through the intercept of a simple linear fit in the plane $\left(\ln(\frac{q}{n-1}),\ln\left(\frac{\sum_j|X_{j+q}-X_j|^k}{\nu-q+1}\right)\right)$ for increasing q values (and possibly different k values).

The algorithm is effective only when $S_q(H^*,\rho)$ is large enough to perform the log-linear regression in a reliable way; thus, the filtering procedure requires a large amount of data, and consequently, it is time-consuming. On the other hand, the main advantage is that when $H(t)=\frac{1}{2}$ and $q=1$, $Var(\hat{H}_{\nu,1,n,K}(t))=(2\nu\ln^2(n-1))^{-1}$ (proof in Ref. 25, pp. 266–267); hence, the rate of convergence of estimator (5) is $\mathcal{O}(\nu^{-\frac{1}{2}}(\ln n)^{-1})$, which ensures great accuracy even for small estimation windows ν and is also useful to preserve the rationality of the assumption of local Gaussianity.

Another way to get rid of the parameter K is described in Refs. 2, 22, and 31, where the numerator of (4) is calculated using different resolutions (for example, by halving the points into the estimation window), that is,

$$M_2(t,a)=\frac{1}{\nu}\sum_{j=0}^{\nu-1}|X_{j+1}-X_j|^2,\quad\text{and}$$

$$M_2'(t,a)=\frac{2}{\nu}\sum_{j=0}^{\nu/2-1}|X_{2(j+1)}-X_{2j}|^2$$

for $a=(1,-1)$, or

$$M_2(t,a)=\frac{1}{\nu-1}\sum_{j=0}^{\nu-2}|X_{j+1}-2X_j+X_{j-1}|^2,\quad\text{and}$$

$$M_2'(t,a)=\frac{2}{\nu-1}\sum_{j=0}^{\nu/2-2}|X_{2(j+1)}-2X_{2j}+X_{2(j-1)}|^2$$

for $a=(1,-2,1)$.

Since as n tends to ∞, $M_2(t,a)$ and $M_2'(t,a)$ tend to $K^2(n-1)^{-2H(t)}$ and $K^2(\frac{n-1}{2})^{-2H(t)}$, respectively, their ratio tends to $2^{2H(t)}$, from which an estimate of the Hurst exponent which converges almost surely to $H(t)$ is

$$\hat{H}_{\nu,n}(t,a)=\frac{1}{2}\log_2\left(\frac{M_2'(t,a)}{M_2(t,a)}\right). \tag{8}$$

With the rate of convergence discussed in (a), this technique leads to erratic estimates, to the point that recently Garcin[31] proposed a non-parametric smoothing technique to reduce the noise.

Remark 1. The two approaches cannot eliminate trends in the data. If this is not a problem for the mBm, where the asymptotic behavior in (2) holds, for more general processes, the presence of trends could be identified by comparing the estimates $\hat{H}_{\nu,n}(t,a)$ for different filters a. For example, if $\hat{H}_{\nu,n}(t,(1,-1))$ and $\hat{H}_{\nu,n}(t,(1,-2,1))$ agree, trends are probably not significant since they usually affect first and second order derivatives.

Remark 2. Comparing fluctuations only on very short timescales could be misleading, as correlations on these scales can be affected by additional short-term correlations. Thus, a further analysis would be needed to evaluate the effect on the estimator of possible short-term correlations as well as the crossover in the scaling behavior of the data.

B. Improved estimator

As discussed above, estimator (8) is unbiased but affected by a large variance. On the other hand, estimator (5) exhibits a low variance, but the bias due to the unknown K needs a computationally intensive correction that slows down the estimation. Figure 1 displays an example of the estimated values $\hat{H}_{\nu,q,n,K^*}(t)$ (black), the unbiased $\hat{H}_{\nu,q,n,K}(t)$ (blue), and $\hat{H}_{\nu,n}(t)$ (grey) for an mBm simulated with sinusoidal functional parameter $H(t)$ (red). The toilsome procedure **P1** used to remove the bias of a wrong K^* can be

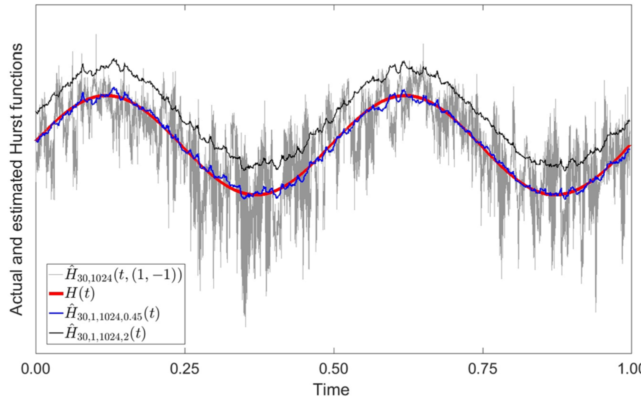

FIG. 1. Actual (red line) and estimated Hurst parameters.

avoided in a very simple and effective way. By exploiting the unbiasedness of estimator (8), denoted by $H(t)$ the functional parameter to be estimated and by ξ a zero-mean random variable, one can write

$$\hat{H}_{\nu,n}(t,a) = H(t) + \xi(t). \tag{9}$$

On the other side, from (7), it is also

$$\hat{H}_{\nu,q,n,K^*}(t) = H(t) - \frac{\ln(K^*/K)}{\ln(q/(n-1))}. \tag{10}$$

Therefore

$$\frac{\ln(K^*/K)}{\ln(q/(n-1))} = \hat{H}_{\nu,n}(t,a) - \xi(t) - \hat{H}_{\nu,q,n,K^*}(t),$$

from which, by averaging with respect to t, it immediately follows:

$$h := \frac{\ln(K^*/K)}{\ln(q/(n-1))} = \frac{1}{n}\sum_{t=1}^{n}\Big(\hat{H}_{\nu,n}(t,a) - \hat{H}_{\nu,q,n,K^*}(t)\Big).$$

Notice that since K^* is chosen arbitrarily, the corrected estimate

$$\hat{H}_{\nu,q,n}(t,a) = \hat{H}_{\nu,q,n,K^*}(t) + h \tag{11}$$

does not depend on K any longer.

III. SIMULATIONS AND RESULTS

A. Simulations of mBm

The advantage of using the correction defined in (11) can be proved by a Monte Carlo simulation of mBm with different functional parameters and lengths, in order to stress the estimation algorithms. In this regard, with respect to the support $[0,1]$, we have considered three functional parameters (linear ($H_L \in [0.2, 0.8]$), sinusoidal ($H_S \in [0.2, 0.8]$), and a Brownian motion ($H_B \in [0.25, 0.75]$)) and five lengths ($n = 2^p$, for $p = 10, 11, 12, 13, 14$). For each functional parameter and length, 1,000 samples of mBm were simulated using the Chan and Wood algorithm.[36] All the computations were run using Matlab 2017b under Windows 10 Pro, with processor Intel Core i3-350M, (3 Mb Cache, 2.26 GHz).

TABLE I. Root mean squared error (RMSE) and mean absolute percentage error (MAPE) of the simulations. Bold face denotes the best estimate with respect to RMSE or MAPE

		RMSE			MAPE		
			P2			P2	
	p	P1	$(1,-1)$	$(1,-2,1)$	P1	$(1,-1)$	$(1,-2,1)$
H_L	10	0.0511	**0.0059**	0.0239	11.7211	**1.3362**	5.4753
	11	0.0220	**0.0042**	0.0246	5.0574	**0.9316**	5.6599
	12	0.0068	**0.0037**	0.0254	1.5541	**0.8274**	5.8474
	13	0.0035	**0.0027**	0.0257	0.7818	**0.6025**	5.9275
	14	0.0148	**0.0025**	0.0261	3.4158	**0.5008**	5.9905
H_S	10	0.0218	**0.0074**	0.0294	5.0884	**1.8885**	7.0354
	11	0.0092	**0.0067**	0.0283	2.2006	**1.6812**	7.0118
	12	**0.0012**	0.0079	0.0260	**0.1943**	1.9922	6.4694
	13	0.0111	**0.0074**	0.0257	2.7618	**1.8324**	6.4389
	14	0.0155	**0.0079**	0.0257	3.8681	**1.9683**	6.4224
H_B	10	0.0596	**0.0324**	0.0625	10.7261	**5.1473**	11.3086
	11	0.0328	**0.0137**	0.0285	6.6526	**2.3564**	5.7133
	12	**0.0151**	0.0188	0.0364	**2.2151**	2.9367	6.9215
	13	**0.0215**	0.0300	0.0416	**2.6564**	4.2071	6.9641
	14	**0.0161**	0.0257	0.0391	**2.6524**	3.6956	6.8349

If the linear and the sinusoidal functions appear as standard choices, assuming a rescaled Brownian motion deserves a few comments: (a) the jaggedness of the parameter represents a stress test for the estimator as compared to the very smooth linear or sinusoidal functions; (b) it allows us to study the error of the estimator with respect to real-world situations, where the empirical estimates for several financial time series display highly irregular and mean reverting Hurst functions ranging from about 0.25 to 0.75.[19,37–39]

For each surrogated series, we have estimated $\hat{H}_{\nu,q,n,K^*}(t)$ and $\hat{H}_{\nu,n}(t,a)$ with $\nu = 30, q = 1, K^* = 2$, and $a \in \{(1,-1),(1,-2,1)\}$ (for the choice of these parameters, see, e.g., Refs. 40 and 31). Finally, we have applied procedure **P1** to calculate $\hat{H}_{\nu,q,n,K}(t)$ and procedure **P2**, with both $a = (1,-1)$ and $a = (1,-2,1)$, to calculate $\hat{H}_{\nu,q,n}(t,a)$.

B. Discussion of results

The results of the Monte Carlo simulations are summarized in Tables I and II, which display the averages over the samples of the Root Mean Squared Error (RMSE) and the Mean Absolute Percentage Error (MAPE) and the computational times, respectively. We observe the following:

(a) As shown in Table I, procedure **P2** outperforms significantly the estimates in almost all the cases when

TABLE II. Computational times (in seconds).

		$p=10$	$p=11$	$p=12$	$p=13$	$p=14$
H_L	P1	1.6164	2.4637	4.7125	8.6164	23.4242
	$\mathrm{P2}_{(1,-1)}$	0.0434	0.0648	0.1094	0.1976	0.3764
H_S	P1	1.2357	2.3985	4.2874	8.3943	15.7677
	$\mathrm{P2}_{(1,-1)}$	0.0426	0.0686	0.1076	0.2102	0.3804
H_B	P1	1.2805	2.3418	4.3117	8.4367	30.4982
	$\mathrm{P2}_{(1,-1)}$	0.0434	0.0651	0.1119	0.2013	0.3857

$a = (1, -1)$. On the other hand, when $a = (1, -2, 1)$, procedure **P1** reveals higher performance. The improvement of $\hat{H}_{\nu,q,n}(t, (1, -1))$ with respect to $\hat{H}_{\nu,q,n,K}(t)$ is particularly effective for small lengths (when $n = 1{,}024$, the MAPE reduces of almost 9 (H_L), 2.7 (H_S), and 2.1 (H_B) times; when $n = 2{,}048$, the MAPE reduces of almost 5.4 (H_L), 1.3 (H_S), and 2.8 (H_B) times). As an example, Fig. 2 displays the gain obtained by estimating $H_B(t)$ through $\hat{H}_{\nu,q,n}(t, (1, -1))$ (green line), which fits the actual functional parameter (red line) much better than $\hat{H}_{\nu,q,n,K}(t)$ (blue line). Only for very large lengths and mean reverting functional parameters, procedure **P1** outperforms procedure **P2**. This is reasonable and can be explained by considering that, in these cases, the number of points needed to rebuild the conditional distributions of procedure **P1** is large enough to ensure a reliable estimation of the (constant) parameters;

(b) As shown in Table II, in all cases, the computational times become essentially negligible when procedure **P2** with $a = (1, -1)$ is considered. The reduction, increasing with the length n, is very large even for small samples: on average, the computational time is about 32 times less when $n = 1{,}024$ and grows to about 61 times less when $n = 16{,}384$. Since the filter $a = (1, -1)$ outperforms the filter $a = (1, -2, 1)$, only the case $\mathbf{P2}_{(1,-1)}$ was chosen as the benchmark with respect to **P1** in the analysis of the time reduction.

IV. CONCLUDING REMARKS AND FUTURE DEVELOPMENTS

In order to improve the estimation of the Hurst functional exponent of an mBm, we have proposed to merge two existing estimators, one unbiased, but with a large variance and the other biased but with a low variance. The bias of the latter estimator can be removed at the cost of a toilsome and time-consuming procedure defined in Ref. 26. By minimizing the quadratic mean error of the sequence of differences between the two estimates, in this paper, we have built a fast, unbiased, and low-variance estimator. We have performed a Monte Carlo simulation to test the improvement with different sample sizes and functional parameters. As for the computational speed, our algorithm significantly reduces the times required by the previous correction procedure proposed to remove the bias. As to the goodness of fit, the new technique outperforms the previous estimates for short sample sizes and for any functional parameter, while it is essentially equivalent to the low-variance estimator for long sample sizes. Possible future developments concern:

- the reduction of the echo effect due to the size of the moving window used to estimate the functional parameter "pointwise," by properly weighting the observations to reduce the impact of the most far data with respect to the most recent ones;
- the elimination of possible trends in the data, which could be identified by comparing the estimates $\hat{H}_{\nu,n}(t, a)$ for different filters a;
- the exploration of possible short-term correlations or crossover in the scaling behavior of the data which cannot be accounted for by comparing fluctuations only on very short timescales, as in (8).

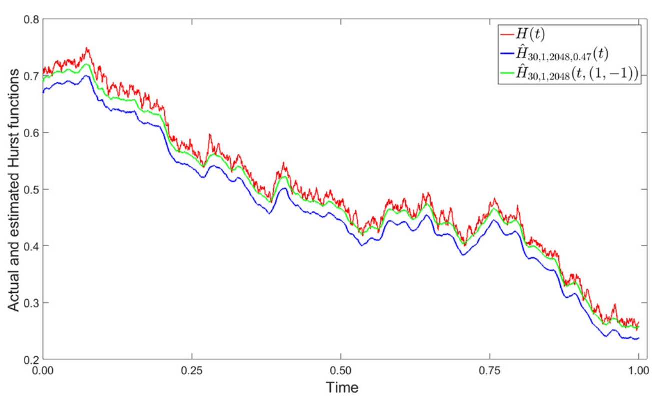

FIG. 2. Improvement obtained by replacing $\hat{H}_{\nu,q,n,K}(t)$ (**P1**) with $\hat{H}_{\nu,q,n}(t, (1, -1))$ (**P2**).

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