Repricing the Cross Smile: An Analytic Joint Density

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First draft: 14th March 2009. This draft: 17th May 2011.

Abstract

Derivative contracts on multiple foreign exchange rates must be priced to avoid arbitrage by contracts on the cross-rates. Given the triangle of smiles for two underlyings and their cross, we provide an analytic formula for a joint probability density such that all three vanilla markets are repriced. The method extends to N dimensions and leads to simple necessary conditions for a triangle of smiles to be arbitrage-free in the model.

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1 Introduction

When valuing a derivative contract whose payoff depends on two assets, the correlation between the random processes followed by those two assets must be taken into account. In most asset classes, there is no liquid instrument to determine that correlation. This makes the exposure to correlation hard to hedge, but straightforward from a modelling point of view since a single number, perhaps calculated from a historic time series of spot returns, can be used.

Foreign exchange is different. Lets consider the concrete case of a contract involving the euro-dollar and sterling-dollar exchange rates at a given expiry, T. We denote the spot rates S_1 and S_2 respectively. Then since the euro-sterling exchange rate $S_3 = S_1/S_2$ is also liquidly traded, any model we use would need to correctly re-price euro-sterling vanilla options in order to avoid arbitrage. This places a heavy constraint on the choice of correlation between the two driving assets. We will call the currency pairs which, for the purpose of our modelling, we use as a basis for exchange rates the *driving pairs* and the other exchange rates which can be determined from them the *crosses*. In this case, euro-dollar and sterling-dollar are the drivers, and euro-sterling is the cross.

In the Black-Scholes model [1], the condition relating the correlation to the volatilities is well known to all FX analysts. We assume that the drivers all have a common domestic currency; dollars in our example. Then, choosing the dollar bond as numeraire, the Black-Scholes processes are

$$S_1 = F_1 e^{-\frac{1}{2}\sigma_1^2 T + \sigma_1 \sqrt{T} X_1} \tag{1}$$

$$S_2 = F_2 e^{-\frac{1}{2}\sigma_2^2 T + \sigma_2 \sqrt{T} X_2} \tag{2}$$

$$E[X_1 X_2] = \rho \tag{3}$$

where F_i are the forwards, σ_i the volatilities, and X_i normal variables with correlation ρ.

The risk-neutral process for the cross is given by the quotient $S_3 = S_1/S_2$ which is log normal with volatility σ_3 as long as

$$\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2 = \sigma_3^2. \tag{4}$$

This formula is known as the triangle rule. As long as it holds, the two asset Black-Scholes model (1)-(3) will give correct Black-Scholes prices for European contracts that depend only on the cross rate S_3.

It is standard practice among market participants to back correlations out from at-the-money volatilities using the triangle rule. As long as options on the cross are traded liquidly, this allows us to replace exposure to correlation risk with a vega that can be hedged easily.

Once we have a correlation, all of the standard multi-asset methods are available to us. For example, we could simulate S_1 and S_2 with two correlated local volatility [2] processes. Or, for a European payout, we could construct a Gaussian copula. We can then do the experiment of using our chosen method to value the contract

$$(S_1/S_2 - K)_+ S_2 \tag{5}$$

which is actually just a vanilla option on the cross, with the payout converted into USD at expiry. We can do this for a number of strikes, and so plot the smile implied for the cross by our model. It will not match the true market smile for the cross, being typically too flat. Worse, there is no reason it would re-price even the at-the-money option correctly.

Thus we have an arbitrage which is likely to lead to a bleed of money through the life of the trade as we vega-hedge the cross. If some part of the trade, perhaps hidden to us, amounts to a European payout on the cross alone, we will be in danger of selling a trade that can be arbitraged at inception.

In order to address this issue, we would like to construct a joint probability density with the property that vanillas on drivers and cross are correctly repriced. This problem has been tackled by the authors of [3] by constructing a copula that can be calibrated numerically to the cross smile (and see [4] for a review of copula methods in FX). The purpose of the present article is to construct such a density analytically. Our construction provides natural no-arbitrage conditions on the triangle of smiles, and it extends to N dimensions providing a joint probability distribution that matches all FX asset- and cross-smiles. Certain contracts (best-ofs, worst-ofs and multi-asset digitals) can be valued analytically in the model. Others can be priced with numerical integration, including for example quanto options whose valuation can be rather elusive [5].

2 Special properties of best-of options

In order to construct the pdf, we are going to study best-of options. A best-of has the property that, depending on spot movements, its value can converge to that of a vanilla option on any of the assets in the triangle. We will be able to postulate a valuation formula for best-ofs that respects the known vanilla prices in its limits, and use this to construct our pdf.

The payoff of a best-of option is

$$P = \max\left\{\frac{(S_1 - K_1)_+}{K_1}, \frac{(S_2 - K_2)_+}{K_2}\right\}. \tag{6}$$

At expiry, we check which asset performed best, and pay a vanilla contract on it.

If at any time S_1 becomes strongly out-of-the-money while S_2 does not, then the contract reduces to a vanilla on S_2 with strike K_2. Likewise, in opposite circumstances, it can become a vanilla on S_1 with strike K_1.

What, though, if both S_1 and S_2 become strongly in-the-money? Then the optionality becomes a choice between S_1 and S_2. In that case, the payout can be expressed as

$$\begin{aligned}
\max\left\{\frac{(S_1 - K_1)_+}{K_1}, \frac{(S_2 - K_2)_+}{K_2}\right\} &= \max\left\{\frac{(S_1 - K_1)}{K_1}, \frac{(S_2 - K_2)}{K_2}\right\} &(7)\\
&= \frac{S_2}{K_1}\max\left\{\frac{S_1}{S_2}, \frac{K_1}{K_2}\right\} - 1 &(8)\\
&= \frac{S_2}{K_1}\left(\frac{S_1}{S_2} - \frac{K_1}{K_2}\right)_+ + \frac{S_2}{K_2} - 1, &(9)
\end{aligned}$$

which is a call option on the cross with strike $K_3 = K_1/K_2$ together with a forward contract.

Then depending on spot movements, a best-of option can become arbitrarily close in value to a vanilla on either asset or on the cross. The conclusion is that any model used to value a best-of option must correctly re-price vanillas on the drivers and the cross, otherwise the valuation will be directly arbitragable with a vanilla contract.

Suppose we know the value of best-of options for all possible strikes K_1 and K_2. Does this also imply complete knowledge of the risk-neutral joint pdf? As we will now see, it certainly does.

By differentiating the best-of payoff P (6), we obtain the dual digital payoff

$$\left[1 + K_1 \frac{\partial}{\partial K_1} + K_2 \frac{\partial}{\partial K_2}\right] P + 1 = 1_{\{S_1 < K_1, S_2 < K_2\}}, \tag{10}$$

whose undiscounted value is the bivariate cumulative distribution. Two more differentiations give us the probability density. Then, if we denote the undiscounted value of the best-of with strikes K_1, K_2 by $B(K_1, K_2)$, and the joint pdf by f, they are related by

$$\frac{\partial^2}{\partial K_1 \partial K_2} \left[K_1 \frac{\partial}{\partial K_1} + K_2 \frac{\partial}{\partial K_2} + 1\right] B(K_1, K_2) = f(K_1, K_2). \tag{11}$$

In summary, there are two properties making best-of options special

1. A best-of with strikes K_1, K_2 can reduce into a vanilla on either of the drivers, or on the cross with strike $K_3 = K_1/K_2$.
2. The set of all possible best-of options completely defines the market for European options.

3 Construction of a joint pdf

If we could find a smooth function $B(K_1, K_2)$ which reproduces the market values of the vanillas in the appropriate limits, then equation (11) tells us this is equivalent to finding a candidate joint pdf $f(K_1, K_2)$. As long as f satisfies all the usual necessities of a probability density function, we will have a mechanism for valuing multi-asset European contracts that cannot be arbitraged against the vanilla market.

In order to write down an appropriate function B, we make use of the fact that best-ofs can be valued analytically in Black-Scholes. The (undiscounted) value is [6, 7]

$$\begin{aligned} B(K_1, K_2; \sigma_1, \sigma_2, \sigma_3) &= \frac{F_1}{K_1} N(d_1^+, d_3^+; \rho_{13}) \\ &+ \frac{F_2}{K_2} N(d_2^+, -d_3^-; \rho_{23}) \\ &+ N(-d_1^-, -d_2^-; \rho_{12}) - 1 \end{aligned} \tag{12}$$

where N is the bivariate normal function, and the correlations are given by

$$\rho_{12} = \frac{\sigma_1^2 + \sigma_2^2 - \sigma_3^2}{2\sigma_1\sigma_2} \tag{13}$$

and cyclic permutations. The parameters d_i are defined by

$$d_i^{\pm} = \frac{\log(F_i/K_i) \pm \sigma_i^2 T/2}{\sigma_i\sqrt{T}}, \quad i = 1, 2, 3, \tag{14}$$

where $K_3 = K_1/K_2$ is the cross strike. A compact derivation of equation (12) is given in [8].

We can now set up our function $B(K_1, K_2)$ that matches all three cross smiles in the limits. We simply plug into equation (12) volatilities taken from the smile curve of each asset

$$\sigma_i = \sigma_i(K_i). \tag{15}$$

By choosing

$$B(K_1, K_2) = B(K_1, K_2; \sigma_1(K_1), \sigma_2(K_2), \sigma_3(K_3)), \tag{16}$$

Black-Scholes ensures for us that $B(K_1, K_2)$ correctly reprices vanillas on all three assets. It is easy to check this by taking the appropriate limits of equation (12) and using the no-arbitrage condition for individual smiles [9]

$$\sigma_i(K_i)^2 = o(|\log K_i/F_i|) \text{ as } K_i \to 0, \infty \tag{17}$$

to ensure that the $d_i^{\pm}$ behave.

Then our pdf is given by

$$f(K_1, K_2) = \frac{\partial^2}{\partial K_1 \partial K_2} \left[K_1 \frac{\partial}{\partial K_1} + K_2 \frac{\partial}{\partial K_2} + 1 \right] B(K_1, K_2) \tag{18}$$

and, by construction, it correctly re-prices vanillas at all strikes on all three assets.[2]

[2] One can also go the other way and check that integrating the payoff (6) against the pdf (18) reproduces $B(K_1, K_2)$. Those who wish to do so are recommended to begin instead by integrating against the *worst-of* payout which makes handling boundary terms easier when integrating by parts.

4 Extension to N Dimensions

The construction extends readily to more than 2 dimensions. When there are N driving assets, we consider the best-of payoff

$$P_N = \max_{1 \le i \le N} \left\{ \frac{(S_i - K_i)_+}{K_i} \right\}. \tag{19}$$

For the method to work, we need to check that the two special properties of best-ofs continue to apply: they must be reducible to a vanilla on any asset or cross, and they must completely define the market for European options.

To check the first property, we can pick any two assets, say S_1 and S_2. The N dimensional payoff (19) reduces to a best-of on those two assets by taking the other strikes large. Then we are back to the 2 dimensional case, and the payoff can be further reduced to a vanilla on either asset or the cross as before. In this way we see that a vanilla on any asset or cross can be obtained by taking appropriate limits.

For the second property, we need to check that knowledge of all N-asset best-of values is equivalent to knowledge of the probability distribution. Differentiating,

$$\left[1 + \sum_{i=1}^{N} K_i \frac{\partial}{\partial K_i} \right] P_N + 1 = 1_{\{S_1 < K_1\}} \cdots 1_{\{S_N < K_N\}} \tag{20}$$

we obtain the N-asset digital payoff, whose expected value is equal to the N-variate cumulative distribution function, giving the result.

Then, as before, we plug smile volatilities into the Black-Scholes formula for the N-asset best-of[3] at strikes K_i for the ith asset and K_i/K_j for the ij cross. Black-Scholes ensures for us that in the limits all vanillas are correctly repriced. Therefore when we differentiate we obtain a probability distribution that reprices all vanillas correctly.

[3] The best-of formula (12) generalises naturally to N assets. In practice, a high quality approximation for the trivariate cumulative normal function is known [10], but for $N > 3$, numerical techniques may be needed.

5 Example: Two-asset Basket

As an example, we will look at a call option on a basket of euros (EUR) and Japanese yen (JPY) against dollars (USD). The basket is initially struck at-the-money-forward on 14th April 2008 so that the payoff on expiry date 14th April 2009 is

$$P = N_{\mathrm{USD}}(0.5S_1/K_1 + 0.5K_2/S_2 - 1)_+ \tag{21}$$

where S_1 and S_2 are respectively the EURUSD and USDJPY spots[4] at expiry, and the strikes $K_1 = 1.56$ and $K_2 = 99.2$ are set equal to the one year forwards on the inception date in order to make the trade at-the-money. N_{USD} is the notional of the trade, and has been set equal to 100,000,000 USD in this example.

We can simplify the valuation of a two-asset basket by integrating the payoff $(K - w_1S_1 - w_2S_2)_+$ against the density (18) by parts twice. The second derivative of the payoff becomes a delta function, and so we can do one of the integrals analytically giving undiscounted value

$$V_{Basket} = \int_0^K dU\, C\left(\frac{U}{w_1}, \frac{K-U}{w_2}\right) \tag{22}$$

where $C(S_1, S_2)$ is the joint cumulative distribution.

It is market standard to quote foreign exchange smiles in delta space since one can use the Black-Scholes formula to map between strike and delta for a given volatility. Figure 1 shows the volatility smiles at inception plotted against (absolute value of) put delta, and figure 2 shows the joint probability density given by equation (18).

The proper test of a model is whether it works in practice. Figure 3 shows the result of running the trade through a back-tester that simulates hedging for the life of the trade. Each day, closing market data is loaded from the Barclays Capital database, and vanilla options put into a portfolio at 10- and 25-delta call and put strikes and at-the-money, to eliminate smile and vega risk. Then the residual delta is hedged with forward contracts which, since the trade is European, also hedge the interest rate risk. Finally the cash is

[4] I have chosen to quote USDJPY in its market convention, rather than as JPYUSD which would be more natural for this basket.

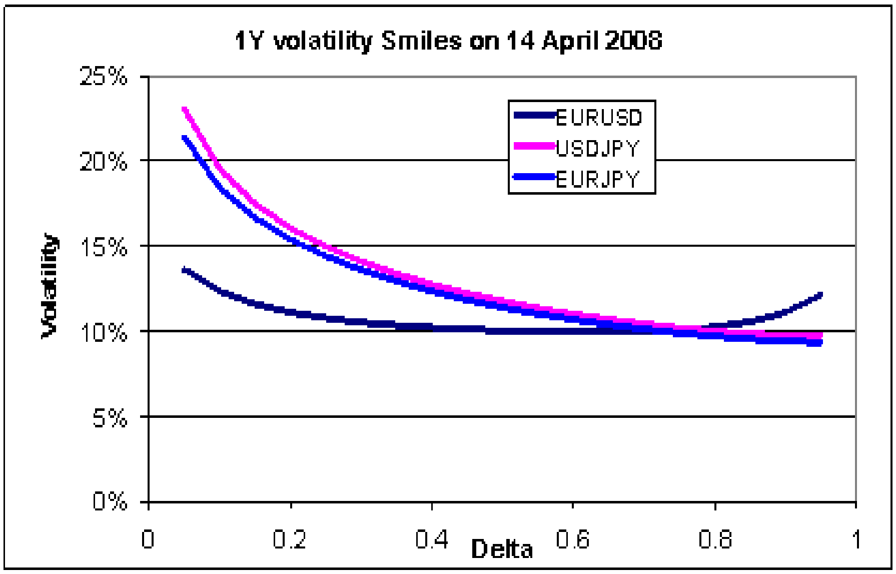

Figure 1: 1Y smiles on 14th April 2008.

put in a bank account and earns interest over night. Then next day the old hedges are unwound and new hedges put on and any profit or loss recorded.

The feint lines show what happened to the EURUSD and USDJPY spot rates over the year, the right hand axis showing their percentage change. The three bold lines show the value of the portfolio over the life of the trade. The aim of a good model and perfect hedging is to keep these lines as close to zero as possible. As a simple measure of the performance of each model, we can look at the final value of the portfolio, indicating the amount of money lost (or unintentionally made) during the life of the trade. Ignoring smile and using Black-Scholes to model the basket caused the strategy to make around 870,000 USD (or 87 basis points) during its life. Using a Gaussian copula with correlation calibrated to reprice the at-the-money cross-vanilla performed better at around 37 basis points, but the implied probability density we have constructed performed significantly better at 12 basis points.

By using a model that correctly reprices all three smiles, the risks from the model tell us to put on hedges that are much closer to a replicating portfolio than otherwise, and so we have been better protected from market moves through the life of the trade.

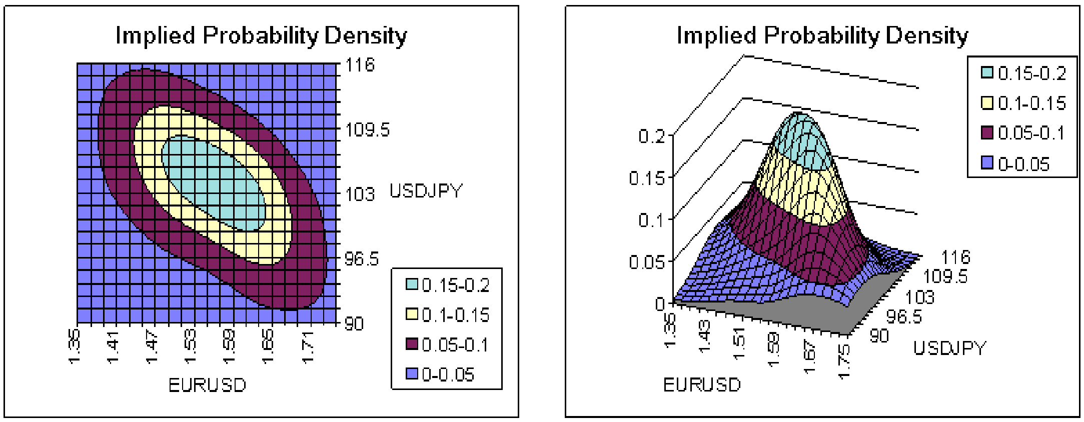

Figure 2: 1Y implied pdf on 14th April 2008.

6 Discussion

Given a triangle of foreign exchange assets, we have constructed an analytic formula for a joint probability density with the property that the vanilla markets on all three assets are repriced. The pdf has been constructed by using special properties of best-of contracts.

While, since it reprices vanillas, the density is guaranteed to integrate to 1, it is not certain to be real and positive for all input smiles. This is not surprising as the functional form of any two smiles in a triangle will impose some constraints on the smile possible for the cross. The condition for f to be real is that the correlation

$$\rho_{12}(K_1, K_2) = \frac{\sigma_1^2(K_1) + \sigma_2^2(K_2) - \sigma_3^2(K_3)}{2\sigma_1(K_1)\sigma_2(K_2)} \tag{23}$$

must satisfy

$$-1 < \rho_{12}(K_1, K_2) < 1\,, \quad \text{for all } K_1, K_2, \tag{24}$$

and as long as this constraint is true then both ρ_{23} and ρ_{13} are also good. In terms of volatilities, this constraint becomes

$$\sigma_1(K_1) + \sigma_2(K_2) \;>\; \sigma_3(K_3) \tag{25}$$
$$\sigma_2(K_2) + \sigma_3(K_3) \;>\; \sigma_1(K_1) \tag{26}$$
$$\sigma_1(K_1) + \sigma_3(K_3) \;>\; \sigma_2(K_2) \tag{27}$$

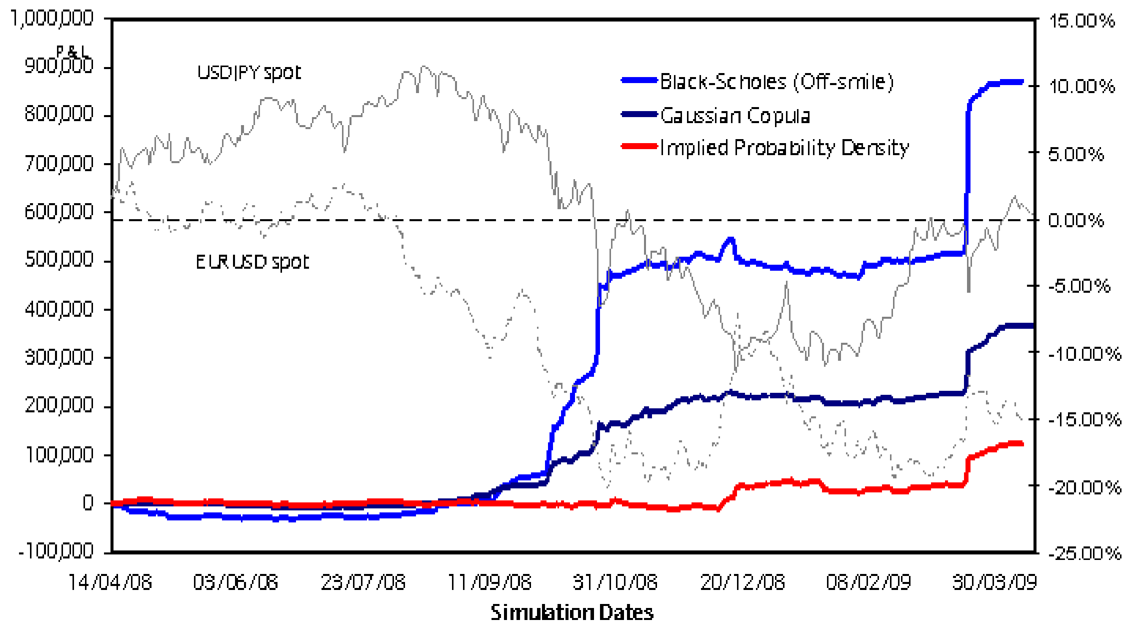

Figure 3: Hedging a one year call option on a basket of EUR and JPY versus USD over the course of its lifetime

for all K_1, K_2 and $K_3 = K_1/K_2$.

Equations (25)-(27) are a necessary condition for the model to be arbitrage free, and provide a tool for vanilla traders to avoid arbitrage in a triangle of currency pairs.

Certain two- and three-asset derivative contracts can be valued analytically in the model. Best-of and worst-of options are examples, as are multi-asset digitals since we have an analytic formula for the cumulative probability distribution. Other contracts can be valued semi-analytically by integrating against the pdf.

The model can be compared with a copula, since it joins two probability distributions, defined by the two driver smiles. Unlike standard copulas, rather than having a number of parameters that define the underlying distribution, it can fit to an entire additional dimension, in our case the cross-smile.

The joint probability density (18) can be thought of as an extension of the

Breeden-Litzenberger formula [11] to N dimensions. An important difference is that with knowledge only of vanilla and cross vanilla smiles, the joint density is not unique. It is, however, entirely analytical, i.e., without any numerical calibration, and, *by construction*, consistent with both base smiles as well as the cross-smile, which is a rare combination of features in exotic derivatives models for multiple underlyings.

7 Acknowledgements

I am grateful to Arthur Mountain for ideas that led to the development of this work and to Oleg Butkovsky for the calculation of equation (10). Also to Peter Jäckel, Vladimir Piterbarg and Tino Senge for comments on the text that have greatly improved this article, to Rashmi Tank for discussion of the N-asset case, and to Johnson Han for generating figure 3.

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