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A Universal Model for Pricing All Options

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ABSTRACT

This paper presents a new option pricing approach for all underlying assets that precisely fits the market data. We obtain the probability density function of the underlying asset without any external parameter. The density function for a given expiration date is uniquely determined by the prices of three options with different strikes but the same expiration. Our approach allows for the calculation of path dependent options as we are able to calculate the contingent density function of the underlying asset between subsequent expiries as a function of the underlying asset price. The new model accurately matches market option prices in all asset classes (currencies, interest rates, equities and commodities) including exotic options. The data validates that all liquid financial assets behave according to this new three parameter probability density function, yet typically each asset class corresponds to a different region of the parameter space.

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Since the publication of the revolutionary Black Scholes (BS) (1973) and Merton (1973) models, option pricing has witnessed an explosion in new models, each one differently describing the stochastic process of the underlying asset. In the past two decades, option markets have become increasingly liquid and transparent making testing of various models with actual market data relatively simple. Many empirical papers, such as Bakshi, Cao and Chen (1997) and Bates (2003) provide broad analysis of different models with direct comparison to the market. Even with these advances, until now, more than four decades after the invention of the BS model, there has been no model that is able to produce a realistic volatility smile.

Because of this reason, Vanilla option prices are quoted via their "implied volatility", which is the specific BS volatility for each strike. When the underlying asset is a forward paying asset post expiry, such as options on swaps, then the implied volatility is determined via the Black (1976) model. The implied volatilities of Vanilla options with the same underlying asset and expiration for a large range of strikes make up what is called the "volatility smile" a la Rubinstein (1983). The reason for the word "smile" is that typically the volatility at very high and low strikes is higher than the volatility for strikes around the current underlying asset price or forward rate. Some recent papers about the volatility smile include Carr and Lee (2010), Carr and Wu (2016), Carr and Madan (2013), Madan (2014), Gatheral (2008), Gatheral, Matic, Radoicic and Stefanica (2017), El Euch, Gatheral and Rosenbaum (2017).

In this paper, we develop a new approach to price options that accurately generates the volatility smile. The same model applies to all asset classes including interest rates, and hence we view it as a universal model. The model was fully tested in all asset classes- currencies, equities, commodities and interest rates- for liquid assets/rates where the bid-ask price spreads in the market are tight, and there is little uncertainty about the "market price". Our volatility smile is obtained from deriving the probability density function of the underlying asset/rate. Unlike the BS (lognormal) model where the probability density function is dictated by one variable (the volatility), in our model, the probability function is dictated by three potentially correlated variables. Hence, for example, the prices of options of three strikes with the same expiration determine the full volatility smile for this expiration. In deriving the model, we show that the three quantities that determine the volatility smile include the “pivot” volatility for the expiration (which can be thought of as the At The Money (ATM) volatility), the expected variance of the pivot volatility from inception to the expiration, and the expected covariance of the pivot volatility and the underlying asset/rate from inception to the expiration. After testing against a wide selection of asset across all asset classes, we conclude that the options market considers all liquid financial assets as though they obey the same type of new probability density function generated by this model. Moreover, we see that different asset classes are governed by different regions of the parameter zone leading to the varying shape of volatility smiles across asset classes.

In deriving our volatility smile we assume absolutely nothing about the stochastic process of the underlying asset. We start by valuing certain European option structures (butterflies and risk reversals) using the risk neutral ‘no arbitrage’ approach, as the expected profit generated from instantaneously re-hedging during the option lifetime against value fluctuations due to the instantaneous changes in the underlying asset price and the expected volatility of the underlying asset return until maturity. We then make a natural assumption that the (correlated) instantaneous changes in the underlying asset price and volatility are essentially independent of the current underlying asset price and volatility. In this case, the volatility smile emerges as a path integral over all the paths of the underlying asset price from inception to expiry. When considering all these structures for the same maturity, the equality between the path integral expression and the value of the butterflies and risk reversals provides a consistency condition for the density function to maturity to satisfy. We describe an iterative method to solve for the (unique) density function that satisfies the consistency condition. This allows us to obtain the density function of the underlying asset from just the no arbitrage condition. As an example, we show that if we assume that the expected volatility of the underlying asset is constant from inception to maturity then we obtain the BS price without the pre-assumption that the underlying asset follows a simple Brownian process. Neglecting all third and higher order contributions in the instantaneous changes of the underlying asset and volatility, we then obtain a general expression for the volatility smile that depends only on the expected volatility of the underlying asset return at inception (essentially the at the money volatility), the variance of the instantaneous changes of the underlying asset, and the covariance of the instantaneous changes of the underlying asset and the volatility.

Over the past two decades, many new types of path dependent options, known as exotic options and embedded option products have become popular. So far, no model has been able to accurately price these exotic options universally in the market, even when calibrated with prices from the vanilla option market. For example, the stochastic volatility model of Hull and White (1987), Heston (1993); the interest rates term structure model of Cox, Ingersol and Ross (1985), the two-factor short rate model of Longstaff and Schwartz (1992); the stochastic volatility jump diffusion models of Bates (1996) and Scott (1997), the local volatility model of Dupire (1994), Derman and Kani (1994); the stochastic local volatility of Schonbucher (1999), Lediot (2002) and Lipton (2002), the Libor market models (LMM) of Brace, Gatarek and Musiela (known as BGM) (1997) and the SABR model of Hagan, Kumar, Lensienwski and Woodward (2002) all cannot match the prices for exotic options. These models usually have a range of option parameters at which they perform reasonably well, but elsewhere they are off market. Our formalism, which requires no external parameters or calibration, allows us to obtain the conditional transfer probability from the vanilla term structure. Therefore, we can calculate the price of exotic options. We compare the prices of a large sample of four types of liquid exotic options quoted in the interbank market to those generated by our model. In all cases, the market and model prices are remarkably close.

The paper is organized as follows. In Section 1, we create a generic representation of the volatility smile through the introduction of two new functions while simultaneously solving a system of two equations. We then develop consistency condition equations in Section 2 that stem from the path integral representation of the two equations from Section 1. Section 3 develops an iterative process to solve these consistency condition equations while Section 4 applies this methodology to the simple case of constant volatility to show that the Black-Scholes model is a specific case of this general theory. In section 5, we analyze the no-arbitrage zone implied by the model and show that different asset classes correspond to different regions of the parameter space of the model. The model is tested empirically against market prices in all asset classes in Section 6. In Section 7, we derive the probability transfer density function (the contingent density function) between two volatility smiles, and then we show in Section 8 how this density function can be used to accurately price exotic options. Section 9 concludes.

1. Generic Representation of The Volatility Smile

In this section, we propose a new representation for the volatility smile involving the introduction of two functions. We explain the motivation for this representation and discuss their asymptotic behavior. These two functions will be self-determined in the following sections and offer a very useful way to obtain volatility smiles.

We start by deriving the one smile model for all asset classes. We need a terminology that will apply to all asset classes regardless of their different financial market conventions. We thus start with the fact that in all markets the BS/Black model provides the mapping between option prices and volatilities and vice versa. (For simplicity from now whenever we use BS we also include the Black model.) In the BS model, the prices of the European vanilla call and put options are

$$P_{call} = df\ (F\ N(d_1) - K\ N(d_2)) \quad (1)$$

$$P_{put} = df\ (K\ N(-d_2) - F\ N(-d_1)) \quad (2)$$

where

$$d_1= \frac{\log\left(\frac{F}{K}\right)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T} \quad ; \quad d_2= d_1 - \sigma\sqrt{T} \tag{3}$$

F is the forward price of the underlying asset at expiry, df is the domestic discount factor and N(x) is the Normal cumulative distribution function.

When the underlying asset of the option has a forward payment post-expiry, for example a future contract with settlement date after the expiry of the option or a swap with payments after the expiry of the option, then the BS model is modified by the Black model with the introduction of an "annuity". The annuity serves to discount the value of the asset to the expiry date. For example, the Black formula to calculate an option on swaps (called a swaption) is

$$P_{receiver}= An\ df\ (F\ N(d_1) - K\ N(d_2)) \tag{4}$$

$$P_{payer}= An\ df\ (K\ N(-d2) - F\ N(-d_1)) \tag{5}$$

where F is the forward rate of the swap or the current fixed rate of the underlying forward started swap and An is its annuity which can be approximated using the forward price F as follows

$$An=(df\ (T) - df\ (T+L))/F= (1-1/(1+F/m)^{m}L)/F \tag{6}$$

Here L is the tenor/length of the swap in years and m is the compounding per year in the swap rate (e.g. if the swap pays two semi-annual coupons per year m=2). The terms Payer/receiver replace call/put and refer to the counterparty that pays/receives the fixed rate of the swap vs. the floating (market rate) rate of the coupons of the swap.

Regardless of the pricing model used to calculate the option price, having option prices for all strikes P(K) enables us to obtain the distribution/density function of the price of the underlying asset on the expiration day. By definition the price of a call option with strike K and expiry T is

$$P_{call}(K,T)= df \int (S-K)^{+} g(S,T)\, dS \qquad (7)$$

where g (S,T) is the density of the underlying asset at time T and spot price S(T) and $(S-K)^+$ is S-K for S>K and 0 otherwise. Therefore, we have that

$$g(S,T) = df^{-1} \left.\frac{\partial^2 P(K,T)}{\partial K^2}\right|_{K=S} \qquad (8)$$

Given a European vanilla option, we define its BS implied volatility as the **intrinsic volatility** of this option. Intrinsic volatility is defined as the volatility there would have been in the absence of a volatility smile. Given the prices of European vanilla options with expiry T for all strikes K, P(K,T), we obtain the implied volatility smile σ(K,T) by solving

$$P(K,T) = BS\,(K,T,\sigma(K,T)) \qquad (9)$$

or more generally, the price of the option at time t when the underlying asset price is s is P(s,t,K,T)= BS(s,t,K,T, σ(s,t,k,T)). Notice that whenever F, K > 0 the function σ(s,t,K,T) exists. If F and/or K are negative, we can use the shifted volatility scheme for the mapping where X is such that F+X, K+X>0

$$P(K,T) = BS\,(K+X,T,F+X, \sigma_{shifted}\,(K,T)) \qquad (10)$$

We will return to this in section 5. For simplicity we omit t and s in some places.

One of the reasons we selected the representation in (1) is because it automatically satisfies some required conditions. For example, the difference between the call and the put with the same strike satisfies Call (K) – Put (K) = df(F – K) and so the volatility of call and put options with the same strike is the same.

We started with a general option pricing function P and calculated its intrinsic (implied) volatility via the BS, so if we want to learn how the price P changes with respect to the intrinsic volatility, we can simply use the BS derivatives.

$$\Delta P(K,T) = \Delta\sigma(K,T)\, d\, BS(K,T,\sigma(K,T)) / d\,\sigma(K,T)$$
$$d\,P(K,T) / d\,\sigma(K,T) = Vega(K,T,\sigma(K,T)) = df\, F\sqrt{T}\, n(d_1(\sigma(K,T))) \quad (11)$$

and similarly with higher derivatives of P(K,T). To make the equations more compact we omit the dependency of σ(K,T) on T, but it clearly exists. In (11) we used the fact that the BS Vega of call and put options are

$$\mathrm{Vega}_{\mathrm{call}} = \frac{d\,\mathrm{Pcall}(\sigma)}{d\,\sigma} = \mathrm{Vega}_{\mathrm{put}} = \frac{d\,\mathrm{Pput}(\sigma)}{d\,\sigma} = df\, F\sqrt{T}\, n(d_1), \quad (12)$$

where $n(d_1)$ is the standard normal density function.

Therefore, the strike for which the Vega is maximal satisfies d1=0 and is denoted K_0.

We define a **pivot** volatility as the volatility that corresponds to d1=0 and denote it as σ_0.

For any option with strike K, we define ζ(K) as the difference between the price of the option and the BS price when the pivot volatility σ_0 is used instead of the intrinsic volatility

$$\zeta(K) = P(K) - BS(K,\sigma_0) = BS(\sigma(K)) - BS(\sigma_0) \quad (13)$$

Of course at the pivot strike $\zeta(K_0)=0$. While in (9) it depends on whether the option is a call or a put, ζ(K) is the same for both a call and put with the same strike K.

d_1 **Strike Duality:** Since Vega depends on ${d_1}^2$, it means that there are two different strikes with the same Vega. Taking the higher strike for a call option and the lower for a put option $K_{put} < K_0 < K_{call}$ they satisfy

$$d_1(K_{call}, \sigma(K_{call})) = -d_1(K_{put}, \sigma(K_{put})) \quad (14)$$

We define the two strikes that satisfy (14) as **dual** strikes. Hence same Vega strangle and risk reversals are defined as

$$d_1\mathrm{Strangle} = \mathrm{Call\ options}\ (d_1) + \mathrm{Put\ option}(-d_1) \quad (15)$$
$$d_1\mathrm{Risk\ Reversal} = \mathrm{Call\ options}\ (d_1) - \mathrm{Put\ option}(-d_1) \quad (16)$$

where for a given strike of the call option K_{call} the strike of the put options K_{put} is its dual strike. In the BS model the call and the put have opposite Deltas.

Market makers hedge the volatility risk in their option portfolio along three axes.

The first hedge is against changes in the volatility. The hedge is achieved by having a Vega neutral (i.e. zero in total) portfolio. To neutralize the Vega, traders buy/sell At The Money (ATM) straddles which are the most liquid.

The second hedge is against changes in the Vega as a result of changes in the volatility. This means that $\frac{\mathrm{d\,Vega}}{\mathrm{d}\,\sigma}$ for the portfolio should be close to zero. (The worst case is when $\frac{\mathrm{d\,Vega}}{\mathrm{d}\,\sigma}$ is very negative. In this case, fluctuations in the ATM volatility will cause serious losses in re-hedging the Vega.) The primary tool to offset $\frac{\mathrm{d\,Vega}}{\mathrm{d}\,\sigma}$ for their portfolio is Vega neutral butterflies.

The third hedge is against changes in the Vega as a result of changes in the underlying asset price. This means that the portfolio has to have a $\frac{\mathrm{d\,Vega}}{\mathrm{d\,S}}$ close to zero. The primary tool to offset $\frac{\mathrm{d\,Vega}}{\mathrm{d\,S}}$ for the portfolio is risk reversals (which are Vega neutral). Notice that $\frac{\mathrm{d\,Vega}}{\mathrm{d\,S}}$ is the same as $\frac{\mathrm{d\,Delta}}{\mathrm{d}\,\sigma}$. Hence, hedging the Vega from the underlying spot movement is the same as hedging the Delta from fluctuations in the volatility.

The Vega derivatives of the d_1 strangle are:

$$\frac{\mathrm{d\,Vega}}{\mathrm{d}\,\sigma}(\mathrm{Call}\,(\mathrm{d}_1) + \mathrm{Put}\,(\text{-}\mathrm{d}_1)) = df\ \mathrm{F}\sqrt{\mathrm{T}}\mathrm{n}(\mathrm{d}_1)\mathrm{d}_1{}^2\left(\frac{1}{\sigma_\mathrm{c}}+\frac{1}{\sigma_\mathrm{p}}\right) \qquad (17)$$

$$\frac{\mathrm{d\,Vega}}{\mathrm{d\,S}}(\mathrm{Call}\,(\mathrm{d}_1) + \mathrm{Put}\,(\text{-}\mathrm{d}_1)) = df\,\frac{F}{S}\mathrm{n}(\mathrm{d}_1)\left(\mathrm{d}_1\left(\frac{1}{\sigma_\mathrm{p}}-\frac{1}{\sigma_\mathrm{c}}\right)+2\sqrt{\mathrm{T}}\right) \qquad (18)$$

The Vega derivatives of the d_1 risk reversal are:

$$\frac{\mathrm{d\,Vega}}{\mathrm{d\,S}}(\text{Call }(\mathrm{d}_1) - \text{Put }(-\mathrm{d}_1)) = -df\frac{F}{S}\mathrm{n}(\mathrm{d}_1)\mathrm{d}_1\left(\frac{1}{\sigma_\mathrm{c}}+\frac{1}{\sigma_\mathrm{p}}\right) \tag{19}$$

$$\frac{\mathrm{d\,Vega}}{\mathrm{d}\,\sigma}(\text{Call }(\mathrm{d}_1) - \text{Put }(-\mathrm{d}_1)) = -df\mathrm{F}\sqrt{\mathrm{T}}\mathrm{n}(\mathrm{d}_1)\mathrm{d}_1\left(\mathrm{d}_1\left(\frac{1}{\sigma_\mathrm{p}}-\frac{1}{\sigma_\mathrm{c}}\right)+2\sqrt{\mathrm{T}}\right) \tag{20}$$

We define the Vega neutral butterfly as

$$\text{Vega Neutral Butterfly }(\mathrm{d}_1) = \text{strangle}(\mathrm{d}_1) - \mathrm{e}^{-\frac{\mathrm{d}_1^2}{2}} \times \text{straddle }(\mathrm{d}_1 = 0) \tag{21}$$

Notice that the pivot ATM straddle has zero $\frac{\mathrm{d\,Vega}}{\mathrm{d}\,\sigma}$ but non-zero $\frac{\mathrm{d\,Vega}}{\mathrm{d\,S}}$.

In the case of the **Black** model for swaptions (or any forward payment underlying asset)

$$\text{Vega}_{\text{Black}} = \text{An } df\,\mathrm{F}\sqrt{\mathrm{T}}\,\mathrm{e}^{-\frac{\mathrm{d}_1^2}{2}} = \text{An Vega}_{\text{BS}} \tag{22}$$

and we replace the derivatives by the spot price S with the derivatives by the forward price F. The right hand side of equations (17) and (20) is multiplied by the annuity An(F) and equations (18) and (19) become

$$\frac{\mathrm{d\,Vega}}{\mathrm{d\,F}}(\text{Call }(\mathrm{d}_1) + \text{Put }(-\mathrm{d}_1)) = \mathrm{An}\ \mathrm{n}(\mathrm{d}_1)df\left(\mathrm{d}_1\left(\frac{1}{\sigma_\mathrm{p}}-\frac{1}{\sigma_\mathrm{c}}\right)+2\left(\frac{\frac{\mathrm{dAn}}{\mathrm{dF}}}{\mathrm{An}}\mathrm{F}+\sqrt{\mathrm{T}}\right)\right)$$

$$\frac{\mathrm{d\,Vega}}{\mathrm{d\,F}}(\text{Call }(\mathrm{d}_1) - \text{Put }(-\mathrm{d}_1)) = -\mathrm{An}\ \mathrm{n}(\mathrm{d}_1)df\mathrm{d}_1\left(\frac{1}{\sigma_\mathrm{c}}+\frac{1}{\sigma_\mathrm{p}}\right) \tag{23}$$

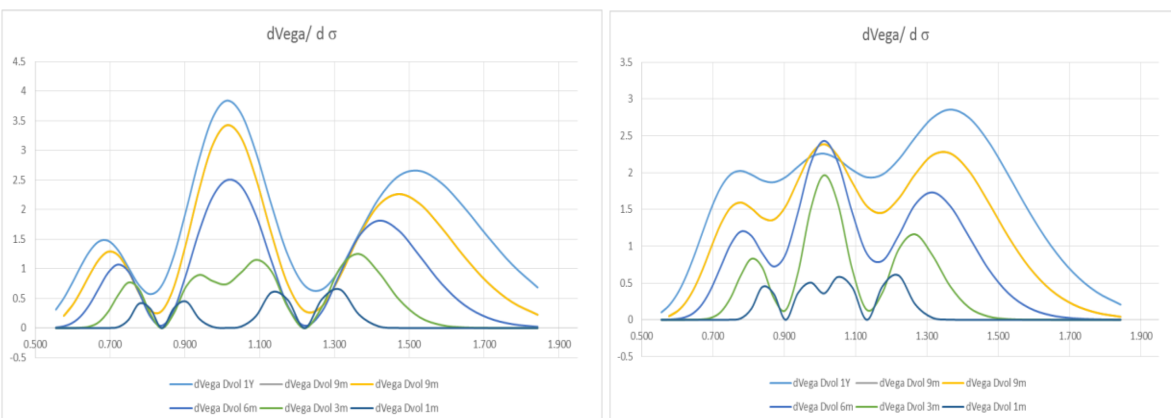
We now look at all the d_1 risk reversal and d_1Vega neutral butterflies where the strikes of the call and the put are d_1-dual. These structures offer a hedge against the impact of the shape of the smile, but the effectiveness of the hedge depends on d_1. We want to find the relative value between risk reversals with different d_1's and some relative values between butterflies with different d_1's that will ultimately define the smile in a unique way.

We start with a heuristic argument: How would a D_1 butterfly (i.e. the call and the put that correspond to d_1 = D_1) be evaluated versus a D_2 butterfly? From a hedging perspective, we will directly compare $\frac{\mathrm{d\,Vega}}{\mathrm{d}\,\sigma}$, but may potentially also compare other elements like the time decay of $\frac{\mathrm{d\,Vega}}{\mathrm{d}\,\sigma}$ and the range of the spot (in interest rates the range of the forward) where the $\frac{\mathrm{d\,Vega}}{\mathrm{d}\,\sigma}$ will still be significant.

Similarly, we ask how would a D_1 risk reversal be evaluated versus a D_2 risk reversal? Again, from a hedging perspective, we will compare $\frac{d\,Vega}{d\,S}$ but may also compare other elements like the time decay of $\frac{\mathrm{d\,Vega}}{\mathrm{d\,S}}$ and the range of the spot (forward) where $\frac{\mathrm{d\,Vega}}{\mathrm{d\,S}}$ is significant.

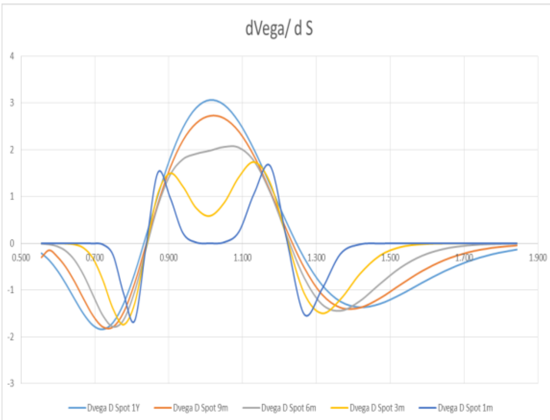
Chart 1a illustrates the behavior over spot and time of $\frac{\mathrm{d\,Vega}}{\mathrm{d}\,\sigma}$ for D_1 and D_2 strangles corresponding to d_1=.75 and d_1=1.25 for expiry 1 year. In this chart we use a constant volatility σ=15%.

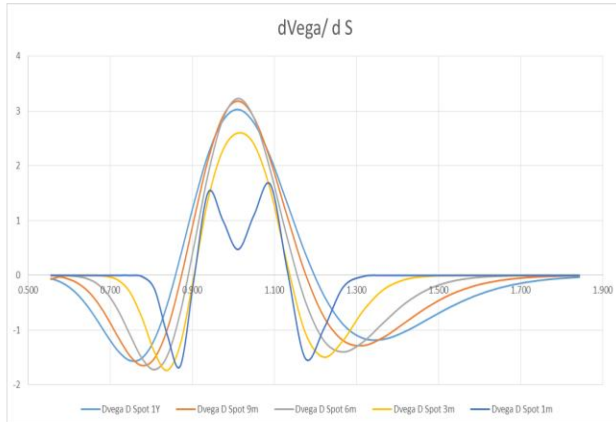
Chart 1b illustrates the behavior over spot and time of $\frac{\mathrm{d\,Vega}}{\mathrm{d\,S}}$ for D_1 and D_2 strangles corresponding to d_1=.75 and d_1=1.25 for expiry 1 year and σ=15%.



d1=1.25 **d1=0.75**

Chart 1a: dVega/dσ for d_1 strangles over spot and time. σ=15% and the expiry is 1 year. Each line represents to time to maturity. The 1Y line correspond to the time at inception, the 9m line corresponds to 3 months after the inception, etc. The d_1=1.25 strangle has a faster time decal and a narrower spread of the dVega/dσ.





d1=1.25 **d1=0.75**

Chart 1b: dVega/ds for d_1 Risk Reversal over spot and time. σ=15% and the expiry is 1 year. Each line represents to time to maturity. The 1Y line correspond to the time at inception, the 9m line corresponds to 3 months after the inception, etc. The d_1=1.25 Risk Reversal has a faster time decal of the dVega/ds.

In order to make this approach more rigorous, we first need to "generalize" these quantities in order to make them orthogonal in the Vega derivatives. We can define a **generalized butterfly** with zero $\frac{\mathrm{d\,Vega}}{\mathrm{d\,S}}$ and a **generalized risk reversal** with zero $\frac{\mathrm{d\,Vega}}{\mathrm{d}\,\sigma}$ as follows.

If we add an ATM straddle to the Vega neutral butterfly, we can have a generalized butterfly that has zero $\frac{\mathrm{d\,Vega}}{\mathrm{d\,S}}$. Notice that the ATM straddle does not change $\frac{\mathrm{d\,Vega}}{\mathrm{d}\,\sigma}$ for the butterfly.

The structure of the generalized butterfly would then be

$$\text{Butterfly}'(d_1) = \text{butterfly}\ (d_1) + \alpha \text{ ATM straddle} \tag{24}$$

where

$$\alpha = e^{-\frac{d_1^2}{2}} \frac{d_1}{2\sqrt{T}} \left(\frac{1}{\sigma_p} - \frac{1}{\sigma_c}\right) \tag{25}$$

satisfies

$$\frac{\mathrm{d\,Vega}}{\mathrm{d}\,\sigma}\ (\text{Butterfly'}\ (\mathrm{d}_1)) = \frac{\mathrm{d\,Vega}}{\mathrm{d}\,\sigma}\ (\text{Butterfly}\ (\mathrm{d}_1)) = \frac{\mathrm{d\,Vega}}{\mathrm{d}\,\sigma} (\text{Strangle}\ (\mathrm{d}_1)) \qquad (26)$$

$$\frac{\mathrm{d\,Vega}}{\mathrm{d\,S}} (\text{Butterfly'}\ (\mathrm{d}_1)) = 0 \qquad (27)$$

Similarly we define a generalized risk reversal (RR)

$$\begin{aligned} \mathrm{RR'}(\mathrm{d}_1) &= \mathrm{RR}\ (\mathrm{d}_1) - \omega\ \text{Butterfly'}(\mathrm{d}_1) \\ &= \mathrm{RR}\ (\mathrm{d}_1) - \omega\ \text{Vega neutral butterfly}(\mathrm{d}_1) - \omega\ \alpha\ \text{ATM straddle}(0) \end{aligned} \qquad (28)$$

where

$$\omega = -\frac{\sigma_c - \sigma_p + 2\sqrt{\mathrm{T}}\sigma_c\sigma_p/d_1}{\sigma_c + \sigma_p} \qquad (29)$$

satisfies

$$\frac{\mathrm{d\,Vega}}{\mathrm{d\,S}} (\mathrm{RR'}\ (\mathrm{d}_1)) = \frac{\mathrm{d\,Vega}}{\mathrm{d\,S}} (\mathrm{RR}\ (\mathrm{d}_1)) \qquad (30)$$

$$\frac{\mathrm{d\,Vega}}{\mathrm{d}\,\sigma} (\mathrm{RR'}\ (\mathrm{d}_1)) = 0 \qquad (31)$$

We now represent the volatility smile with 2 equations:

$$\zeta_{\text{butterfly'}}\,(\mathrm{d}_1) = \zeta_{\text{strangle}}\,(\mathrm{d}_1) = \lambda\ (\mathrm{d}_1, \mathrm{T}, \sigma_0)\,\frac{\mathrm{d\,Vega}}{\mathrm{d}\,\sigma}(\text{strangle}(\mathrm{d}_1)) \qquad (32)$$

$$\zeta_{\mathrm{RR'}}\,(\mathrm{d}_1) = \zeta_{\mathrm{RR}}(\mathrm{d}_1) - \omega\ \zeta_{\text{strangle}}\,(\mathrm{d}_1) = \chi\ (\mathrm{d}_1, \mathrm{T}, \sigma_0)\,\frac{\mathrm{d\,Vega}}{\mathrm{d\,S}}\ (\mathrm{RR}(\mathrm{d}_1)) \qquad (33)$$

where $\lambda\ (\mathrm{d}_1, \mathrm{T}, \sigma_0)$ and $\chi\ (\mathrm{d}_1, \mathrm{T}, \sigma_0)$ are two functions to be determined in Section 4. At this stage they can be viewed as the ratio between the generalized Vega derivatives and their zeta's. Equation (33) can also be written as

$$\zeta_{\mathrm{RR}}(\mathrm{d}_1) = \lambda\ (\mathrm{d}_1, \mathrm{T}, \sigma_0)\,\frac{\mathrm{d\,Vega}}{\mathrm{d}\,\sigma}(\mathrm{RR}(\mathrm{d}_1)) + \chi\ (\mathrm{d}_1, \mathrm{T}, \sigma_0)\,\frac{\mathrm{d\,Vega}}{\mathrm{d\,S}}\ (\mathrm{RR}(\mathrm{d}_1)) \qquad (34)$$

From equations (32)-(33), we get a set of two coupled equations.

For the higher strike K_c:

$$\zeta_c = df\ \mathrm{F}\ \mathrm{n}(d_1)\ d_1\left(\frac{\lambda(d_1)\sqrt{\mathrm{T}}d_1}{\sigma_c} - \lambda(d_1)T - \frac{\chi(d_1)}{2\mathrm{F}}\left(\frac{1}{\sigma_c}+\frac{1}{\sigma_p}\right)\right) \tag{35}$$

and for the lower strike K_p (we use $K_c > K_0 > K_p$):

$$\zeta_p = df\ \mathrm{F}\ \mathrm{n}(d_1)\ d_1\left(\frac{\lambda(d_1)\sqrt{\mathrm{T}}d_1}{\sigma_p} + \lambda(d_1)T + \frac{\chi(d_1)}{2\mathrm{F}}\left(\frac{1}{\sigma_c}+\frac{1}{\sigma_p}\right)\right) \tag{36}$$

Equations (35) and (36) can be written as functions of d_1 since $\zeta_c = \zeta(d_1)$;

$$\zeta_p=\zeta(-d_1)\ ;\ \sigma_c=\sigma(d_1)\ ;\ \sigma_p=\sigma(-d_1) \tag{37}$$

By plugging equations (1), (2) and (24) into the expressions (35) and (36), we can derive the asymptotic behavior of $\lambda(d_1)$ and $\chi(d_1)$ using the asymptotic behavior of N(x) at infinity and the moment formula for implied volatility derived by Roger Lee (2003). In Appendix 1 we show explicitly that

$$\lambda(d_1) \rightarrow O({d_1}^{-2}),\ \chi(d_1) \rightarrow O({d_1}^{-1}),\quad d_1 \rightarrow \infty \tag{38}$$

Finally, for an underlying asset with a forward payment (like swaptions) where the Black model is applicable instead of BS, there is a minor modification for α in (24), as follows

$$\alpha_{\mathrm{Black}} = e^{-\frac{d_1^2}{2}}\ \frac{\frac{d_1}{2\sqrt{T}}\left(\frac{1}{\sigma_p}-\frac{1}{\sigma_c}\right)}{1+\ \frac{1}{A_n}\ \mathrm{F}\frac{dA_n}{d\mathrm{F}}} \tag{39}$$

and ω is the same as in (29). Therefore, in this situation, equations (35) and (36) get scaled by the annuity value An leaving us with equations (40) and (41).

$$\zeta_c = \mathrm{An\ F\ n}(d_1)d_1\left(\frac{\lambda(d_1)\sqrt{\mathrm{T}}d_1}{\sigma_c} - \lambda(d_1)T - \frac{\chi(d_1)}{2\mathrm{F}}\left(\frac{1}{\sigma_c}+\frac{1}{\sigma_p}\right)\right) \tag{40}$$

$$\zeta_p = \mathrm{An\ F\ n}(d_1)d_1\left(\frac{\lambda(d_1)\sqrt{\mathrm{T}}d_1}{\sigma_p} + \lambda\,(d_1)T + \frac{\chi(d_1)}{2\mathrm{F}}\left(\frac{1}{\sigma_c}+\frac{1}{\sigma_p}\right)\right) \tag{41}$$

2. Path Integral representation of the Volatility Smile

In this section, we derive a new approach to pricing options. We formulate the risk neutral valuation of the d_1 butterfly and d_1 risk reversal as their expected profit from continuous re-hedging against instantaneous changes in the underlying asset price and its expected volatility until maturity, without any pre-assumption about the stochastic process they follow. The expressions for the valuations coincide with the representation of the volatility smile suggested in section 1 reformulated in terms of path integrals and provide explanations for the well-known shape of the volatility smile.

Let us first explain why there must be a volatility smile. Denote the ATM volatility for expiry T at inception by σ_0. Obviously, the ATM volatility is expected to fluctuate until expiry. Let us divide the time to expiry T into N time intervals $t_i = iT/N$. At each time i, the ATM volatility until the expiry T is $\sigma(t_i,T)$. We define the stochastic variable $\delta\sigma_{i.} = \sigma(t_i, T) - \sigma(t_{i-1}, T)$, and similarly the change in the underlying asset price from time t_{i-1} to t_i is denoted $\delta s_i = s_i - s_{i-1}$.

We first consider a specific Vega neutral d_1 butterfly with expiry T (i.e. Vega hedged d_1 strangle). We denote the strikes of the butterfly as K_{call}, K_{put}, K_0. The change in the value of the butterfly (up to a second order) from time t_{i-1} to time t_i due to the change in the volatility is

$$\delta \Pi i\,(\text{butterfly}, t_i) = \text{Vega}(t_{i-1})\,\delta\sigma_i + \delta\,\text{Vega}(t_i)\,\delta\sigma_i/2 = \text{Vega}(\text{butterfly}, t_{i-1})\,\delta\sigma_i + \frac{1}{2}\frac{\text{d Vega}}{\text{d}\,\sigma}(\text{butterfly}, t_{i-1})\,\delta\sigma_i^2 + \frac{1}{2}\frac{\text{d Vega}}{\text{d s}}(\text{butterfly}, t_{i-1})\,\delta\sigma_i\,\delta s_i \qquad (42)$$

In equation (42), we assume that in the infinitesimal time interval the smile moves almost parallel to the ATM volatility and

$$\text{Vega}(\text{butterfly}, t_{i-1}) = \text{Vega}(K_{call}, t_{i-1}) + \text{Vega}(K_{put}, t_{i-1}) - 2\,e^{-\frac{d_1^2}{2}}\,\text{Vega}(K_0, t_{i-1}) \qquad (43)$$

The re-hedging strategy for changes in the ATM volatility $\sigma(t_i, T)$ is as follows. At time t_i the hedger buys or sells ATM options whose ATM strike is K_0^i so that the total amount of Vega of the butterfly and the ATM hedge is zero. Since the previous ATM hedge had a strike k_0^{i-1} which may be different than K_0^i, the hedger replaces the previous ATM option hedge with the current ATM option. Therefore at each time t_i the only hedge is the current ATM option so that

$$\text{Vega(ATM hedge}, t_i) = - \text{Vega (butterfly}, t_i) \quad (44)$$

The ATM option has $\frac{\text{d Vega}}{\text{d}\,\sigma}=0$ and $\frac{\text{d Vega}}{\text{d s}} = -\frac{\text{Vega}}{\text{s}}$. Hence the replacement of the ATM strike hedge with K_0^{i-1} at time t_i by strike K_0^i generates a profit/loss of

$$\text{Vega (ATM hedge}, t_{i-1})\, \delta\sigma_i + \frac{1}{2}\frac{\text{d Vega}}{\text{d s}}\text{(ATM hedge}, t_{i-1})\, \delta s_i\, \delta\sigma_i =$$
$$- \text{Vega (butterfly}, t_{i-1})\, \delta\sigma_i - \frac{1}{2}\text{Vega (butterfly}, t_{i-1})/s_{i-1}\, \delta s_i\, \delta\sigma_i \quad (45)$$

Therefore up to a second order, the total profit/loss from the butterfly and the hedge is

$$\delta\, \Pi i\text{ (butterfly + ATM hedge}, t_i) = \frac{1}{2}\frac{\text{d Vega}}{\text{d}\,\sigma}\text{(butterfly}, t_{i-1})\, \delta\sigma_i^2 +$$
$$\frac{1}{2}\left(\frac{\text{d Vega}}{\text{d s}}\text{(butterfly}, t_{i-1}) - \text{Vega (butterfly}, t_{i-1})/s_{i-1}\right) \delta\sigma_i\, \delta s_i \quad (46)$$

The profit/loss from time i to i+1 is realized at time i+1 (i.e. on a cash basis). In the re-hedging process, the hedger either borrows money at interest rate r_i to buy the ATM option or lends money at interest rate r_i after selling the ATM option. Therefore when taking into account the funding cost, up to second order the profit/loss in (46) becomes

$df_i\, \delta\, \Pi i$ (butterfly + ATM hedge, t_i)

where df_i is the discount factor from inception to time i.

Therefore the expected profit from holding the butterfly until maturity while re-hedging with the ATM option is

$$E\left(\sum_{i=1}^{N} df_i\ \delta\Pi i(\text{butterfly} + \text{ATM hedge}\right) = \frac{1}{2} E\left(\sum_{i=1}^{N} df_i \frac{\text{d Vega}}{\text{d}\,\sigma} \text{(butterfly, } t_{i-1}) \delta\sigma_i^2 + \left(s_{i-1} \frac{\text{d Vega}}{\text{d s}} \text{(butterfly, } t_{i-1}) - \text{Vega (butterfly, } t_{i-1})\right) \delta\sigma_i \frac{\delta s_i}{s_{i-1}}\right) \quad (47)$$

Now, if $\delta\sigma_i$ and $\frac{\delta s_i}{s_{i-1}}$ are independent (or approximately independent) of

Vega (d_1 strangle, t_{i-1}) and Vega (K_0, t_{i-1}) then the last equation can be written as

$$E\left(\sum_{i=1}^{N} \delta\ \Pi i\right) \approx \frac{1}{2} E\left(\sum_{i=1}^{N} df_i \frac{\text{d Vega}}{\text{d}\,\sigma} \text{(butterfly, } t_{i-1})\right) E\left(\sum_{i=1}^{N} \delta\sigma_i^2\right)/N + \frac{1}{2} E\left(\sum_{i=1}^{N} df_i \left(s_{i-1} \frac{\text{d Vega}}{\text{d s}} \text{(butterfly, } t_{i-1}) - \text{Vega (butterfly, } t_{i-1})\right)\right) E\left(\sum_{i=1}^{N} \delta\sigma_i \frac{\delta s_i}{s_{i-1}}\right)/N \quad (48)$$

We denote

$$\text{Var}(\sigma_{ATM}) = E\left(\sum_{i=1}^{N} \delta\sigma_i^2\right)/N = ; \qquad \text{Cov}(\sigma_{ATM}, s) = E\left(\sum_{i=1}^{N} \delta\sigma_i \frac{\delta s_i}{s_{i-1}}\right)/N \quad (49)$$

where Var (σ_{ATM}) is the expected variance of the ATM volatility in the period from now until maturity and Cov (σ_{ATM},s) is the expected covariance of the ATM volatility and the return of the underlying asset price in the period from now until maturity (notice that if $0 \neq E \sum_{i=1}^{N} \delta\sigma_i$ then this is not exactly the covariance).

Similarly we can calculate the profit for the d_1 risk reversal with re-hedging using ATM options

$$E\left(\sum_{i=1}^{N} \delta\ \Pi i\right) \approx \frac{1}{2} E\left(\sum_{i=1}^{N} df_i \frac{\text{d Vega}}{\text{d}\,\sigma} \text{(risk reversal, } t_{i-1})\right) \text{Var}(\sigma_{ATM}) + \frac{1}{2} E\left(\sum_{i=1}^{N} df_i \left(s_{i-1} \frac{\text{d Vega}}{\text{d s}} \text{(risk reversal, } t_{i-1}) - \text{Vega(risk reversal, } t_{i-1})\right)\right) \text{Cov}(\sigma_{ATM}, s) \quad (50)$$

Now we can connect equation (48) to the price of the d_1 butterfly. Up to a second order, the price of the d_1 Vega neutral butterfly is made of two components: the first is the BS price with some constant volatility σ'_0 which takes into account the re-hedging of the options with the underlying asset and the time decay of the options, and the second is expected profit from continuous re-hedging of the Vega due to the changes in the ATM volatility. Hence

$$P(d_1 \text{ butterfly}) = BS(K^{call}, \sigma_0') + BS(K^{put}, \sigma_0') + e^{-\frac{d_1^2}{2}} BS(K_0, \sigma_0') + \frac{1}{2}E(\sum_{i=1}^{N} df_i \frac{\text{d Vega}}{\text{d}\,\sigma}(\text{butterfly}, t_{i-1})) \, Var(\sigma_{ATM}) + \frac{1}{2}E(\sum_{i=1}^{N} df_i (s_{i-1} \frac{\text{d Vega}}{\text{d s}}(\text{butterfly}, t_{i-1}) - Vega(\text{butterfly}, t_{i-1})) \, Cov(\sigma_{ATM}, s) \quad (51)$$

and similarly the price of the d_1 risk reversal is

$$P(d_1 \text{ risk reversal}) = BS(K_{call}, \sigma_0') - BS(K_{put}, \sigma_0') + \frac{1}{2}E(\sum_{i=1}^{N} df_i \frac{\text{d Vega}}{\text{d}\,\sigma}(\text{risk reversal}, t_{i-1})) \, Var(\sigma_{ATM}) + \frac{1}{2}E(\sum_{i=1}^{N} df_i (s_{i-1} \frac{\text{d Vega}}{\text{d s}}(\text{risk reversal}, t_{i-1}) - Vega(\text{risk reversal}, t_{i-1})) Cov(\sigma_{ATM}, s) \quad (52)$$

Since the combination of all butterflies and risk reversals define the whole volatility smile, equations (51)-(52) describe the volatility smile under 3 assumptions:

(i) The option price is approximated by decomposing it into the constant volatility element and the varying volatility element

(ii) The varying volatility element is calculated up to a second order. Clearly contributions from higher order terms are smaller than the bid-ask spread.

(iii) The changes of the ATM volatility and the underlying asset return at time t_i are independent of the Vega of the option at time t_{i-1}.

The last assumption can be easily removed.

Under these assumptions, we can express the whole smile with only 3 variables: Var (σ_{ATM}) , Cov (σ_{ATM},s) and $\sigma_0{}'$.

Moreover, equations (51)-(52) explain the shape of the volatility smile. At d_1=0 by definition the butterfly becomes zero. For very large d1 values, the contribution of the summations goes to zero because the Vega of the butterfly goes to zero. Therefore, as d1 increases from zero, the summation contribution shape has at least one maximum which we denote d_1^{max}. This implies that zeta must be growing at least in some area from d_1=0 to d_1^{max}. Thus, the “average” volatility of the call and put of the strangles is increasing. We conclude that in this range the volatility must have a smile shape.

In order to obtain σ_0', we can look at the ATM option with d_1=0 and strike K_0 at inception. Consider the d_1=0 straddle, i.e Call and Put with strike K_0. Applying the same re-hedging strategy with the concurrent ATM options at each time t_i we obtain

$$E(\sum_{i=1}^{N} \delta\, \Pi i\ (K_0)) = \frac{1}{2} E(\sum_{i=1}^{N} df_i \frac{\mathrm{d\,Vega}}{\mathrm{d}\,\sigma} \text{ (Straddle } K_0, t_{i-1}) \,\delta\sigma_i^2 + (s_{i-1}(\frac{\mathrm{d\,Vega}}{\mathrm{d\,s}} \text{(Straddle } K_0, t_{i-1}) - \text{ Vega (Straddle } K_0, t_{i-1})\,)\,\delta\sigma_i \frac{\delta s_i}{s_{i-1}}) \tag{53}$$

Decomposing the price of the straddle with strike K_0 into the BS price and re-hedging profits as we did before and taking into account the fact that by definition ζ (K_0) = 0 gives us

$$\text{P(Straddle } K_0) = \text{BS (Straddle } k_0, \sigma_0) = \text{ BS (Straddle } k_0, \sigma'_0) + \frac{1}{2} E(\sum_{i=1}^{N} df_i (\frac{\mathrm{d\,Vega}}{\mathrm{d}\,\sigma} \text{ (Straddle } k_0, t_{i-1})\, \delta\sigma_i^2 + (s_{i-1} \frac{\mathrm{d\,Vega}}{\mathrm{d\,s}} \text{(Straddle } K_0, t_{i-1}) - \text{Vega (Straddle } K_0, t_{i-1}))\, \delta\sigma_i \frac{\delta s_i}{s_{i-1}}) \tag{54}$$

We can get a simple expression for σ_0' using

$$0 = \text{BS}(K_0, \sigma'_0) - \text{BS}(K_0, \sigma_0) \approx \text{Vega } (K_0, \sigma_0)(\sigma'_0 - \sigma_0) + \frac{\mathrm{d\,Vega}}{\mathrm{d}\,\sigma} (K_0, \sigma_0)(\sigma'_0 - \sigma_0)^2/2 \tag{55}$$

and $\frac{\mathrm{d\,Vega}}{\mathrm{d}\,\sigma}(K_0, \sigma_0)$=0

Then up to second order

$$\sigma'_0 \approx \sigma_0 - \frac{1}{2}E(\sum_{i=1}^{N} df_i \frac{\mathrm{d\,Vega}}{\mathrm{d}\,\sigma} \text{ (Straddle } k_0, t_{i-1}) \text{ Var } (\sigma_{ATM}) + (s_{i-1}\frac{\mathrm{d\,Vega}}{\mathrm{d\,s}} \text{ (Straddle } k_0, t_{i-1}) - \text{Vega (Straddle } k_0, t_{i-1})) \text{ Cov } (\sigma_{ATM}, s)) / (df\, F_{i-1}\sqrt{T/2\Pi}) \quad (56)$$

Therefore we can say that the volatility smile is determined by the following expected values

$$\sigma_0, \ \ \text{Var } (\sigma_0), \ \ \text{Cov } (\sigma_0, s) \quad (57)$$

We now follow the same steps we took in order to reach equations (32)-(33) and define the generalized butterfly and risk reversal as follows:

Generalized (d_1 butterfly) ≡ d_1 butterfly' = (d_1 butterfly) - α (d_1=0 strangle)

such that

$$\zeta_{\text{strangle}}(\mathrm{d}_1) = \zeta\,(d_1 \text{ butterfly}) = \frac{1}{2}E(\sum_{i=1}^{N} df_i \frac{\mathrm{d\,Vega}}{\mathrm{d}\,\sigma} \text{ (butterfly', } t_{i-1}) \ \delta\sigma_i^2) \approx \frac{1}{2}\text{E}(\sum_{i=1}^{N} df_i \frac{\mathrm{d\,Vega}}{\mathrm{d}\,\sigma} \text{ (butterfly', } t_{i-1})) \text{ Var } (\sigma_{ATM}) \quad (58)$$

where α satisfies equation (59)

$$0 = \text{BS } (d_1 \text{ butterfly}, \sigma'_0) - \text{BS } (d_1 \text{ butterfly}, \sigma_0) - \alpha(\text{BS (Straddle } k_0, \sigma_0) - \text{BS (Straddle } k_0, \sigma'_0)) + \frac{1}{2}E\sum_{i=1}^{N} df_i[\frac{\mathrm{d\,Vega}}{\mathrm{d\,s}} (d_1 \text{ butterfly, } t_{i-1}) - \text{Vega } (d_1 \text{ butterfly, } t_{i-1})/s_{i-1} - \alpha\,(\frac{\mathrm{d\,Vega}}{\mathrm{d\,s}} (d_1{=}0 \text{ strangle, } t_{i-1}) - \text{Vega } (d_1{=}0 \text{ strangle, } t_{i-1})/s_{i-1})]\ \delta\sigma_i\, \delta s_i \quad (59)$$

Generalized (d_1 risk reversal) ≡ d_1 risk reversal' = d_1 risk reversal + ω (d_1 butterfly') +

+ β (d_1=0 strangle)

where the amount of the butterfly' ω and the amount of the d_1=0 strangle β are set to offset the $\frac{\mathrm{d\,Vega}}{\mathrm{d}\,\sigma}$ of the risk reversal and the $\frac{\mathrm{d\,Vega}}{\mathrm{d\,s}}$ of the re-hedging with ATM options and the contribution from σ'_0 (for simplicity we omit df_i from the following expressions).

$$\zeta_{RR'}(\mathrm{d}_1) = \mathrm{E}\Big(\sum_{i=1}^{N} s_{i-1} \frac{\mathrm{d\,Vega}}{\mathrm{d\,s}} (d_1 \text{ risk reversal}, t_{i-1})\Big) \frac{\delta s_i}{s_{i-1}} \delta\sigma_i /2$$
$$= \mathrm{E}\Big(\sum_{i=0}^{N-1} s_i \frac{\mathrm{d\,Vega}}{\mathrm{d\,s}} (d_1 \text{ risk reversal}, t_i)\Big) \mathrm{Cov}(\sigma_{ATM}, s)/2 \tag{60}$$

where ω and β are selected such that

$$0 = \omega\, \mathrm{E}\Big(\sum_{i=1}^{N} \frac{\mathrm{d\,Vega}}{\mathrm{d}\,\sigma} (d_1 \text{ butterfly'}, t_i)\Big) + \beta\, \mathrm{E}\Big(\sum_{i=1}^{N} \frac{\mathrm{d\,Vega}}{\mathrm{d}\,\sigma} (d_1{=}0 \text{ strangle}, t_i)\Big) +$$
$$+ \mathrm{E}\Big(\sum_{i=1}^{N} \frac{\mathrm{d\,Vega}}{\mathrm{d}\,\sigma} (d_1 \text{ Risk Reversal}, t_i)\Big) \tag{61}$$

and

$$0 = \mathrm{BS}(d_1 \text{ risk reversal}, \sigma'_0) - \mathrm{BS}(d_1 \text{ risk reversal}, \sigma_0) -$$
$$\beta(\mathrm{BS}(\text{Straddle } \mathrm{k}_0, \sigma_0) - \mathrm{BS}(\text{Straddle } \mathrm{k}_0, \sigma'_0)) +$$
$$(\beta\, E\Big(\sum_{i=1}^{N} s_{i-1} \frac{\mathrm{d\,Vega}}{\mathrm{d\,s}} (d_1{=}0 \text{ strangle}, t_i) - \mathrm{Vega}(d_1{=}0 \text{ strangle}, t_i)\Big) -$$
$$\mathrm{E}\Big(\textstyle\sum_{i=1}^{N} \mathrm{Vega}(d_1 \text{ Risk Reversal}, t_i)\Big)) \mathrm{Cov}(\sigma_{ATM}, s)/2 \tag{62}$$

Equation (60) can also be written in a similar style as equation (34):

$$\zeta_{RR}(\mathrm{d}_1) = \frac{1}{2}\mathrm{E}\Big(\sum_{i=0}^{N-1} \frac{\mathrm{d\,Vega}}{\mathrm{d}\,\sigma} (\text{risk reversal}, t_i)\Big) \mathrm{Var}(\sigma_{ATM})$$
$$+ \frac{1}{2}\mathrm{E}\Big(\sum_{i=0}^{N-1} s_i \frac{\mathrm{d\,Vega}}{\mathrm{d\,s}} (d_1 \text{ risk reversal}, t_i)\Big) \mathrm{Cov}(\sigma_{ATM}, s)/2 \tag{63}$$

In the following, we obtain the functions $\lambda(d_1)$ and $\chi(d_1)$ for options on all underlying assets except for forward paying assets such as interest rates swaps. Later we obtain them for options on swaps (swaptions) for which we have to take into account the forward rate and the annuity.

The power of the methodology suggested below is that we do not need to know the probability density function to go from the underlying asset price at time t_1 to another price at time t_2 g ($(s_1, t_1 \rightarrow s_2, t_2)$ implied from the vanilla options prices.

The implementation of (58) and (60) in conjunction with (32) and (33) as the time interval between periods goes to 0 is done via the translation of the summation to integrals using the density function of the underlying asset g(s,t) and then eventually solving for the density function.

We now define the following path integrals for the d_1 call and put options with expiry T and ATM volatility σ_0

$$E\left(\frac{d\,Vega}{d\,\sigma}\text{ strangle }(d_1)\right) = \int_0^T dt \int_0^\infty ds\ g(s,t)df(t)\left[\frac{dVega}{d\sigma}(s,t,T,\sigma_t(K_{call},T),K_{call}) + \frac{dVega}{d\sigma}(s,t,T,\sigma_t(K_{put},T),K_{put})\right]$$

$$E\left(\frac{d\,Vega}{d\,\sigma}\text{ straddle }(d_1=0)\right) = 2\int_0^T dt \int_0^\infty ds\ g(s,t)df(t)\frac{dVega}{d\sigma}(s,t,T,\sigma_t(K_0,T),K_0)$$

$$E\left(s\frac{d\,Vega}{d\,s}\text{ strangle }(d_1)\right) = \int_0^T dt \int_0^\infty ds\ g(s,t)df(t)s\left[\frac{dVega}{ds}(s,t,T,\sigma_t(K_{call},T),K_{call}) + \frac{dVega}{d\sigma}(s,t,T,\sigma_t(K_{put},T),K_{put})\right]$$

$$E\left(Vega\text{ (strangle }(d_1)\right) = \int_0^T dt \int_0^\infty ds\ g(s,t)\,df(t)\left[Vega\,(s,t,T,\sigma_t(K_{call},T),K_{call}) + Vega\,(s,t,T,\sigma_t(K_{put},T),K_{put})\right]$$

and similarly for risk reversal

$$E\left(s\frac{d\,Vega}{d\,s}\text{ risk reversal }(d_1)\right) = \int_0^T dt \int_0^\infty ds\ g(s,t)s\left[\frac{dVega}{ds}(s,t,T,\sigma_t(K_{call},T),K_{call}) - \frac{dVega}{d\sigma}(s,t,T,\sigma_t(K_{put},T),K_{put})\right]$$

etc. (64)

where K_c and K_p are the current strikes of the call and put options that correspond to d_1, and K_0 is the strike of the current ATM straddle. The integral over time is from today to the expiry T, the density function g(s,t) is the same as in (8), and *df*(t) is the discount factor for time t. The calculation of $\frac{dVega}{d\sigma}$ or $\frac{dVega}{ds}$ through the integral at time t and spot s is done with the smile at time t for options with expiry time T-t. We denote the smile at time t for expiration T as $\sigma_t(K,T)$. In other words, we use the expressions for the Vega derivatives with the volatility that corresponds to the strikes determined at inception, i.e. $\sigma_t(K) = \sigma(s,K,t,T)$ where the smile in the integral is calculated via (35) and (36). We now define

$$\tilde{\boldsymbol{\lambda}} \equiv \mathrm{Var}\,(\sigma_{ATM})\,/2 \qquad \text{and} \qquad \tilde{\boldsymbol{\chi}} \equiv \mathrm{Cov}\,(\sigma_{ATM}, s)\,/2$$

Equation (58) becomes

$$\zeta_{\,butterfly}\,(d_1) = \lambda\,(d_1, T, \sigma_0)\,\frac{d\,Vega}{d\,\sigma}\,\big(\mathrm{strangle}(d_1)\big) = \tilde{\boldsymbol{\lambda}}(E\,(\frac{d\,Vega}{d\,\sigma}\,\mathrm{strangle}\,(d_1)) - e^{-\frac{{d_1}^2}{2}}\,E\,(\frac{d\,Vega}{d\,\sigma}\,\mathrm{straddle}\,(d_1{=}0)) + \alpha\,\tilde{\boldsymbol{\lambda}}\,E\,(\frac{d\,Vega}{d\,\sigma}\,\mathrm{straddle}\,(d_1{=}0)) \tag{65}$$

And similarly equation (60) becomes

$$\zeta_{\,RR'}\,(d_1) = \zeta_{\,RR} - \omega\,\zeta_{\,butterfly}\,(d_1) = \chi\,(d_1, T, \sigma_0)\,\frac{d\,Vega}{d\,s}\,\big(\mathrm{risk\ reversal}(d_1)\big) = \tilde{\boldsymbol{\chi}}\,E\,(s\,\frac{d\,Vega}{d\,s}\,\mathrm{risk\ reversal}\,(d_1)) \tag{66}$$

where α and ω can be written with the integrals in (64).

From here, $\lambda(d_1)$ and $\chi(d_1)$ are defined in equations (67) and (68)

$$\lambda(d_1) = [\tilde{\boldsymbol{\lambda}}(E\,(\frac{d\,Vega}{d\,\sigma}\,d_1\,\mathrm{strangle}) - (e^{-\frac{{d_1}^2}{2}} - \alpha)\,E\,(\frac{d\,Vega}{d\,\sigma}\,d_1{=}0\,\mathrm{straddle}))]\,/\,\frac{d\,Vega}{d\,\sigma}\,\big(\mathrm{strangle}(d_1)\big) \tag{67}$$

$$\chi(d_1) = \tilde{\boldsymbol{\chi}}\ E\left(s \frac{\mathrm{d\ Vega}}{\mathrm{d\ s}}\ d_1 \text{ risk reversal}\right) / \frac{\mathrm{d\ Vega}}{\mathrm{d\ S}}\ (\text{risk reversal } (d_1)) \tag{68}$$

To determine α and ω we use equations (59),(61), and (62) along with

$$BS(K_0, \sigma_0) = BS(K_0, \sigma'_0) + \tilde{\boldsymbol{\lambda}}\ E\left(\frac{\mathrm{d\ Vega}}{\mathrm{d}\ \sigma}\ d_1{=}0 \text{ straddle}\right) / 2 + \tilde{\boldsymbol{\chi}}\ \left(E\left(s \frac{\mathrm{d\ Vega}}{\mathrm{d\ s}}\ d_1{=}0 \text{ straddle}\right) - E\left(\text{Vega } d_1{=}0 \text{ straddle}\right)\right)/2 \tag{69}$$

$$\alpha = \Big(BS\ (d_1 \text{ strangle}, \sigma_0) - BS\ (d_1 \text{ strangle}, \sigma'_0) + \tilde{\boldsymbol{\lambda}}\ \Big(E\left(s \frac{\mathrm{d\ Vega}}{\mathrm{d\ s}}\ d_1 \text{ strangle}\right) - e^{-\frac{{d_1}^2}{2}}\ E\left(s \frac{\mathrm{d\ Vega}}{\mathrm{d\ s}}\ d_1{=}0 \text{ straddle}\right) + E\left(\text{Vega } d_1 \text{ strangle}\right) - e^{-\frac{{d_1}^2}{2}}\ E\left(\text{Vega } d_1{=}0 \text{ straddle}\right)\Big) / \Big(2\ BS(K_0, \sigma_0) - 2\ BS(K_0, \sigma'_0) + \tilde{\boldsymbol{\lambda}}\ E\left(s \frac{\mathrm{d\ Vega}}{\mathrm{d\ s}}\ d_1{=}0 \text{ straddle}\right) - E\left(\text{Vega } d_1{=}0 \text{ straddle}\right)\Big) \tag{70}$$

$$\beta = \Big[BS\ (d_1 \text{ risk reversal}, \sigma'_0) - BS\ (d_1 \text{ risk reversal}, \sigma_0) - \beta(BS(K_0, \sigma'_0) - BS(K_0, \sigma_0)) + \tilde{\boldsymbol{\chi}} E\left(\text{Vega } d_1 \text{ risk reversal}\right) / \tilde{\boldsymbol{\chi}}\ \Big(E\left(s \frac{\mathrm{d\ Vega}}{\mathrm{d\ s}}\ d_1{=}0 \text{ straddle}\right) - E\left(\text{Vega } d_1{=}0 \text{ straddle}\right)\Big) \tag{71}$$

$$\omega = -\left(E\left(\frac{\mathrm{d\ Vega}}{\mathrm{d}\ \sigma}\ d_1 \text{ risk reversal}\right) + \beta\ E\left(\frac{\mathrm{d\ Vega}}{\mathrm{d}\ \sigma}\ d_1{=}0 \text{ straddle}\right)\right) / \left(E\left(\frac{\mathrm{d\ Vega}}{\mathrm{d}\ \sigma}\ d_1 \text{ strangle}\right) - \left(e^{-\frac{{d_1}^2}{2}} - \alpha\right) E\left(\frac{\mathrm{d\ Vega}}{\mathrm{d}\ \sigma}\ d_1{=}0 \text{ straddle}\right)\right) \tag{72}$$

Calculation of λ(d_1) and χ(d_1) for swaptions and other forward starting underlying assets

The implementation of equations (58),(60), and (64) to options on forward starting assets is straight forward. In options on forward starting assets such as swaps, the variable to consider is the underlying forward rate F instead of the underlying spot price, and of course one has to take into account the annuity in the calculations. The integral over the spot price in (64) is replaced by the integral over the forward rate, $\frac{\mathrm{d\ Vega}}{\mathrm{d\ S}}$ is replaced by $\frac{\mathrm{d\ Vega}}{\mathrm{d\ F}}$, and the density function in (8) is replaced by g(F,t). Instead of call and put, we now have receiver and payer of the fixed rate.

$$\tilde{\boldsymbol{\lambda}} \equiv \text{Var}\,(\sigma_{ATM})\,/2 \qquad \text{and} \qquad \tilde{\boldsymbol{\chi}} \equiv \text{Cov}\,(\sigma_{ATM},F)\,/2 \tag{73}$$

For example

$$\mathrm{E}\left(\frac{\mathrm{d\ Vega}}{\mathrm{d}\ \sigma}\ \text{strangle (d1)}\right) \equiv \int_0^T \mathrm{dt} \int_0^\infty \mathrm{dF}\ \mathrm{g(F,t)}\ df(\mathrm{t}) \tag{74}$$

$$\left[\frac{\mathrm{dVega}}{\mathrm{d}\sigma}(\mathrm{F,t,T},\sigma_{\mathrm{t}}(\mathrm{K_{receiver}},\mathrm{T}),\mathrm{K_{receiver}}) + \frac{\mathrm{dVega}}{\mathrm{d}\sigma}\left(\mathrm{F,t,T},\sigma_{\mathrm{t}}\left(\mathrm{K_{payer}},\mathrm{T}\right),\mathrm{K_{payer}}\right)\right]$$

$$\mathrm{E}\left(F\,\frac{\mathrm{d\ Vega}}{\mathrm{d\ F}}\ \text{strangle (d1)}\right) \equiv \tag{75}$$

$$\int_0^T \mathrm{dt} \int_0^\infty \mathrm{dF}\ \mathrm{g(F,t)}\ df(\mathrm{t})\mathrm{F}\left[\frac{\mathrm{dVega}}{\mathrm{dF}}(\mathrm{F,t,T},\sigma_{\mathrm{t}}(\mathrm{K_{receiver}},\mathrm{T}),\mathrm{K_{receiver}}) - \right.$$

$$\left.\frac{\mathrm{dVega}}{\mathrm{dF}}(\mathrm{F,t,T},\sigma_{\mathrm{t}}\left(\mathrm{K_{payer}},\mathrm{T}\right),\mathrm{K_{payer}})\right]$$

where $\frac{d\mathrm{Vega}}{dF}$ includes a $\frac{d\mathrm{An(F,t,T)}}{dF}$ term and the annuity An in the integral changes with the time t, $\mathrm{An} = \mathrm{An(F,T-t)}$

3. Calculating The Probability Density Function

In order to calculate the path integrals of the butterflies and risk reversals and obtain $\zeta_{\text{butterfly}}$ (d_1) and ζ_{RR} (d_1), we need to obtain g(s,t) for $0 \leq t \leq T$. Equations (65) and (66) are consistency equations on the probability density function g(s,T). This is because the set of $\zeta_{\text{butterfly}}$ (d_1) and $\zeta_{RR'}$ (d_1) for all $\{d_1\}$ and σ_0 define the probability density function to maturity as in (8), which is used in the path integrals. In this section, we show how to calculate the density function g(s,T) via an iterative process.

Since we are calculating the valuation of European options, the path integrals should be independent of the path to expiry. Hence, when we solve for g(s,T), we have some freedom in selecting a consistent g(s,t) for t<T. We divide the time to expiry T into very small equal time

steps Δt=T/N. This generates the time series t_1,.., t_N=T. Let us denote by $g_1(s_0$,0 →$s, t_1)$ the probability density function from time 0 to time t_1. s_0 is the underlying asset spot price at time 0 and s is the underlying asset spot price at time t_1. We will now create a term structure so that at any time t_i the forward term structure from t_i to t_{i+1} {t,t+ Δt } is the same and is translational invariant in the spot axis . Therefore, the probability density function g(s,t→S,t+ Δt) is the same for any t and g(s,t→S,t+ Δt)= g(γs,t→ γS,t+ Δt) for all γ. The density function from time 0 to t_2 $g_2(s_0$,0 →s,t_2) is a convolution integral of g_1

$$g_2\ (s_0, 0\ \rightarrow s2, t_2) = \int ds\ g_1(s_0, 0 \rightarrow s, t_1)\ g_1(s, t_1 \rightarrow s2, t_2) \tag{76}$$

and similarly

$$g_n\ (s_0, 0\ \rightarrow s_n, t_n) = \int ds\ g_{n-1}(s_0, 0 \rightarrow s, t_{n-1}) g_1(s, t_{n-1} \rightarrow s_n, t_n) \tag{77}$$

In this methodology, by definition, we preserve the property that the probability to reach time T is independent of the path, provided that the time step is small enough. Hence, for any j we can write the integral in (77) as

$$g_n\ (s_0, 0\ \rightarrow s_n, t_n) = \int ds\ g_{n-j}\left(s_0, 0 \rightarrow s, t_{n-j}\right) g_j\left(s, t_{n-j} \rightarrow s_n, t_n\right) \tag{78}$$

We refer to $g_1(0, t_1)$ as the **kernel density** since all the density functions from 0 to T will be calculated from it. Naturally, the forward density from t_j to t_{j+n} is the same as from 0 to t_n

$$g_n\ (s_0, 0\ \rightarrow s, t_n) = g_{j,j+n}\ (s_0, t_j\ \rightarrow s, t_{j+n}) \tag{79}$$

We use the following method. From a given g(s,T), we calculate the kernel density. Then we obtain $g_n\ (s_0, 0\ \rightarrow s_n, t_n)$ for all n=2,…,N-1. Equation (7) can then be used to calculate the whole volatility smile from inception to t_n for all n=1,…,N in order to obtain g(s,t) for all s,t. This set of volatility smiles automatically gives the **forward** volatility smiles from t_n to T-t_n for all n.

We use the forward volatility smiles to calculate $\mathrm{Vega}(s, t, T, \sigma_t(K_{call}, T), K_{call})$, $\frac{dVega}{d\sigma}(s, t, T, \sigma_t(K_{call}, T), K_c all)$ and $\frac{dVega}{dS}(s, t, T, \sigma_t(K_{call}, T), K_{call})$ and similarly for K_{put} and K_0 in the integral representation in (64). We calculate the integrals (65) and (66) for a wide range of $\{d_1\}$ (e.g. from 0.1 to 5) and obtain from the set of $\zeta_{butterfly}(d_1)$ and $\zeta_{RR'}(d_1)$ a new density function g(s,T). We repeat the process until convergence.

We now develop a recursive process to calculate $g_1(s_0, 0 \rightarrow s, t_1)$ from $g_T(s_0, 0 \rightarrow S, T)$. To reduce the number of calculations, we select $N=2^m$ for some integer m. We use

$$g_{2j}(s_0, 0 \rightarrow s_{2j}, t_{2j}) = \int ds\, g_j(s_0, 0 \rightarrow s, t_j) g_j(s, t_j \rightarrow s_{2j}, t_{2j}) \tag{80}$$

Since the density function is the same for each of the two halves of the period, we start with the known terminal distribution of the asset at expiry $T = 2^m t_1$ and calculate the density for half the period $t = 2^{m-1} t_1$, continuing recursively until we obtain g_1 from g_2.

We start by defining $G_j(s_0, 0 \rightarrow s_j, t_j)$ as the cumulative density function for $g_j(s_0, 0 \rightarrow s_j, t_j)$. We define a one-to-one mapping of G_j to the normal cumulative distribution function N(x) so that

$$G_j(\log(s_j/s_0)) \equiv N(X_j) \tag{81}$$

or

$$X_j \equiv N^{-1}\left(G_j(\log(s_j/s_0))\right) \tag{82}$$

Denote the inverse function as $s_j(X_j)$. Now we define the function $V_j(X)$

$$V_j(X_j) = d \log s_j(X_j) / dX_j \tag{83}$$

$V_j(X)$ must be strictly positive as it is a mapping between two density functions. Therefore,

$$P_{Call}(K, 2jt, s_0) = \int_{-\infty}^{\infty} dX''_j \int_{-\infty}^{\infty} dX'_j\, n(X''_j)\, n(X'_j) \left(e^{\log S_{2j}(X'_j, X''_j)} - K\right)^+ \tag{84}$$

where

$$\log S_{2j}(X'_j, X''_j) = \int_0^{X'_j} V_j(x)dx + \int_0^{X''_j} V_j(x)dx + \log F \tag{85}$$

For j<N/2 the mapping of the function $G_{2j}\big(\log(S_{2j}/s_0)\big)$ to the normal distribution

$$X_{2j} \equiv N^{-1}(G_{2j}\big(\log(S_{2j}/s_0)\big)) \tag{86}$$

is given from the previous step when we calculated g_{2j} from g_{4j}. We express the price of a call option as

$$P_{Call}(K, 2jt, s_0) = \int_{-\infty}^{\infty} dX_{2j} \left(e^{\log S_{2j}(X_{2j})} - K\right)^+ n\big(X_{2j}\big) \tag{87}$$

where

$$\log S_{2j}(X_{2j}) = \int_0^{X_{2j}} V_{2j}(x)dx + \log F_{2j} \tag{88}$$

and

$$F_{2j} = s_0\, e^{(r_l - r_r)_{2j} 2jt} \tag{89}$$

In the first iteration 2j=N, so $G_{2j} = G_T$ is obtained from the first guess smile at time T. We find $V_j(x)$ by equating (84) and (87) for a large set of strikes. We use an optimization method developed by Levenberg (1944), Marquardt (1963) known as LMA, and Kanzow, Yamashita, Fukushima (2004). We define the price of the option (87) with expiry time 2jt as P(K,2jt, s_0) and the price derived from the convolution (84) as $\widehat{P}$(K, 2jt, V_j) and solve for the V which minimizes the target function S(V) defined as

$$S(V) = \sum_{K_i} [(P(K_i, 2jt, , s_0) - \widehat{P}(K_i, 2jt, V))\ Vega(K_i, T)]^2 \tag{90}$$

The summation is over the set of strikes $\{K_i\}$ selected to cover a wide range around the ATM strike. Since $P(K_i, 2jt, , s_0)$ is known, we can select the strikes using d_1. For example, we can select K from d1=-2.5 to +2.5, on both sides of the ATM strike. We multiply by Vega(K_i,T) as a weight function in order to give a higher weight to the area of the ATM strike with higher Delta. Vega is calculated from the known smile P(K,T).

Hence we showed how to obtain the density function of the underlying asset when the smile is determined by the 3 inputs of equation (57). Obviously this means that any three option prices on the smile determine the smile since we can solve for the three variables of (57) from the three given prices. Once the density function g(s,T) is obtained, the volatility smile is determined and the functions $\lambda(d_1,T)$ and $\chi(d_1,T)$ are obtained. Hence $\lambda(d_1,T)$ and $\chi(d_1,T)$ are self-generated. The importance of these functions is clear: it is a lot easier to calculate option prices by using equations (32) and (33) then by integrating the density function in (7). The volatility smile model can be expressed through tables of $\lambda(d_1,T)$ and $\chi(d_1,T)$ for different sets of the underlying three variables. Therefore we want to characterize the smile using the three variables defined now.

The BS Delta of the call and put options are

$$\Delta_{call} = \frac{dPcall}{dS} = e^{-r_f T} N(d_1) \quad ; \quad \Delta_{put} = -e^{-r_f T} N(-d_1) \qquad (95)$$

For example, a 25 delta call corresponds to a d_1 such that $25\% = e^{-r_f T} N(d_1)$ or

$$d_1 25\Delta = N^{-1}(0.25\, e^{r_f T}) \qquad (96)$$

When the interest rate $r_f = 0$, then $d_1 25\Delta = 0.674$.

$$25 \text{ Delta Risk Reversal} = \sigma\,(d_1 25\Delta) - \sigma\,(-d_1 25\Delta) \equiv 25\Delta RR$$

$$\text{Delta Neutral ATM volatility} = \sigma\,(d_1 = 0) \equiv \sigma_0 \qquad (97)$$

$$25 \text{ Delta butterfly} = (\sigma\,(d_1 25\Delta) + \sigma\,(-d_1 25\Delta))/2 - \sigma_0 \equiv 25\Delta Fly$$

We use the three variables of equation (97) to characterize a smile. For each set of { σ_0 , 25ΔRR, 25ΔFly} we have $\lambda(d_1,T) = \lambda(d_1,T, \sigma_0, 25\Delta RR, 25\Delta Fly)$ and $\chi(d_1,T)= \chi(d_1,T, \sigma_0, 25\Delta RR, 25\Delta Fly)$. For simplicity we omit σ_0 , 25ΔRR , and 25ΔFly going forward.

While the process we described to obtain g(s,T) did not directly involve the functions $\lambda(d_1,T)$ and $\chi(d_1,T)$, there is an alternative process to obtain g(s,T) by actually solving for the self-consistent $\lambda(d_1,T)$ and $\chi(d_1,T)$ that satisfy equations (65) and (66). $\lambda(d_1,T)$ and $\chi(d_1,T)$ are also solved via an iterative process. Using a first guess for $\lambda(d_1,T)$ and $\chi(d_1,T)$, the corresponding density function $g_T(s_0,0\rightarrow S, T)$ is calculated. Like before, we can then recursively calculate the kernel density.

Following the calculation of all the density functions $g_1, \ldots, g_N$, we calculate the implied term structure that corresponds to each of the density functions from 1 to N: $\lambda(d1, t_j)$, $\chi(d_1, t_j)$, $\sigma_0(t_j)$, $25\Delta\ RR(t_j)$, $25\Delta\ Fly(t_j)$ for $1\leq j \leq N$ and also automatically obtain the forward term structure from time t_j to T: $\lambda_{t_j}(d_1, T-t_j)$, $\chi_{t_j}(d_1, T-t_j)$, $\sigma_{t_j}(T-t_j)$, $25\Delta\ RR_{t_j}(T-t_j)$, $25\Delta\ Fly_{t_j}(T-t_j)$ for $1\leq j \leq N-1$. We use this term structure in the integral representation and calculate the new $\lambda(d_1,T)$ and $\chi(d_1,T)$ via (65) and (66).

$$\lambda(d_1, T) = \tilde{\boldsymbol{\lambda}}(E(\frac{d\,Vega}{d\,\sigma}\text{strangle}(d_1)) - (e^{-\frac{d_1^{\,2}}{2}} - \alpha)\ E(\frac{d\,Vega}{d\,\sigma}\text{straddle}(d_1{=}0))) / \frac{d\,Vega}{d\,\sigma}(\text{strangle}(d_1)) \quad (98)$$

$$\chi(d_1, T) = \tilde{\boldsymbol{\chi}}\ E(\frac{d\,Vega}{d\,s}\text{risk reversal}(d_1)) / \frac{d\,Vega}{d\,s}(\text{risk reversal}(d_1)) \quad (99)$$

We now take the new $\lambda(d_1,T)$, $\chi(d_1,T)$ and recalculate the corresponding g(s,T) and repeat the whole process. We continue with these iterations until we reach convergence in $\lambda(d_1,T)$ and $\chi(d_1,T)$.. We define the convergence in the Mth iteration by the condition on the shape functions

$$\left|F_{\lambda,\chi}^{M+1}(d_1) - F_{\lambda,\chi}^{M}(d_1)\right| < 0.001 \quad (100)$$

Finally, when we calculate g(s,T) or $\lambda(d_1,T)$ and $\chi(d_1,T)$ for forward paying assets such as swaptions, we apply the same process on equations (74) and (75) and calculate the kernel density function $g_1(F_0, 0 \rightarrow F_1, t_1)$ followed by the density functions $g_n(F_0, 0 \rightarrow F_n, t_n)$ n=2,…,N-1 with F_n corresponding to the forward rate of the underlying swap at time t_n.

4. Example: Option Pricing at Constant Volatility

In this section, we demonstrate the methodology from section 2 to derive European Vanilla option prices in the case of constant volatility. We derive the path integral obtained from the risk neutral valuation of the expected profit generated from continuous re-hedging against instantaneous changes in the price of the underlying asset and show that the resulting density function g(s,t) is the lognormal distribution function. Hence, our method leads to the BS model without any pre-determined assumption on the probability distribution of the underlying asset. At the end of the section we expand the method to the general case.

As we did in section 2, let us divide the time to expiry T into N time intervals $t_i = iT/N$. At each time i, the change in the underlying asset price from time t_{i-1} to t_i is denoted $\delta s_i = s_i - s_{i-1}$.

We first consider a Delta neutral d_1 strangle with expiry T. We denote the strikes of the butterfly as K_{call}, K_{put}. The change in the value of the strangle (up to a second order) from time t_{i-1} to time t_i is

$$\delta \Pi i\,(\text{strangle}, t_i) = \text{Theta}(t_{i-1})(t_i - t_{i-1}) + \text{Delta}(t_{i-1})\,\delta s_i + \frac{1}{2}\,\delta\,\text{Delta}(t_{i-1})\,\delta s_i = \text{Theta}(t_{i-1})\,\delta t_i + \text{Delta}(\text{strangle}, t_{i-1})\,\delta s_i + \frac{1}{2}\text{Gamma}(\text{strangle}, t_{i-1})\,\delta s_i^2 \quad (101)$$

Where as usual Gamma(strangle ,t_{i-1})= Gamma(K_{call} , t_{i-1}) + Gamma(K_{put} ,t_{i-1}) and

$$\text{Theta}(t_i) = \frac{d\,\Pi\,(\text{strangle}, t_i)}{d\,t} \;;\quad \text{Gamma}(t_i) = \frac{d\,\text{Delta}\,(\text{strangle}, t_i)}{d\,s} \quad (102)$$

The re-hedging strategy for changes in the underlying asset is as follows. At time t_i , the hedger buys or sells a certain amount of the underlying asset at the market price s_i so that the total amount of Delta of the strangle and the hedge is zero. Therefore, the amount of the hedge is the opposite of the Delta of the strangle. Up to a second order, the total profit/loss from the strangle and the hedge is

$$\delta \Pi i \text{ (strangle + hedge, } t_i) = \text{Theta}(t_{i-1})(t_i - t_{i-1}) + \frac{1}{2}\text{Gamma (strangle, } t_{i-1}) \delta s_i^2 \qquad (103)$$

For simplicity we assume that the interest rates are zero.

Therefore the expected profit from holding the strangle until maturity while re-hedging with the underlying asset is

$$E(\sum_{i=1}^{N} \delta \Pi i(\text{butterfly} + \text{hedge}) = E(\sum_{i=1}^{N} \text{Gamma(strangle, } t_{i-1}) \delta s_i^2/2 + E(\sum_{i=1}^{N} \text{Theta}(t_{i-1}) \delta t_i) = E(\sum_{i=1}^{N} \frac{1}{2} s_{i-1}^2 \text{Gamma(strangle, } t_{i-1}) (\frac{\delta s_i}{s_{i-1}})^2 + \text{Theta}(t_{i-1}) \delta t_i \qquad (104)$$

Now, if δs_i is independent of s_i then

$$E(\sum_{i=1}^{N} \delta \Pi i) = \frac{1}{2} E(\sum_{i=1}^{N} s_{i-1}^2 \text{Gamma (strangle, } t_{i-1})) E(\sum_{i=1}^{N} (\frac{\delta s_i}{s_{i-1}})^2/N) + E(\sum_{i=1}^{N} \text{Theta } (t_{i-1}) \delta t_i) \qquad (105)$$

$$= \frac{1}{2}\sigma^2 E(\sum_{i=1}^{N} s_{i-1}^2 \text{Gamma (strangle, } t_{i-1})) + E(\sum_{i=1}^{N} \text{Theta } (t_{i-1}) \delta t_i)$$

where $\sigma^2 \equiv \frac{1}{N} E \sum_{i=1}^{N} (\frac{\delta s_i}{s_{i-1}})^2$

σ^2 is the expected variance of the underlying asset's return in the period from inception (the time of calculating the option) until maturity. Hence the price of the strangle at inception is

$$\Pi(\text{strangle}, t = 0) = \frac{1}{2}\sigma^2 E(\sum_{i=0}^{N-1} s_i^2 \text{Gamma (strangle, } t_i)) + E(\sum_{i=0}^{N-1} \text{Theta } (t_i) \delta t_{i+1}) \qquad (106)$$

Similarly we can calculate the profit for the d_1 risk reversal with re-hedging using the underlying asset. However, since the risk reversal has non-zero delta we need to consider a delta hedged risk reversal. In this case

$\Pi(\text{Delta hedged } d_1 \text{ risk reversal}, \mathrm{t}=0) = \text{Call}(K_{call})$- $\text{Put}(K_{put}) - s_0\text{Delta}(K_{call}) + s_0\text{Delta}(K_{put})$

$$=\frac{1}{2}\sigma^2 \text{E}\left(\sum_{i=0}^{N-1} s_i^2 \text{ Gamma(risk reversal, } t_i)\right)+ \text{ E}\left(\sum_{i=0}^{N-1} \text{Theta } (t_i)\ \delta t_{i+1} \right) \qquad (107)$$

(and $\text{Delta}(K_{put}) = -\text{Delta}(K_{call})$)

When we translate (106) and (107) to the path integral form we obtain

$$\text{P (strangle } (d_1)) = \int_0^{\mathrm{T}} \mathrm{dt} \int_0^{\infty} \mathrm{ds}\ \ \mathrm{g(s,t)}$$
$$\left[\frac{1}{2}\ \sigma^2 \mathrm{s}^2\ \left(\mathrm{Gamma}(\mathrm{s,t,T},\sigma_{\mathrm{t}}(\mathrm{K_{call}},\mathrm{T}),\mathrm{K_{call}}) + \mathrm{Gamma}\left(\mathrm{s,t,T},\sigma_{\mathrm{t}}\left(\mathrm{K_{put}},\mathrm{T}\right),\mathrm{K_{put}}\right)\right) + \mathrm{Theta}(\mathrm{s,t,T},\sigma_{\mathrm{t}}(\mathrm{K_{call}},\mathrm{T}),\mathrm{K_{call}}) + \mathrm{Theta\ Gamma}\left(\mathrm{s,t,T},\sigma_{\mathrm{t}}\left(\mathrm{K_{put}},\mathrm{T}\right),\mathrm{K_{put}}\right)\right] \qquad (108)$$

$$\text{P (risk reversal } (d_1)) - s_0\text{Delta (risk reversal } d_1)) = \int_0^{\mathrm{T}} \mathrm{dt} \int_0^{\infty} \mathrm{ds}\ \ \mathrm{g(s,t)}$$
$$\left[\frac{1}{2}\ \sigma^2\ \mathrm{s}^2 \left(\mathrm{Gamma}(\mathrm{s,t,T},\sigma_{\mathrm{t}}(\mathrm{K_{call}},\mathrm{T}),\mathrm{K_{call}}) - \mathrm{Gamma}\left(\mathrm{s,t,T},\sigma_{\mathrm{t}}\left(\mathrm{K_{put}},\mathrm{T}\right),\mathrm{K_{put}}\right)\right) + \mathrm{Theta}(\mathrm{s,t,T},\sigma_{\mathrm{t}}(\mathrm{K_{call}},\mathrm{T}),\mathrm{K_{call}}) - \mathrm{Theta\ Gamma}\left(\mathrm{s,t,T},\sigma_{\mathrm{t}}\left(\mathrm{K_{put}},\mathrm{T}\right),\mathrm{K_{put}}\right)\right] \qquad (109)$$

Where P(strangle) and P(risk reversal) are the prices of the strangle and risk reversal respectively, at inception.

Now we follow the procedure in section 3 to solve for g(s,T). It is easy to see that if the density function g(s,T) is

$$\mathrm{g(s,T)} = \frac{1}{\sigma \mathrm{s}\sqrt{2\pi T}} e^{-(\log\frac{s}{s_0} - T\sigma^2/2)^2/(2\sigma^2 T)} \qquad (110)$$

then for every t_j the density function that satisfies equations (76)-(80) is

$$g(s,t_j) = \frac{1}{\sigma s\sqrt{2\pi t_j}} e^{-(\log\frac{s}{s_0} - t_j\sigma^2/2)^2/(2\sigma^2 t_j)} \quad (111)$$

$$g(s,t_j, s_T,T_,) = \frac{1}{\sigma s_T\sqrt{2\pi(T-t_j)}} e^{-(\log\frac{s_T}{s} -(T-t_j)\sigma^2/2)^2/(2\sigma^2(T-t_j))} \quad (112)$$

where $g(s,t_j, s_T,T_,)$ is the forward density function at the underlying asset price s from time t_j to T. When substituting (111) and (112) into the integrals in (108) and (109) and using

$$P_{call}(s,K_{call},t,T_,) = \int_0^\infty ds_T\ (s_T - K)^+ \frac{1}{\sigma s_T\sqrt{2\pi(T-t)}} e^{-(\log\frac{s_T}{s} -(T-t)\sigma^2/2)^2/(2\sigma^2(T-t))} \quad (113)$$

$$P_{put}(s,K_{put},t,T_,) = \int_0^\infty ds_T\ (K - s_T)^+ \frac{1}{\sigma s_T\sqrt{2\pi(T-t)}} e^{-(\log\frac{s_T}{s} -(T-t)\sigma^2/2)^2/(2\sigma^2(T-t))} \quad (114)$$

we obtain the BS prices of the d_1 strangles and d_1 risk reversals. Therefore, without any assumption on the stochastic behavior of the underlying asset, we reached the conclusion that when the underlying asset's return $\delta s_i/s_i$ is independent of the underlying asset price s_i and no additional factor affects the price of the option (e.g. the variance of the return of the price of the underlying asset does not change), then up to second order, the option price is the BS price.

It should be mentioned that rather than calculating the correction to the BS model due to non-constant volatility, which we did in order to coincide with the volatility smile representation, we can derive the price of butterflies and risk reversals by considering simultaneously the gains from all hedges. In order to calculate the price of a d_1 butterfly, we calculate the expected value of the butterfly under the following hedging strategy: at each time t_i , we re-hedge the Delta with the underlying asset and re-hedge the Vega with the ATM (d_1=0) straddle which has zero Delta. Hence, the Vega re-hedging does not affect the Delta hedging of the butterfly, but it adds to the Gamma and Theta of the butterfly in (104).

Since at each time t_i the Vega (ATM hedge) = -Vega (butterfly), the notional (amount) of the ATM straddle at time t_i is

$$-\text{Vega(butterfly)}/\text{Vega}(d_1=0 \text{ straddle}) = -\text{Vega(butterfly)}\sqrt{2\Pi}/(2F df\sqrt{T-t_i}) \quad (115)$$

The Gamma and Theta of the ATM straddle hedge satisfy

$$\text{Gamma(ATM hedge)} = -\text{Vega(butterfly)}/\ \sigma_i {s_i}^2(\mathrm{T}-t_i) \quad (116)$$

$$\text{Theta(ATM hedge)} = \frac{1}{2}\text{Vega(butterfly)}\frac{\sigma_i}{\mathrm{T}-t_i} \quad (117)$$

Therefore when we take all the contributions into account we obtain

$$\text{P}(d_1 \text{ butterfly}) = \frac{1}{2}\sigma^2\text{E}\Big(\sum_{i=0}^{N-1} s_i^2 \text{Gamma (butterfly, } t_i) - \text{Vega(butterfly, } t_i)/\sigma_i(\mathrm{T}-t_i)\Big) + \text{E}\Big(\sum_{i=0}^{N-1}(\text{Theta (butterfly, } t_i) + \frac{1}{2}\text{Vega(butterfly, } t_i)\frac{\sigma_i}{\mathrm{T}-t_i}(1-2\text{N}(\sigma_i\sqrt{T-t_i}))\delta t_i\Big) + \frac{1}{2}\text{Var}(\sigma_{ATM})\,\text{E}\Big(\sum_{i=0}^{N-1}\frac{\text{d Vega}}{\text{d}\,\sigma}(\text{butterfly, } t_i)\Big) + \quad (118)$$

$$\frac{1}{2}\text{Cov}(\sigma_{ATM}, s)\ \text{E}\Big(\sum_{i=0}^{N-1}\Big(s_i\frac{\text{d Vega}}{\text{d s}}(\text{butterfly, } t_i) - \text{Vega(butterfly, } t_i)\Big)\Big)$$

And similarly for the delta hedged d_1 risk reversal

$$\text{P}(d_1 \text{ risk reversal}) - 2s_0\text{Delta}(d_1) = \frac{1}{2}\sigma^2\text{E}\Big(\sum_{i=0}^{N-1} s_i^2\text{Gamma (risk reversal, } t_i) - \text{Vega(risk reversal, } t_i)/\sigma_i(\mathrm{T}-t_i)\Big) + \text{E}\Big(\sum_{i=0}^{N-1}(\text{Theta (risk reversal, } t_i) + \frac{1}{2}\text{Vega(risk reversal, } t_i)\frac{\sigma_i}{\mathrm{T}-t_i}(1-2\text{N}(\sigma_i\sqrt{T-t_i}))\delta t_i\Big) + \quad (119)$$

$$\frac{1}{2}\text{Var}(\sigma_{ATM})\,\text{E}\Big(\sum_{i=0}^{N-1}\frac{\text{d Vega}}{\text{d}\,\sigma}(\text{risk reversal, } t_i)\Big) +$$

$$\frac{1}{2}\text{Cov}(\sigma_{ATM}, s)\ \text{E}\Big(\sum_{i=0}^{N-1}\Big(s_i\frac{\text{d Vega}}{\text{d s}}(\text{risk reversal, } t_i) - \text{Vega(risk reversal, } t_i)\Big)\Big)$$

(108) and (109) should be modified accordingly. It is easy to see that when the volatility is constant (i.e. $\sigma_i \equiv \sigma$ for all i and the density function is (110)) then, as required, (118) and (119) become (106) and (107) respectively. Hence, instead of solving for g(s,T) in the equations for $\zeta\ (d_1\ \text{butterfly})$ and $\zeta\ (d_1\ \text{risk reversal})$ in (58) and (60), we can solve for it directly from the path integrals of (118) and (119).

5. Analysis of The Volatility Smile Model

In this section, we analyze the properties of the volatility smile by using the smile representation of section 1 with the functions $\lambda(d_1,T)$ and $\chi(d_1,T)$ that are calculated in section 3. As discussed, these functions will depend on three inputs which we will choose in this section. In addition, we analyze the density function g(s,T) that is obtained from the model.

We represent $\lambda(d_1,T) = \lambda(d_1,T, \sigma_0, 25\Delta\text{RR}, 25\Delta\text{Fly})$ and $\chi(d_1,T)= \chi(d_1,T, \sigma_0, 25\Delta\text{RR}, 25\Delta\text{Fly})$ in the form of a scale factor and a shape function that contains the dependency on d_1 and is normalized to 1 at $d_1 25\Delta = N^{-1}(0.25\ e^{r_f T})$.

$$\lambda(T, d_1) = \lambda_0(T)\ F_\lambda(T,d_1) \qquad (120)$$

$$\chi(T, d_1) = \chi_0(T)\ F_\chi(T,d_1) \qquad (121)$$

$$F_\lambda(d_1 25\Delta, T)= F_\chi(d_1 25\Delta, T)=1 \qquad (122)$$

$\lambda_0(0,t)$ and $\chi_0(0,t)$ can be determined directly from equation (32) and (33)

$$\zeta_{\text{strangle}}\ (d_1 25\Delta,T) = \lambda_0(T)\ \frac{\partial \text{Strangle}}{\partial \sigma}\ (d_1 25\Delta) \qquad (123)$$

$$\zeta_{\text{RR}'}\ (d_1 25\Delta,T) = \chi_0(T)\ \frac{\partial \text{RR}}{\partial \sigma}\ (d_1 25\Delta) \qquad (124)$$

In order to gain better insight into the functions $\lambda(d_1,T, \sigma_0, 25\Delta\text{RR}, 25\Delta\text{Fly})$ and $\chi(d_1,T, \sigma_0, 25\Delta\text{RR}, 25\Delta\text{Fly})$, we will plot the shape functions for different sets of $\{\sigma_0, 25\Delta\text{RR}, 25\Delta\text{Fly}\}$ and expiration.

Table 2 shows the shape functions F_λ, F_χ of λ and χ for different market data and expiries. We use the following σ_0, 25Δ RR, and 25Δ Fly: {10,1,0.1}, {12,2,0.25}, {15,3,0.75}, {18,4,1}, {25,5,1.2}, {30,6,1.2}, {45,10,2} for the following maturities: three month, one year, two years and three years.

N=4																		
Expiry	**3 months**																	
σ ATM	**25dRR**	**25dFly**		**d_1**	**0.25**	**0.5**	**0.75**	**1**	**1.25**	**1.5**	**1.75**	**2**	**2.25**	**2.5**	**2.75**	**3**	**3.25**	**3.5**
10.00%	1.000%	0.100%		F_A	0.982	0.993	1.003	1.010	1.013	1.018	1.022	1.011	0.965	0.896	0.833	0.789	0.765	0.746
				F_B	0.984	0.994	1.003	1.008	1.009	1.008	1.001	0.978	0.927	0.868	0.828	0.810	0.782	0.756
				F_A/F_B	**0.9980**	**0.9989**	**1.0005**	**1.0019**	**1.0043**	**1.0099**	**1.0207**	**1.0345**	**1.0420**	**1.0324**	**1.0068**	**0.9734**	**0.9776**	**0.9876**
12.00%	2.000%	0.250%		F_A	0.970	0.995	1.002	0.984	0.959	0.943	0.932	0.905	0.874	0.849	0.829	0.788	0.702	0.593
				F_B	0.971	0.994	1.003	0.990	0.966	0.946	0.921	0.885	0.856	0.844	0.841	0.814	0.733	0.622
				F_A/F_B	**0.9990**	**1.0012**	**0.9995**	**0.9942**	**0.9919**	**0.9975**	**1.0120**	**1.0223**	**1.0212**	**1.0060**	**0.9847**	**0.9678**	**0.9575**	**0.9548**
15.00%	3.000%	0.750%		F_A	1.053	1.057	0.975	0.889	0.830	0.798	0.773	0.744	0.706	0.641	0.566	0.499	0.446	0.406
				F_B	1.078	1.052	0.977	0.904	0.858	0.827	0.792	0.753	0.704	0.631	0.553	0.488	0.436	0.396
				F_A/F_B	**0.9768**	**1.0044**	**0.9980**	**0.9825**	**0.9676**	**0.9659**	**0.9750**	**0.9869**	**1.0029**	**1.0165**	**1.0220**	**1.0229**	**1.0239**	**1.0261**
18.00%	4.000%	1.000%		F_A	1.061	1.074	0.968	0.869	0.806	0.772	0.743	0.716	0.679	0.617	0.545	0.484	0.435	0.399
				F_B	1.112	1.071	0.969	0.886	0.839	0.809	0.774	0.737	0.688	0.616	0.542	0.480	0.431	0.394
				F_A/F_B	**0.9538**	**1.0026**	**0.9988**	**0.9807**	**0.9603**	**0.9544**	**0.9611**	**0.9715**	**0.9873**	**1.0011**	**1.0064**	**1.0074**	**1.0094**	**1.0127**
25.00%	5.000%	1.200%		F_A	1.090	1.074	0.968	0.874	0.815	0.784	0.758	0.730	0.699	0.645	0.573	0.506	0.453	0.415
				F_B	1.131	1.071	0.969	0.892	0.853	0.829	0.799	0.766	0.727	0.665	0.584	0.510	0.451	0.405
				F_A/F_B	**0.9636**	**1.0033**	**0.9984**	**0.9797**	**0.9558**	**0.9448**	**0.9479**	**0.9529**	**0.9610**	**0.9705**	**0.9798**	**0.9905**	**1.0046**	**1.0249**
30.00%	6.000%	1.200%		F_A	1.089	1.061	0.974	0.892	0.838	0.808	0.784	0.758	0.734	0.697	0.630	0.553	0.489	0.444
				F_B	1.105	1.056	0.976	0.910	0.873	0.851	0.823	0.795	0.773	0.735	0.661	0.573	0.498	0.440
				F_A/F_B	**0.9855**	**1.0048**	**0.9977**	**0.9809**	**0.9597**	**0.9493**	**0.9522**	**0.9527**	**0.9505**	**0.9487**	**0.9541**	**0.9651**	**0.9824**	**1.0105**
45.00%	10.000%	2.000%		F_A	1.065	1.056	0.976	0.898	0.843	0.808	0.776	0.749	0.727	0.691	0.628	0.554	0.497	0.475
				F_B	1.093	1.055	0.976	0.908	0.866	0.836	0.806	0.786	0.770	0.741	0.676	0.596	0.521	0.465
				F_A/F_B	**0.9744**	**1.0006**	**0.9997**	**0.9889**	**0.9737**	**0.9658**	**0.9628**	**0.9527**	**0.9436**	**0.9327**	**0.9286**	**0.9306**	**0.9534**	**1.0218**

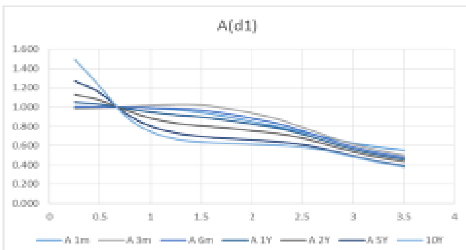
Expiry	**1 year**																	
σ ATM	**25dRR**	**25dFly**		**d_1**	**0.25**	**0.5**	**0.75**	**1**	**1.25**	**1.5**	**1.75**	**2**	**2.25**	**2.5**	**2.75**	**3**	**3.25**	**3.5**
10.00%	1.000%	0.100%		F_A	0.980	0.994	1.003	1.002	0.998	1.001	1.012	1.012	0.978	0.930	0.897	0.860	0.823	0.789
				F_B	0.977	0.991	1.004	1.010	1.010	1.013	1.015	1.002	0.967	0.939	0.908	0.879	0.851	0.823
				F_A/F_B	**1.0028**	**1.0028**	**0.9988**	**0.9920**	**0.9878**	**0.9888**	**0.9968**	**1.0092**	**1.0113**	**0.9903**	**0.9878**	**0.9779**	**0.9680**	**0.9583**
12.00%	2.000%	0.250%		F_A	1.024	1.019	0.992	0.953	0.920	0.903	0.892	0.866	0.839	0.818	0.802	0.775	0.709	0.607
				F_B	1.015	1.011	0.995	0.972	0.951	0.937	0.917	0.884	0.860	0.854	0.859	0.849	0.788	0.676
				F_A/F_B	**1.0081**	**1.0082**	**0.9964**	**0.9805**	**0.9677**	**0.9639**	**0.9726**	**0.9798**	**0.9751**	**0.9576**	**0.9339**	**0.9129**	**0.8995**	**0.8978**
15.00%	3.000%	0.750%		F_A	1.114	1.080	0.966	0.870	0.810	0.778	0.750	0.723	0.692	0.638	0.568	0.504	0.453	0.415
				F_B	1.152	1.077	0.967	0.887	0.848	0.824	0.794	0.762	0.723	0.661	0.583	0.513	0.456	0.412
				F_A/F_B	**0.9668**	**1.0026**	**0.9988**	**0.9807**	**0.9554**	**0.9440**	**0.9454**	**0.9489**	**0.9565**	**0.9662**	**0.9745**	**0.9837**	**0.9943**	**1.0087**
18.00%	4.000%	1.000%		F_A	1.085	1.084	0.964	0.863	0.800	0.762	0.730	0.702	0.671	0.618	0.551	0.490	0.442	0.407
				F_B	1.160	1.089	0.962	0.874	0.830	0.802	0.769	0.741	0.704	0.644	0.570	0.502	0.447	0.402
				F_A/F_B	**0.9351**	**0.9959**	**1.0020**	**0.9874**	**0.9631**	**0.9513**	**0.9493**	**0.9475**	**0.9527**	**0.9599**	**0.9663**	**0.9763**	**0.9886**	**1.0106**
25.00%	5.000%	1.200%		F_A	1.082	1.063	0.973	0.891	0.838	0.802	0.770	0.741	0.713	0.668	0.602	0.537	0.499	0.454
				F_B	1.121	1.065	0.972	0.900	0.859	0.831	0.802	0.782	0.757	0.713	0.642	0.572	0.514	0.488
				F_A/F_B	**0.9651**	**0.9982**	**1.0008**	**0.9904**	**0.9748**	**0.9655**	**0.9600**	**0.9480**	**0.9427**	**0.9373**	**0.9371**	**0.9395**	**0.9700**	**0.9302**
30.00%	6.000%	1.200%		F_A	1.073	1.045	0.981	0.920	0.878	0.847	0.815	0.787	0.764	0.733	0.689	0.628	0.587	0.542
				F_B	1.073	1.037	0.984	0.937	0.903	0.876	0.848	0.836	0.827	0.812	0.772	0.697	0.618	0.597
				F_A/F_B	**1.0002**	**1.0077**	**0.9965**	**0.9826**	**0.9716**	**0.9668**	**0.9606**	**0.9415**	**0.9243**	**0.9020**	**0.8930**	**0.9013**	**0.9501**	**0.9081**
45.00%	10.000%	2.000%		F_A	1.033	1.014	0.994	0.979	0.963	0.931	0.897	0.874	0.847	0.821	0.796	0.771	0.748	0.725
				F_B	0.982	0.988	1.005	1.013	0.999	0.966	0.939	0.920	0.897	0.875	0.854	0.833	0.813	0.793
				F_A/F_B	**1.0524**	**1.0259**	**0.9890**	**0.9662**	**0.9634**	**0.9633**	**0.9556**	**0.9506**	**0.9444**	**0.9381**	**0.9320**	**0.9258**	**0.9197**	**0.9136**

In the ONLINE Appendix 3 we provide the tables for 2 and 3 years

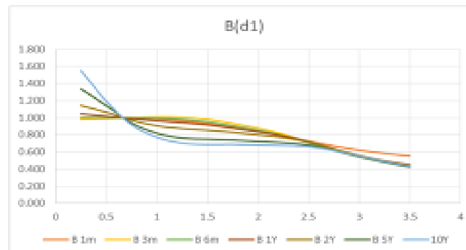
Chart 4 shows the influence of the expiry on the shape functions F_λ, F_χ with σ_0 =18, 25Δ RR=3, 25Δ Fly=0.75 and the following expiries: 1 month, 3 months, 6 months, 1 year, 2 years, 5 years.

Chart 6a: The shape functions for different maturities

σ0 =18, 25Δ RR=3, 25Δ Fly=0.75



$F_\lambda(d_1)$

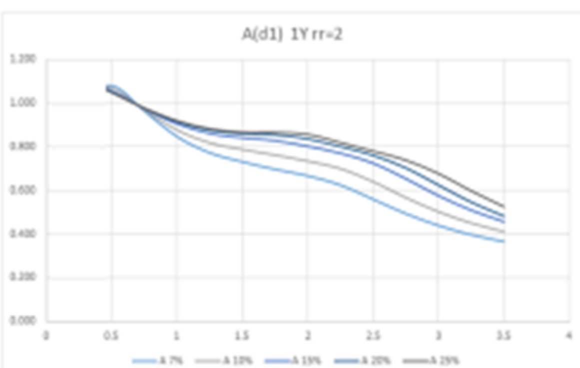


$F_\chi(d_1)$

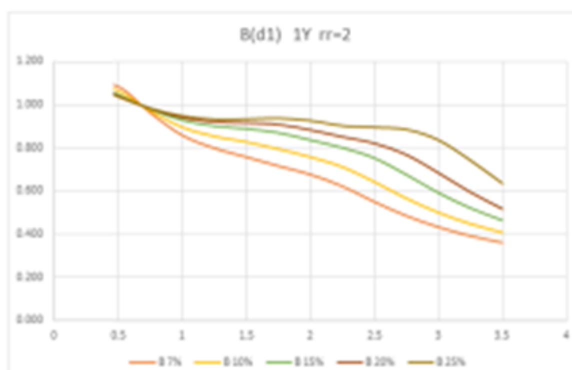
Chart 5 shows the influence of the ATM volatility on the shape functions F_λ, F_χ for expiry 1 year, 25Δ RR=2, 25Δ Fly=0.5 and σ_0 =7%, 10%, 15%, 20%, 25%

25Δ RR=2, 25Δ Fly=0.5
expiry 1 year N=128

Chart 6c: The shape functions for different ATM volatilities



$F_\lambda(d_1)$

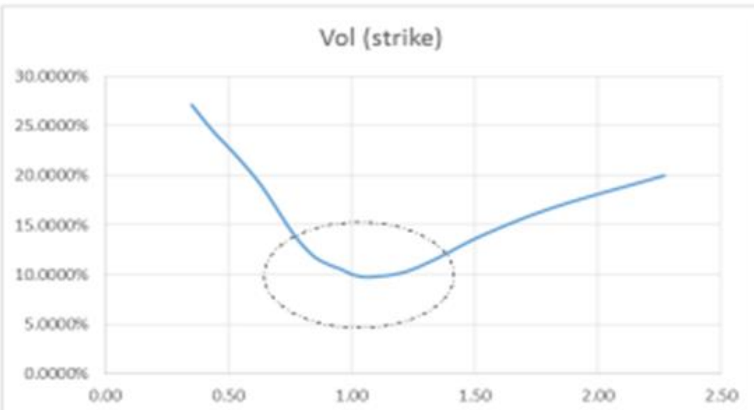


$F_\chi(d_1)$

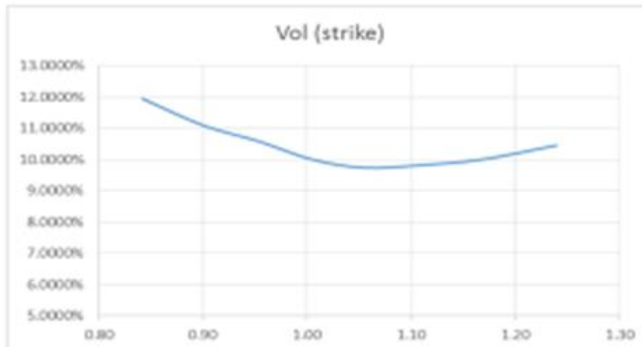
Chart 6 shows the volatility smile in two cases: σ_0 =10, 25Δ RR=1, 25Δ Fly=0.25 with expiry 1 year and σ_0 =15, 25Δ RR=2.5, 25Δ Fly=0.5 with expiry 1 year.

Chart 6d: The volatility smile

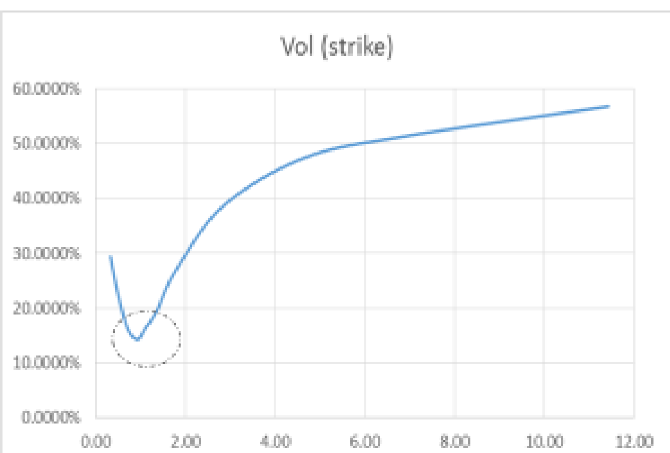
Expiry 1 Year σ_0 =10, 25Δ RR=1, 25Δ Fly=.25



Zoom in



Expiry 1 Year σ_0 =15, 25Δ RR=2.5, 25Δ Fly=.5



Zoom in

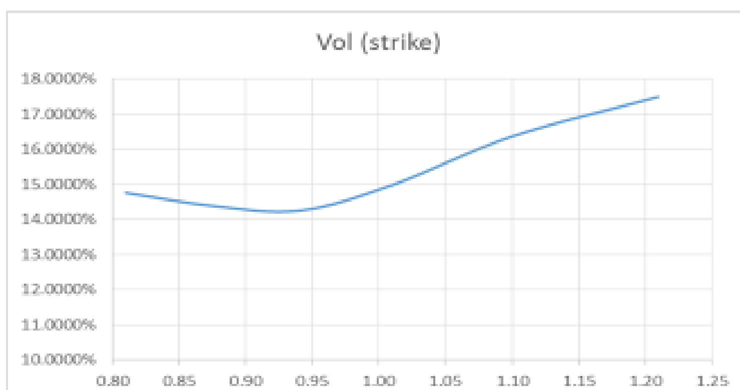


Chart 7 shows the shape functions F_λ, F_χ of λ and χ for five year swaptions for 3 sets of market data { σ_0 =10, 25Δ RR=1, 25Δ Fly=0.25}, {σ_0 =18, 25Δ RR=3, 25Δ Fly=0.75}, {σ_0 =25, 25Δ RR=6, 25Δ Fly=1.5}. The 5-year interest rate is 4% and the 5Y5Y swaption forward rate is 5%. As can be seen, the influence of the annuity on λ and χ is not large.

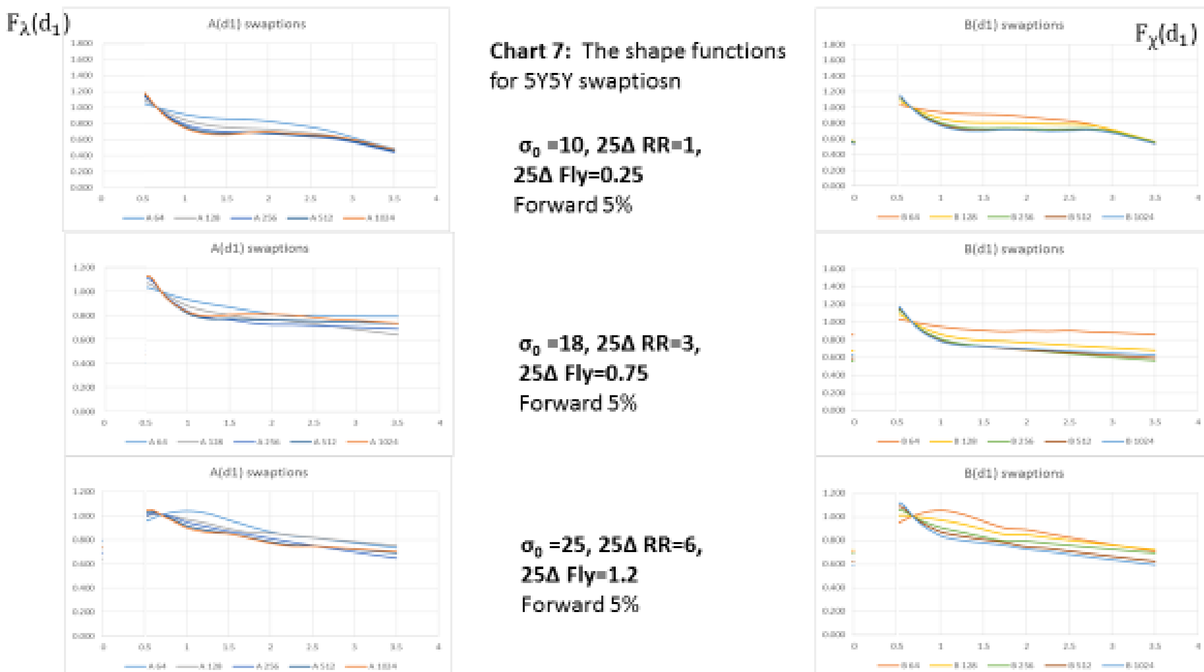
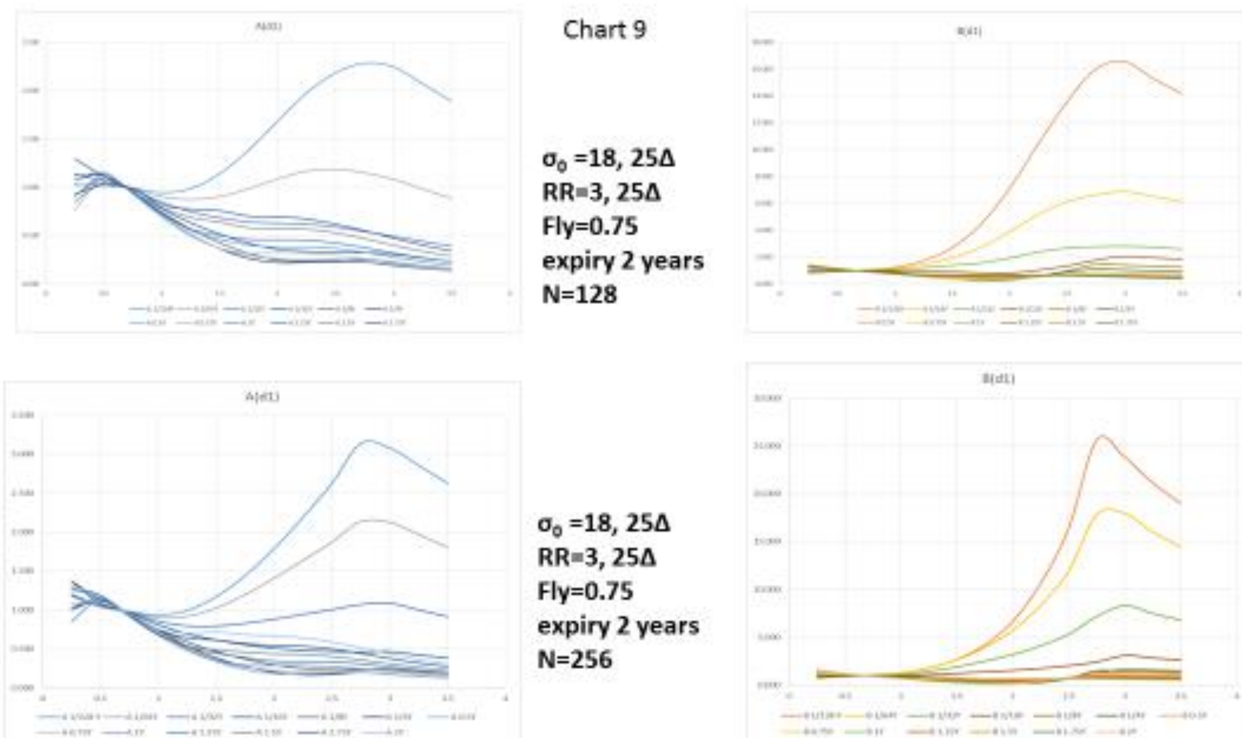


Chart 7: The shape functions for 5Y5Y swaptiosn

In addition we show the resulting term structure of σ_0, 25Δ RR and 25Δ Fly.

As can be seen in chart 9, the kernel compensates for the translational invariance by selecting a steep slope for λ and χ. The slope moderates as the expiry increases. This is why we can produce λ and χ in the translational invariance assumption with very high accuracy.

Chart 8 shows the term structure of the shape functions F_λ, F_χ of λ and χ for N=128 and N=256 for σ_0 =18, 25Δ RR=3, 25Δ Fly=0.75 with expiry 2 years.



$F_\lambda(d_1)$ $F_\chi(d_1)$

Chart 10 shows the density function in our model g(s,T) with the following market parameters for a one year expiry: {σ_0 =20, 25Δ Fly=0.5 and 25Δ RR= -2, 0, 2} and {σ_0 =20, 25Δ Fly=1.5 and 25Δ RR= -2, 0, 2}. In each example we show the resulting volatility smile.

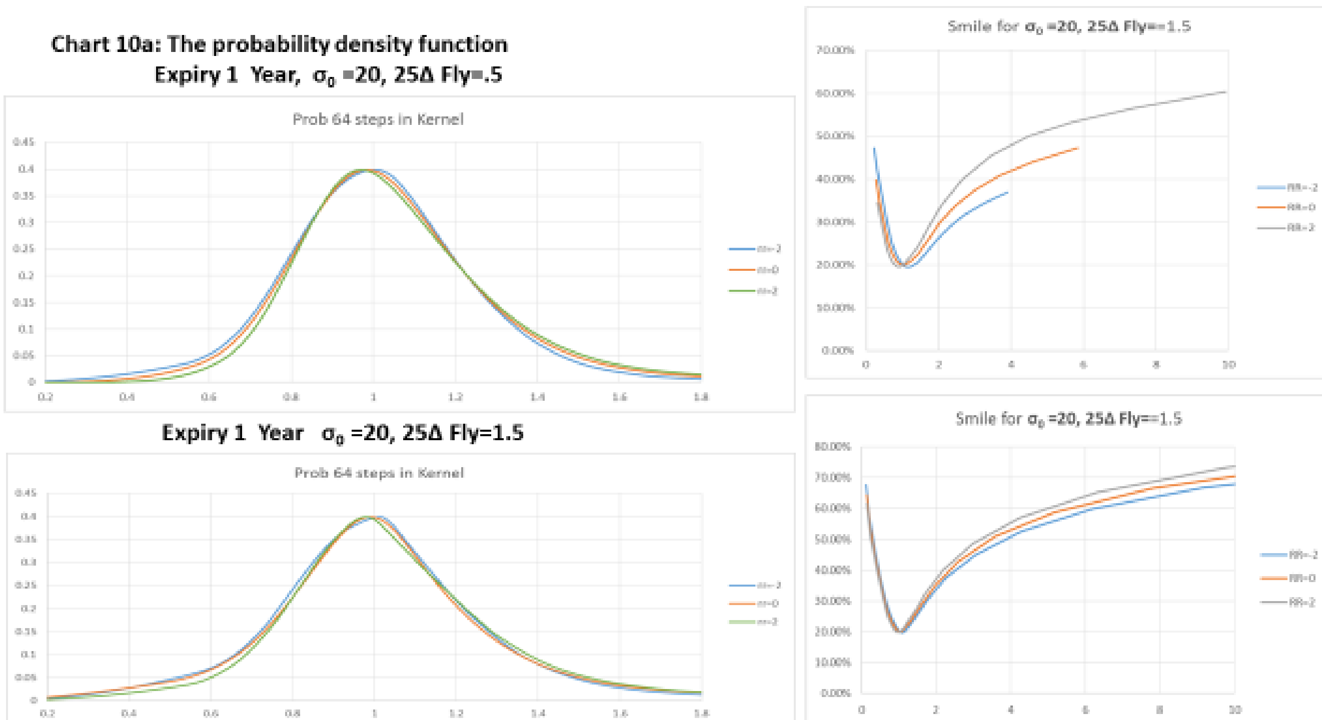


Chart 11a-c shows the mapping between $\log(S_T/S_0)$ and the normal distribution parameter X_T of equation (114) that results from our model (i.e. $\log S_T/S_0\ (X_T)$) for 3 cases: σ_0 =20, 25Δ Fly=0.5 and 25Δ RR= -2, 0, 2 and for an expiry of three months, one year and two years. As a reference, we also display the case where S_T has lognormal density with σ_0 =20.

As can be seen, the graphs are always steeper than the straight line in the lognormal case. Moreover, chart 10b demonstrates that for positive 25Δ RR, the slope steepens for very positive X_T as 25Δ RR gets larger and the graphs slope diminishes for very negative X_T. Similarly, for negative 25Δ RR, the slope steepens for very negative X_T as 25Δ RR gets more negative and the slopes diminish for very positive X_T.

Chart 11d shows the mapping between $\log(S_T/S_0)$ and the normal distribution parameter X_T of equation (114) for 4 cases of 25Δ Fly with fixed 25Δ RR: σ_0 =20, 25Δ RR=0 and 25Δ fly= 0.5, 1, 2, 3 for expiry one year. As can be seen, the larger the 25Δ fly the steeper is the graph for both very negative and very positive X_T.

Chart 11a: Mapping between the probability density function of $\log(S_T/S_0)$ to normal distribution for σ_0 =20, 25Δ Fly=.5, 25Δ RR=-2,0,2 Expiry T= 3 months Year and zero interest rates.

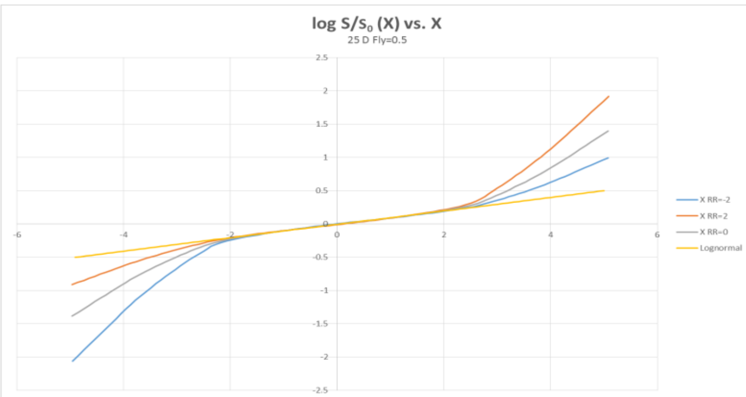


Chart 11b: Mapping between the probability density function of $\log(S_T/S_0)$ to normal distribution for σ_0 =20, 25Δ Fly=.5, 25Δ RR=-2,0,2 Expiry T= 1 Year and zero interest rates.

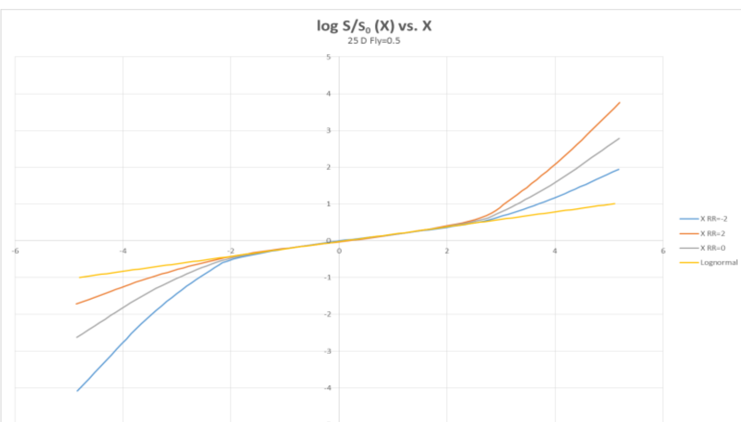


Chart 11c: Mapping between the probability density function of log (S_T/S_0) to normal distribution for σ_0 =20, 25Δ Fly=.5, 25Δ RR=-2,0,2 Expiry T= 2 Years and zero interest rates.

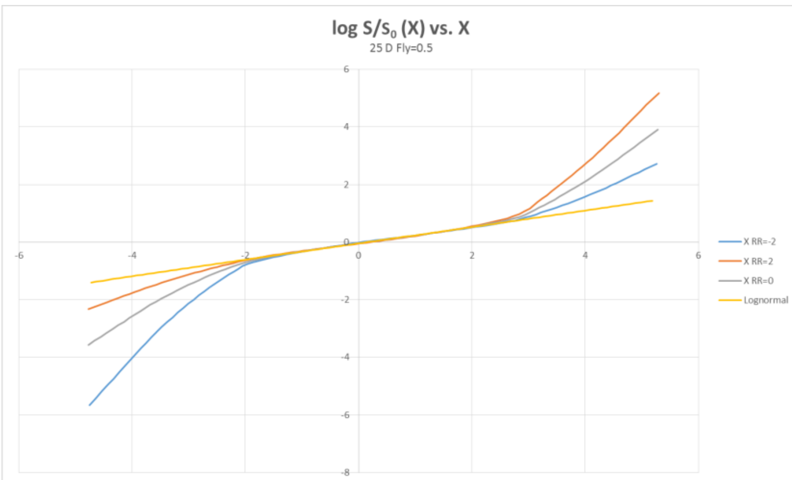
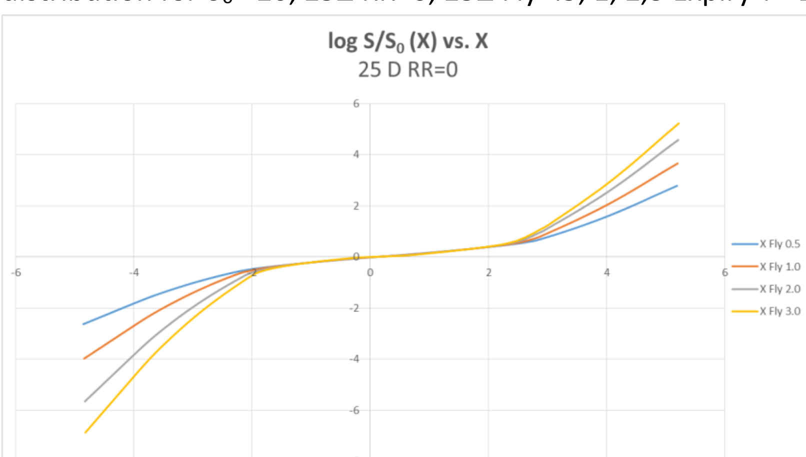


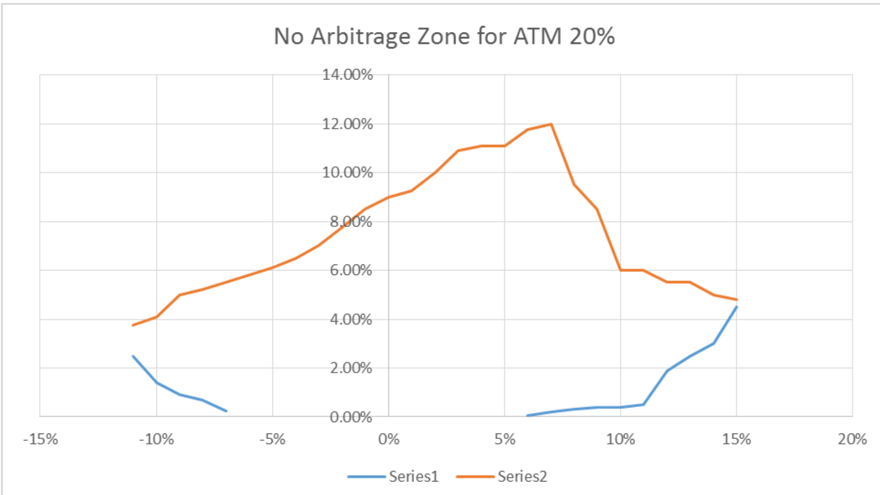
Chart 11d: Mapping between the probability density function of log (S_T/S_0) to normal distribution for σ_0 =20, 25Δ RR=0, 25Δ Fly=.5, 1, 2,3 Expiry T= 1 Year and zero interest rates.



No arbitrage

One important question is to identify the range of market data parameters that is arbitrage free. For example, when we look at (97) obviously 25ΔRR cannot be too large otherwise we may have negative volatility. Hence, given the ATM volatility, we find the region of 25ΔRR and 25Δfly that satisfies the requirement that the density function is strictly positive.

Chart 12 shows the arbitrage free areas for several ATM volatilities for expiry 1 year. In fact we see that for ANY reasonable market data, and especially any market data that ever traded in the market in the past 20 years, there is no arbitrage.



6. Comparison of The Model to Market Prices

In this section, we display the results of a comparison of historical market data to the model suggested in (34) and (35) for FX, equities, and commodities as well.We also compare the model defined in (40) and (41) for swaptions against historical market data. We conclude that our three input model accurately replicates the market prices in all asset classes.

In order to maintain a unified format for all asset classes, we will use the parameterization $\lambda(d_1, T, \sigma_0, 25\Delta RR, 25\Delta Fly)$ and $\chi(d_1, T, \sigma_0, 25\Delta RR, 25\Delta Fly)$ in all cases. Yet, the three natural inputs in each market – FX, equities, commodities and interest rates - are slightly different. When calculating the smile for non-FX markets, it is easier to first solve for the three FX inputs that correspond to the other three inputs. For example, if we are given that the ATMF volatility for 5Y5Y USD swaptions is 27%, the 100bp payer volatility is 27% and the

100 bp volatility receiver is 43%, while the 5Y5Y forward is 2.7%, then we first translate these inputs to Delta neutral ATM volatility 28.8%, 25Delta RR =-9.2%, and 25Delta Fly=4%. With these inputs, we obtain the full smile σ(K) or Price(K) and return to the interest rates convention. Similarly, when we consider exchange prices, for each maturity we find Delta neutral ATM volatility, 25Delta RR, 25Delta Fly and the forward rate that closely replicate the exchange call options.

In the comparison of the model to the market data, we only select liquid assets in order to ensure that we truly represent "the market" without distortions that result from a lack of liquidity. We display the following representatives from all the asset classes here: major currencies as well as liquid emerging market currency pairs in FX options, some very liquid commodity options, interest rate swaptions in highly traded tenors for some major currencies, and liquid stock options that pay no dividend (in order to resemble European options). Moreover, the market data provided by interbank brokers for the last trading date of the year is highly accurate since this data is used for marking to market all the trading books and the annual trading results are generated from this data. Hence we compare the model to data from Dec 31 2014 and Dec 31 2015.

PLEASE NOTICE THAT ALL THE DATA APPEARS IN APPENDIX 3 in the internet. We only bring one example for each asset class in the article itself.

Table 3 compares the model to the FX options market.

We look at nine currency pairs which represent the whole spectrum from the liquidity and interest perspectives: EUR/USD; USD/JPY; EUR/JPY; EUR/GBP; EUR/CHF; GBP/USD; EUR/CHF; USD/KRW; EUR/PLN. The data we have is ATM volatility, the volatility of 25 Delta call, 25 Delta put, 10 Delta call, 10 Delta put. We use the ATM volatility and the volatility of the 25 Delta call and put as the three inputs. We calculate the 10 Delta call and put volatilities in the model and compare them to the market data of 10 Delta calls and puts for 1 month, 3 month, and 1 year expiries.

We need to take into account the market convention of Delta reference when the US Dollar is the left-hand currency (e.g. USD/JPY – US Dollar against the Japanese Yen). In this case, the Delta is calculated such that the premium of the option needs to be hedged as well. As a result, equation (7) is not applicable and even the Delta neutral ATM volatility does not correspond to the d_1=0 strike. In this case, we solve for the Delta neutral ATM volatility, the volatility of the 25 Delta call, and the volatility of the 25 Delta put that correspond to (7) while generating exactly the same output and use them for determining the smile.

Dec 31 2015					
EUR/USD		spot	1.0861		
1 month	**31 days**	**Forward**	**1.08692**		
	ATM	25RR	25Fly	10RR	10Fly
market	9.725	-0.45	0.175	-0.8	0.5
model		-0.45	0.175	**-0.847**	**0.601**
	10d P	25d P	atm	25d call	10 d call
model vol	1.025	10.125	9.725	9.675	9.903
Market vol	10.625	10.125	9.725	9.675	9.825
3 month	**92 days**	**Forward**	**1.08867**		
	ATM	25RR	25Fly	10RR	10Fly
market	9.975	-1.05	0.2	-1.7	0.6
model	9.975	-1.05	0.2	**-1.774**	**0.662**
2	10d P	25d P	atm	25d call	10 d call
model vol	11.524	10.7	9.975	9.65	9.750
Market vol	11.425	10.7	9.975	9.65	9.725
1 Year	**365 days**	**Forward**	**1.10012**		
	ATM	25RR	25Fly	10RR	10Fly
market	10.1	-1.725	0.275	-2.875	0.9
model	10.1	-1.725	0.275	**-2.892**	**0.901**
	10d P	25d P	atm	25d call	10 d call
model vol	12.447	11.2375	10.1	9.5125	9.555
Market vol	12.4375	11.2375	10.1	9.5125	9.5625

USD/JPY		spot	120.32		
1 month	**34 days**	**Forward**	**120.243**		
	ATM	25RR	25Fly	10RR	10Fly
market	7.45	-0.95	0.35	-1.8	1.025
model	7.45	-0.95	0.35	**-1.73**	**1.058**
	10d P	25d P	atm	25d call	10 d call
model vol	9.374	8.275	7.45	7.325	7.642
Market vo	9.375	8.275	7.45	7.325	7.575
3 month	**92 days**	**Forward**	**120.022**		
	ATM	25RR	25Fly	10RR	10Fly
market	8.025	-0.7	0.375	-1.35	1.175
model	8.025	-0.7	0.375	**-1.38**	**1.205**
	10d P	25d P	atm	25d call	10 d call
model vol	9.921	8.75	8.025	8.05	8.540
Market vo	9.875	8.75	8.025	8.05	8.525
1 Year	**369 days**	**Forward**	**119.0774**		
	ATM	25RR	25Fly	10RR	10Fly
market	9.05	-0.275	0.65	-0.45	2.225
model	9.05	-0.275	0.65	**-0.422**	**2.209**
	10d P	25d P	atm	25d call	10 d call
model vol	11.470	9.8375	9.05	9.5625	11.048
Market vo	11.5	9.8375	9.05	9.5625	11.05

Table 4 compares the model to the commodity options market. We consider Brent, Gold, and copper call option prices with maturities one month to 3 years for Brent, 1 year for gold, and 6 months for copper.

Dec 31 2015											
BRENT											
Expiry	**26-Jan-16**			**Expiry**	**24-Mar-16**			**Expiry**	**22-Dec-16**		
Forward	ATM	25dRR	25fFly	Forward	ATM	25dRR	25fFly	Forward	ATM	25dRR	25fFly
37.6569	44.65	-3.71	1.43	39.37	43.26	-2.89	1.22	45.49	33.51	-1.7	0.99
Strike	Exchange	model		Strike	Exchange	model		Strike	Exchange	model	
20	17.68	17.667		20	19.44	19.437		5	40.52	40.498	
21	16.68	16.671		21	18.45	18.451		10	35.54	35.517	
22	15.68	15.677		22	17.46	17.467		15	30.56	30.572	
23	14.68	14.684		23	16.48	16.487		17.5	28.09	28.120	
24	13.69	13.692		24	15.51	15.512		20	25.65	25.686	
25	12.69	12.702		25	14.54	14.544		25	20.93	20.923	
26	11.71	11.714		26	13.58	13.585		27.5	18.65	18.635	
26.5	11.21	11.221		26.5	13.11	13.109		30	16.47	16.440	
27	10.72	10.729		27	12.64	12.637		35	12.42	12.419	
27.5	10.23	10.238		28	11.71	11.705		37.5	10.62	10.629	
28	9.74	9.749		29	10.8	10.791		39	9.62	9.634	
28.5	9.26	9.261		30	9.91	9.900		40	8.99	9.005	
29	8.77	8.776		31	9.04	9.034		41	8.39	8.403	
29.5	8.29	8.294		31.5	8.62	8.613		42	7.82	7.830	
30	7.82	7.816		32	8.2	8.199		43	7.28	7.285	
30.5	7.35	7.342		32.5	7.8	7.794		44	6.77	6.768	
31	6.88	6.875		33	7.4	7.398		45	6.29	6.280	
31.5	6.43	6.413		33.5	7.01	7.012		46	5.83	5.819	
32	5.97	5.960		34	6.63	6.635		46.5	5.61	5.599	
32.5	5.53	5.516		34.5	6.26	6.268		47	5.4	5.386	
33	5.09	5.082		35	5.91	5.912		48	5	4.981	
33.5	4.67	4.659		35.5	5.56	5.567		49	4.62	4.604	
34	4.25	4.250		36	5.23	5.234		50	4.27	4.255	
34.5	3.85	3.856		37	4.6	4.602		51	3.94	3.932	
35	3.47	3.478		37.5	4.3	4.305		52	3.64	3.635	
35.5	3.11	3.118		38	4.02	4.019		53	3.36	3.360	
36	2.77	2.778		38.5	3.75	3.747		54	3.1	3.107	
36.5	2.45	2.459		39	3.49	3.487		55	2.86	2.875	
37	2.16	2.163		39.5	3.24	3.240		56	2.65	2.660	
37.5	1.89	1.890		40	3.01	3.006		57	2.45	2.463	
38	1.64	1.641		40.5	2.79	2.785		58	2.27	2.281	
38.5	1.42	1.418		41	2.58	2.577		59	2.1	2.113	
39	1.22	1.219		41.5	2.39	2.384		60	1.95	1.959	
39.5	1.05	1.045		42	2.2	2.203		61	1.81	1.816	
40	0.9	0.893		42.5	2.03	2.035		62	1.68	1.685	
40.5	0.76	0.761		43	1.88	1.879		63	1.56	1.564	
41	0.65	0.648		43.5	1.73	1.735		64	1.45	1.453	
41.5	0.55	0.552		44	1.6	1.601		65	1.35	1.350	
42	0.47	0.469		44.5	1.47	1.477		66	1.26	1.255	
42.5	0.4	0.399		45	1.36	1.362		67	1.17	1.168	
43	0.34	0.340		45.5	1.25	1.256		68	1.09	1.088	
43.5	0.29	0.291		46	1.16	1.158		69	1.01	1.014	
44	0.25	0.249		46.5	1.06	1.068		70	0.94	0.946	
44.5	0.21	0.213		47	0.98	0.984		71	0.88	0.883	
45	0.18	0.183		47.5	0.9	0.907		72	0.82	0.825	
45.5	0.16	0.158		48	0.83	0.836		73	0.76	0.771	
46	0.14	0.137		48.5	0.77	0.771		74	0.71	0.722	
46.5	0.12	0.120		49	0.71	0.711		75	0.67	0.676	
47	0.11	0.105		49.5	0.65	0.656		76	0.62	0.634	

Table 5 compares the model to the equity options market.

We look at call options on the S&P and Dax indices and Google stock option prices with maturities from about one month to two years. The reason we selected Google is that the company never pays dividends, and therefore the call options are effectively priced as European options

Dec 31 2015													
GOOGLE													
Expiry	**15-Jan-16**				**Expiry**	**13-Mar-16**				**Expiry**	**20-Jan-17**		
Forward	ATM	25dRR	25fFly		Forward	ATM	25dRR	25fFly		Forward	ATM	25dRR	25fFly
778.879	19.8	-4	0.45		779.474	26.33	-4.24	0.4		777.776	27.67	-5.7	0.7
strike	bid	ask	**model**		strike	bid	ask	**model**		strike	bid	ask	**model**
260	517.00	520.5	518.88		330	447.8	451.30	449.71		260	517.50	522.5	520.08
270	507.00	510.7	508.88		340	437.80	441.4	439.73		270	508	512.5	510.27
275	502.1	505.7	503.88		350	427.8	431.4	429.76		280	498	502.5	500.47
280	497.1	500.7	498.88		360	417.6	421.4	419.78		290	488	493	490.68
285	492.2	495.4	493.88		370	407.9	411.5	409.81		300	478.5	483	480.90
290	487.1	490.4	488.88		380	397.9	401.5	399.84		310	468.5	473.5	471.14
295	482.2	485.4	483.88		390	387.7	391.5	389.87		320	459.5	464	461.38
300	477.1	480.4	478.88		400	378	381.5	379.91		330	449.8	454	451.64
305	472	475.4	473.88		410	368	371.5	369.95		340	440.1	444.5	441.92
310	467.1	470.4	468.88		420	357.9	361.5	359.99		350	430	434.5	432.21
315	462.1	465.4	463.88		430	348	351.5	350.04		360	420.8	425.5	422.53
320	457	460.4	458.88		440	338	341.5	340.10		370	411.2	415.5	412.88
325	452.2	455.4	453.88		450	328.3	331.8	330.15		380	401.6	406	403.25
330	447	450.4	448.88		460	318.1	322	320.22		390	391.5	396.5	393.65
335	442.2	445.4	443.88		470	308.5	312	310.29		400	382	387	384.09
340	437.6	440.4	438.88		480	298.5	302	300.37		410	372.5	377.5	374.57
345	432.2	435.4	433.88		490	288.7	292	290.45		420	363.6	368	365.08
350	427.3	430.5	428.88		495	283.4	287	285.50		430	353.5	358.5	355.63
355	422.3	425.5	423.88		500	278.9	282	280.54		440	344.8	349	346.24
360	417.3	420.5	418.88		505	273.5	277.2	275.59		450	335	339.5	336.89
365	412.2	415.5	413.88		510	268.9	272.3	270.65		460	326.3	330.5	327.60
370	407	410.5	408.88		515	264.1	267.3	265.70		470	317.1	321.5	318.39
375	402	405.5	403.88		520	259	262.4	260.76		480	307.5	312	309.24
380	397.6	400.5	398.88		525	253.8	257.4	255.82		490	298	303	300.16
385	392.3	395.5	393.88		530	249.2	252.5	250.89		500	289	294	291.17
390	387.2	390.5	388.88		535	244	247.5	245.96		510	280.5	285	282.26
395	382.2	385.5	383.88		540	239	242.5	241.03		520	272.1	276.5	273.45
400	377.6	380.5	378.88		545	233.9	237.5	236.11		530	263.1	267.5	264.74
405	372.3	375.5	373.88		550	229	232.8	231.19		540	255	259.5	256.13
410	367.1	370.5	368.88		555	224.6	227.9	226.28		550	246.1	251	247.62
415	362.2	365.5	363.88		560	219.8	223	221.37		560	237.5	242	239.23
420	357.7	360.5	358.88		565	214.9	218.1	216.47		570	229	233.5	230.94
425	352.3	355.5	353.88		570	209.5	213	211.58		580	220.5	225	222.78
430	347.2	350.5	348.88		575	205	208.4	206.70		590	213	217.5	214.73
435	342.2	345.5	343.88		580	200.3	203.5	201.83		600	205	209.5	206.81
440	337.4	340.5	338.88		585	195.5	198.7	196.97		610	197.1	202	199.02
445	332.3	335.5	333.88		590	190.6	193.8	192.12		620	189.5	194	191.36
450	327.8	330.5	328.88		595	185.8	189	187.29		630	182	186.3	183.83
455	322.4	325.5	323.88		600	181	184	182.47		640	174.6	178.5	176.44
460	317.7	320.5	318.88		605	175.5	179.4	177.67		650	167.2	171.6	169.19
465	312.8	315.5	313.88		610	171.3	174.6	172.88		660	160	164.5	162.09
470	307.8	310.5	308.88		615	166	169.8	168.12		680	146	150.4	148.32
475	302.5	305.5	303.88		620	161.9	165.1	163.38		700	133.3	137	135.16
480	297.7	300.5	298.88		625	156.6	160.4	158.66		720	120.6	122.8	122.63
485	292.8	295.8	293.88		630	152.4	155.7	153.96		740	108.7	111.2	110.74
490	287.8	290.8	288.88		635	147.8	151	149.30		760	97.8	99.9	99.52
495	282.8	285.8	283.88		640	142.6	146.4	144.66		780	87.5	89.7	88.99
500	277.5	280.5	278.88		645	138.1	141.8	140.06		800	76.8	79.9	79.17
505	272.7	275.5	273.88		650	133.9	137.2	135.49		820	69.2	70.9	70.06

Table 6 compares the model to the in interest rate swaptions market.

Using broker data, we collect the volatilities of swaptions in EUR, USD, JPY and CHF with maturities from one year to 10 years when the underlying swap is five years (1Y5Y; 2Y5Y; 5Y5Y; 10Y5Y). In the case of EUR, the brokers provide both the regular and the shifted volatility. In the case of CHF, the brokers provide only the shifted volatility.

Dec 31 2015																	
	USD																
								ATM	25d RR	25d fly							
	1y5y			forward (%)	2.065			36.7	-10	2							
Strike b.p. from fwd			-150.00	-100	-75	-50	-25	-12.5	0.00	12.5	25	50	75	100	150	200	300
Strike in (%)			0.565	1.065	1.315	1.565	1.815	1.94	2.065	2.19	2.315	2.565	2.815	3.065	3.565	4.065	5.065
Market vol			74.1	55.09	49.3	44.76	41.23	39.75	38.39	37.53	36.73	35.64	35.36	35.14	35.97	37.34	40.45
model			74.42	54.91	49.11	45.02	41.99	39.98	38.43	37.25	36.54	35.52	35.21	35.02	35.88	37.29	40.31
								ATM	25d RR	25d fly							
	2y5y			forward (%)	2.300			33	-9	3.3							
Strike b.p. from fwd			-150	-100	-75	-50	-25	-12.5	0	12.5	25	50	75	100	150	200	300
Strike in (%)			0.800	1.300	1.550	1.800	2.050	2.175	2.300	2.425	2.550	2.800	3.050	3.300	3.800	4.300	5.300
Market vol			62.48	49.32	44.87	41.27	38.34	37.14	36.04	35.09	34.20	32.90	32.21	31.61	31.54	32.12	34.05
model			62.07	48.12	44.63	41.21	38.72	37.11	36.01	35.1	34.11	32.73	32.06	31.98	31.69	32.21	34.31
								ATM	25d RR	25d fly							
	5Y5Y			forward (%)	2.7090			29	-8.65	3.5							
Strike b.p. from fwd		-200	-150	-100	-75	-50	-25	-12.5	0	12.5	25	50	75	100	150	200	300
Strike in (%)		0.709%	1.209%	1.709%	1.959%	2.209%	2.459%	2.584%	2.709%	2.834%	2.959%	3.209%	3.459%	3.709%	4.209%	4.709%	5.709%
Market vol		62.17	49.35	41.93	39.13	36.79	34.76	33.85	32.99	32.23	31.51	30.23	29.21	28.29	27.05	26.37	26.11
model		62.01	49.02	45.51	38.99	36.61	34.64	33.62	32.74	32.21	31.32	30.13	29.02	28.11	27.21	26.51	27.01
								ATM	25d RR	25d fly							
	10Y5Y			forward (%)	2.9840			24	-6.2	4							
Strike b.p. from fwd			-150	-100	-75	-50	-25	-12.5	0	12.5	25	50	75	100	150	200	
Strike in (%)			1.484%	1.984%	2.234%	2.484%	2.734%	2.859%	2.984%	3.109%	3.234%	3.484%	3.734%	3.984%	4.484%	4.984%	
Market vol			39.36	34.02	31.98	30.23	28.72	28.04	27.4	26.82	26.26	25.28	24.47	23.73	22.66	21.98	
model			40.04	34.41	32.11	30.43	28.99	28.21	27.44	26.86	26.37	25.33	24.51	23.81	22.93	22.34	

As can be seen from all the tables in the appendix, it is fair to say that the model universally matches the vanilla market in all asset classes. Our conclusion is that all European vanilla options obey the same model.

7. Calculation of The Probability Transfer Density

In the previous sections, we developed a method to generate the probability density function for a given expiry with three market data inputs. In this section, we develop a method to obtain the transfer density function $g(s_1, t_1, \to s_2, t_2)$ from the probability density function $g(s, t_1)$ and $g(s, t_2)$ for any $t_2 > t_1$. $g(s_1, t_1, \to s2, t_2)$ is often referred to in the industry as the **contingent probability** $g(s_2, t_2 \mid s_1, t_1)$. In our approach, we use the same type of density function with expiry $t_2 - t_1$, and at each underlying spot price s_1, we need to obtain the three inputs that characterize the density function. At every s_1 , these three inputs form the implied **forward local smile** at time t_1. We choose to parameterize the smile at each spot and time with {σ_0, 25Δ RR, 25Δ fly} of equation (97). In order to define the implied forward local smile at time t_1 and the underlying asset spot price s for expiry t_2, we need to find the set of three functions

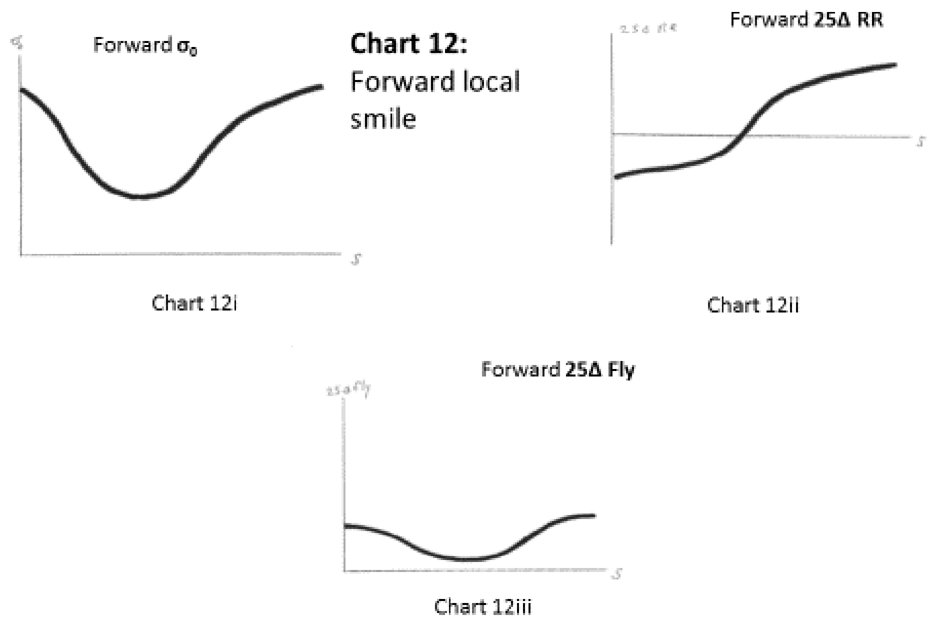
$$\{\sigma_0 (s, t_1, t_2); 25\Delta RR(s, t_1, t_2); 25\Delta Fly (s, t_1, t_2)\} \qquad (125)$$

We can think of the implied functions $\sigma_0 (s, t_1, t_2)$; $25\Delta RR(s, t_1, t_2)$; $25\Delta Fly (s, t_1, t_2)$ as the expected value of these stochastic variables at each spot price s. One should expect that the implied values depend on the probability of the various paths from s_0 to s. For example, if we move in equal steps from $s_0, 0 \to s, t$, then we expect σ_0 to be a lot smaller than if s remains close to s_0 most of the time and jumps to s just a very short time before t (or similarly if the underlying price zigzags sharply on the way to s). The same argument applies to the 25Δ RR and 25Δ Fly. Yet, the forward local implied smile is meaningful when T-t is very small, such as a few days or hours, since a change in the price of the underlying asset that takes place in a day or two will have a very similar impact on the smile, usually with very little influence on the path taken. Since we are interested in creating a dense grid of implied local smiles in order to be able to calculate the prices of exotic options, we will take a very small t_2-t_1 (say a day or hours) which is exactly the limit where the implied smile is most meaningful.

Before we explain how to calculate the implied local smile from one expiry time t_1 to another expiry time t_2 , it is important to explain what our expectation of the qualitative behavior of the implied local smiles are. Let us first describe the economic behavior of the three smile functions (106) at any $0<t_1<t_2$.

Chart 12 depicts the qualitative behavior of $\sigma_0(T,s,t)$; 25ΔRR(T,s,t); 25ΔFly(T,s,t) and F(T,s,t) as a function of the underlying price for a given t and T and as a function of (T-t).

Chart 12: Forward local smile



The behavior of the ATM volatility $\sigma_0(t,T,s)$ as a function of s is a smooth strictly positive function with **one** minimum with turning points on each side, i.e. on the right side of the minimum the function monotonically increases but at a decreasing pace for large s and on the left side of the minimum it is a monotonically decreasing function which decreases at an increasing pace for low s (**Chart 12i)**. Usually when there is a very large move in the price of the underlying asset, financial markets tend to initially align with the move in the options market and the risk reversal will favor the direction of the large move. There is usually some overshoot whenever there is a shock in the market but this overshoot is much smaller for very large moves.

Considering no-arbitrage, we get $|25\Delta RR(s)| << \sigma_0(s)/2$, but it is expected that the ratio $25\Delta RR(s) / \sigma_0(s)$ will decrease in the limits of very large and very small s (**Chart 12ii**).

The behavior of the 25ΔFly(s) is similar to the ATM volatility in the sense that it is positive with one minimum. It tends to have a much lower slope at very large or small spot prices, and we have seen that in the biggest shocks in the past 20 years, the ATM exceeded 100% (**Chart 12iii**).

Even though we will not include the local forward rate in the implied local smile, it can be included as well. The forward rate F(s) is a monotonically increasing function of the spot price and the ratio F(s)/s will increase when s is very large and decrease when s is very small. The ratio is the exponent of the interest rates' differential. For example, it is expected that when a certain currency is devalued, its interest rate will be sharply hiked, when a stock price plunges, its dividend rate is expected to decrease, when the price of a certain commodity collapses, its cost of carry decreases. Therefore, in all asset classes the ratio is expected to go up when there is a drastic increase in the price and sharply down if there is a drastic decrease in the price. However, after the area of the price shock the ratio changes at a slower pace.

The behavior of all the smile variables as well as the forward rate depend on the time to maturity. The longer the maturity the more moderate is the change in the smile variables.

Obtaining the implied local smile:

We use the equation

$$P(K, t_2, s_0) = df_1 \int ds_1\, g(s_0, 0 \rightarrow s_1, t_1))\, P(K, t_2 - t_1, s_1) \quad (126)$$

where

$$P(K, t_2 - t_1, s_1) = P(K, t_2, t_1, s_1, \sigma_0(s_1), 25dRR(s_1), 25dfly(s_1)) \quad (127)$$

and where $g(s_0, 0 \rightarrow s_1, t_1)$ is derived from the smile at t_1, and df_1 is the discount factor from time zero to time t_1.

Now we use the LMA optimization method to solve for the three functions. In order to obtain the solution that satisfies the desired shape of $\sigma_0(s_1)$, $25dRR(s_1)$, $25dfly(s_1)$ (our three economic conditions), we use the Tikhonov (1963) Regularization and Tarantola (2005) method. In essence, we introduce a set of small regularization penalties that ensure stability (uniqueness) of the solution without affecting its precision.

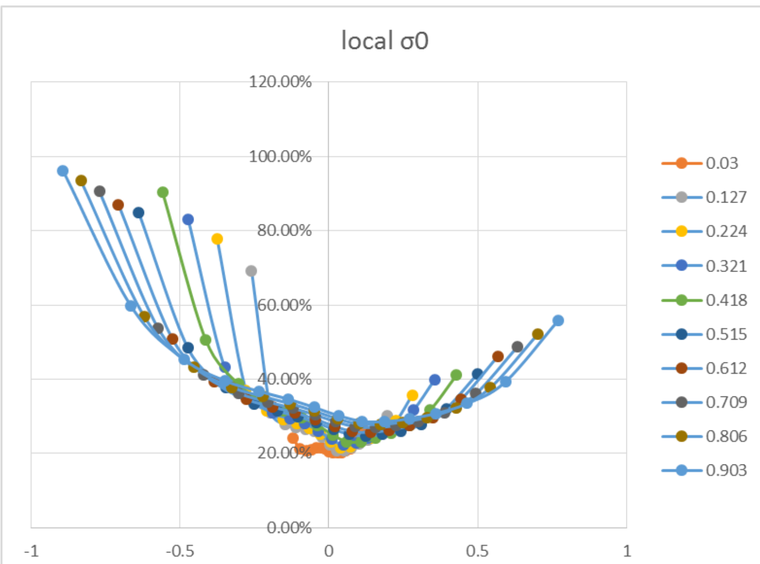
We select N points s_i at time t_1 that range from $-B \le d_1(t_1) \le B$, where $d_1(t_1)$ refers to the smile at t_1 (e.g. B=3.5). At each s_i, we find $\sigma_0(s_i)$, $25dRR(s_i)$, $25dfly(s_i)$, and we interpolate between all the points s_i later. Therefore, we are solving for 3N parameters. We select M strikes K_j from both sides of the ATM strike at t_2 ranging from the ATM strike to very low Delta, $-C \le d_1(t2) \le C$, where $d_1(t_2)$ corresponds to the smile at t_2 (e.g. C=2.5). As the target function, we use the difference between (126) and the known smile at t_2 for a large sample of strikes.

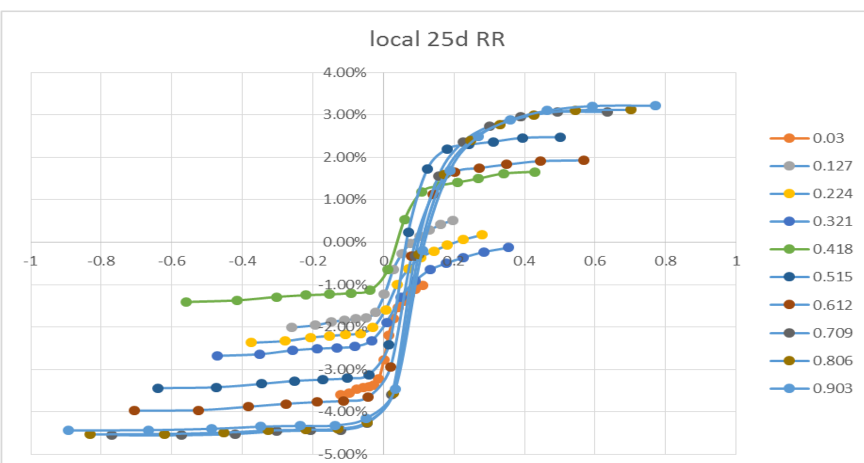
$$\text{Min}\left(\Sigma_j\{(P(K_j, t_2, s_0) - \int ds_1\, g(s_0, 0 \rightarrow s_1, t_1))\, P(K_j, t_2 - t_1, s_1))\; \text{Vega}(K_j, t_2)\}^2 + \sum_i C_i\right) \qquad (128)$$

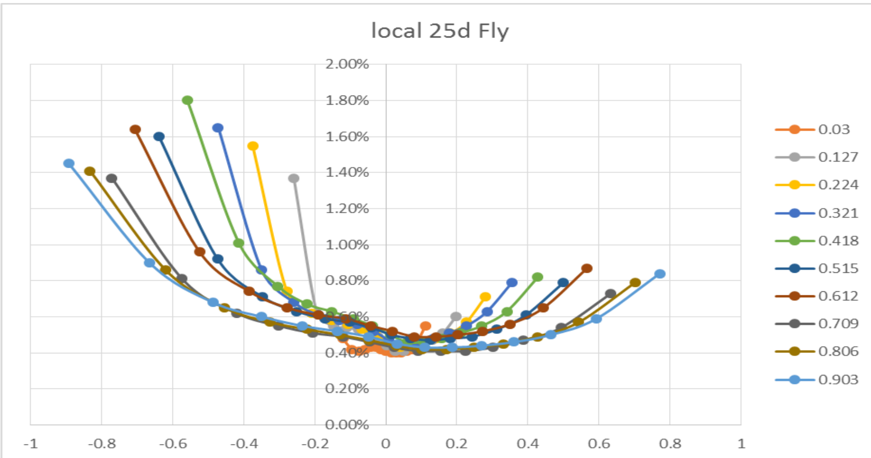
Where the C_i include the three economical regularizations. For example, the condition that $25\Delta RR(s_i)$ is monotonically increasing can be met by construction, by including only positive increments for $25\Delta RR(s_{i+1})$- $25\Delta RR(s_i)$. For the condition that $\sigma_0(s_i)$ and $25\Delta fly(s_i)$ are always positive and have only one minimum, we can use the derivatives of $\sigma_0(s_i)$ and $25\Delta fly(s_i)$) with respect to s_i. One can also include smoothness conditions in the $\sum_i C_i$ in (128). For example, to prevent oscillations we penalize negative curvature for $\sigma_0(s_1)$ and $25dfly(s_1)$ and for $25dRR(s_1)$ we penalize negative curvature of the left wing and positive curvature on the right wing.

Due to standard numerical issues, the larger N is the more we may see little oscillations in the shapes of the three parameters. In order to smooth out these fluctuations, we have to introduce some conditions with very low weight to reduce the wavy/zigzag shape without affecting the accuracy. Practically, we can choose N=9 spot points. In this case, we are solving for 27 variables.

Chart 13 shows the results of the implied local smile of the three parameters in some cases. From 1 day going forward: 1 day, 1 week, 2 weeks, 1 month, 2 month, 3 month, 6 month, 1 year







Given the implied local smile at (s_1, t_1) for expiry date t_2 {σ₀ (s_1), 25ΔRR(s_1), 25Δfly(s_1)}, it is straightforward to derive the transfer density function $g(s_1, t_1 \rightarrow s_{2,} t_2)$ from s_1 to any underlying asset spot price s_2 at time t_2 .

Dynamically replicating a given option:

Having the probability transfer density allows for the creation of a dynamic replicating strategy for a given option. In other words, at each time interval, we can replicate the change of the value of the given option with some dynamically reset "replicating portfolio". Let us parameterize the density function with {σ_0 (s), 25ΔRR(s), 25Δfly(s)}. These three variables are stochastic and correlated, and when we calculate the probability density transfer, we obtain a certain expectation of these variables at a certain spot and time (please see Appendix 2). Since at each time interval we have to replicate the outcome of three stochastic processes, we need to use three options to replicate the given option in addition to using the underlying asset. In Appendix 2, we demonstrate dynamic replication with three strikes with d_1= -D,0,D and discuss optimization of D.

8. Exotic Options and Comparison to The Interbank Market

In this section, we test the procedure of Section 6 to obtain the probability transfer density function $g(s_1,t_1 \rightarrow s_2,t_2)$. Given a term structure of the volatility (e.g. σ_0, 25dRR, 25dFly) for benchmark periods (expiries), we know how to obtain the complete probability transfer density $g(s,t \rightarrow s', t+\Delta t)$ for a time increment Δt as small as we like. This allows us to calculate path-dependent options, known as exotic options. Like before, we can test our model against the market data to validate our procedure. We can also calculate vanilla options via the probability transfer density for different term structures and show that the price is independent of the term structure but only on the market data to maturity.

In the currency (FX) options market for example, knockout and binary options are kinds of exotic options that trade very regularly with relatively high liquidity. This provides us with a great testing field. We can take the live term structure when an exotic option trades in the market and price it via the transfer density and compare the model price to the traded price. Hence we will compare the model price to four kinds of exotic options:

Double no touch (DNT): This binary option has a low barrier and a high barrier. If the underlying spot price stays strictly between the two barriers and never touches either of them from inception to expiry, it pays 1 at expiry, otherwise 0.

One touch (OT): This binary option has one barrier that is either above or below the current spot. If the underlying spot price touches the barrier at least once from inception to expiry it pays 1 at expiry, otherwise 0.

Knockout (KO): These options pay like European vanilla options (put/call with a strike) provided that the barrier is not touched from inception to the expiry of the option.

Double Knockout (DKO): These options have two barriers and pay like European vanilla options (put/call with a strike) provided that neither of the barriers is touched from inception to the expiry of the option.

We start with the DNT with expiry T with low barrier B_l and high barrier B_h. The current spot price is s_0 ($B_l < s_0 < B_h$). The price of the option is the **contingent cumulative distribution** between B_1 and B_h at expiry i.e. contingent on not touching B_l and B_h from inception until the expiry date.

$$P_{DNT}(B_l, B_h, T, s_0) = df \int_{B_l}^{B_h} ds_T\, g\, (s_0, 0 \rightarrow s_T, T\, |B_l < s_t < B_h\ \forall\ t \leq T) \\ \equiv df\ G(s_0, 0 \rightarrow s_T, T \mid B_l < s_t < B_h\ \forall\ t \leq T) \quad (129)$$

We choose the time interval Δt =T/N. From the given market term structure which includes the expiries T_1,T_2,..,T, we interpolate/extrapolate over the term structure in order to have the term structure for every time t_i=i Δt <T. For each t_i, we calculate the density function g(s,t_i) and then using the technique of Section 6, we calculate $g_i(s, t_i \rightarrow s', t_{i+1})$ from the density functions g(s,t_i) and g(s,t_{i+1}). Now

$$G(s_0, 0 \rightarrow s_T, T \mid B_l < s_t < B_h\ \forall\ t \leq T) = \\ \int_{Bl}^{Bh} ds_1 \int_{Bl}^{Bh} ds_2 \ldots \int_{Bl}^{Bh} ds_N\, g_1(s_0, 0 \rightarrow s_1, t_1)\, g_2(s_1, t_1 \rightarrow \\ s_2, t_2) \ldots g_N(s_{N-1}, t_{N-1} \rightarrow s_N, t_N) \quad (130)$$

We divide (B_h - B_l) into M spot points s_j and numerically calculate on a grid of s,t ($B_l < s < B_h$; 0 ≤t≤ T). Similarly, the price of a double knock out call option with strike K is

$$P_{DKO}(T,K,B_l,B_h) = df \int_{Bl}^{Bh} ds_1 \int_{Bl}^{Bh} ds_2 \ldots \int_{Bl}^{Bh} ds_N\, g_1(s_0, 0 \rightarrow s_1, t_1)\, g_2(s_1, t_1 \rightarrow \\ s_2, t_2) \ldots g_N(s_{N-1}, t_{N-1} \rightarrow s_N, t_N)\, (s_N - K)^+ \quad (131)$$

We can use the method in (131) for vanilla options (where B_l=0 and B_h=∞) in order to show that the option price is independent of the term structure.

Table 7 shows the 6-month vanilla smile for three different term structures using the density integral (131) when the lower barrier is zero and the upper barrier is infinity. In addition, we compare to the vanilla smile formula. As can be seen the option prices are very similar in all the term structures.

		6 month vanilla smile using different term structures										
		Term strucuture 1				Term strucuture 2				Term strucuture 3		
	Tenor	**ATM**	**25d RR**	**25d Fly**		**ATM**	**25d RR**	**25d Fly**		**ATM**	**25d RR**	**25d Fly**
	1W	12.00%	2.000%	0.250%		8.00%	1.000%	0.150%		10.00%	2.000%	0.250%
	1M	11.73%	2.00%	0.25%		8.27%	1.13%	0.16%		10.00%	2.000%	0.250%
	3M	11.00%	2.00%	0.25%		8.96%	1.48%	0.20%		10.00%	2.000%	0.250%
	6M	10.00%	2.000%	0.250%		10.00%	2.000%	0.250%		10.00%	2.000%	0.250%
strikes	**0.900**	**0.940**	**0.980**	**1.020**	**1.060**	**1.100**	**1.140**	**1.180**	**1.220**	**1.260**	**1.300**	**1.340**
TS 1	10.81%	1.11%	9.51%	9.09%	9.32%	9.96%	10.74%	11.62%	12.56%	13.56%	14.68%	15.92%
TS 2	10.81%	10.10%	9.53%	9.10%	9.34%	9.97%	10.69%	11.63%	12.55%	13.61%	14.69%	15.92%
TS 3	10.80%	10.11%	9.52%	9.13%	9.33%	10.01%	10.72%	11.63%	12.56%	13.54%	14.71%	15.94%
Market	**10.85%**	**10.15%**	**9.55%**	**9.13%**	**9.31%**	**9.95%**	**10.73%**	**11.60%**	**12.54%**	**13.58%**	**14.74%**	**15.98%**

Table 8 includes examples of prices from the interbank broker market and their model price for Double no touch, One touch and Barrier options. These examples with different barriers, maturities and strikes of four kinds of options help us to verify that the transfer density we developed in the previous section is indeed the transfer density in the options market.

	One Touch						
date	ccy	spot	expiry	barrier	BS	market	model
6-Nov-15	USDJPY	121.8	11 DAYS	125	4.6	7.5	7.6
8-Feb-16	EURUSD	1.117	42 DAYS	1.06	14.3	13.25	13.1
14-Apr-16	USDJPY	109.15	49 DAYS	107.4	68	62.25	61.9
10-Feb-16	EURUSD	1.1295	5M	1.000	7	9.75	9.6

	Double No Touch					
date	ccy	spot	expiry	range	market	model
13-Nov-15	EURUSD	1.079	2M	1.040-1.1200	21	20.6
22-Mar-16	USDJPY	113.35	2M	108.00-115.00	29	28.1
22-Mar-16	USDJPY	111.4	2M	109.50-115.00	17.75	16.9
23-Mar-16	EURUSD	1.1215	2M	1.0750-1.1450	24.75	24.6
9-Nov-15	EURUSD	1.074	3M	1.0250-1.1150	18.5	18.3
11-Nov-15	EURUSD	1.0755	3M	1.050-1.15	19.75	20.1
1-Feb-16	EURUSD	1.0835	3M	1.030-1.1300	12.25	12.1
25-Feb-16	USDJPY	112.05	3M	106.00-118.00	35	33.75
1-Mar-16	EURUSD	1.086	3M	1.055-1.1150	6.75	6.4
5-Feb-16	EURUSD	1.1195	4M	1.060-1.1400	8.25	7.9
17-Mar-16	USDJPY	112.5	4M	106.00-115.00	17.5	16.9
23-Mar-16	USDJPY	112.3	5M	104-115.50	25.25	24.9
9-Feb-16	USDJPY	115.3	6M	110-120	10.5	10.3
24-Feb-16	USDJPY	111.85	6M	102.5-114.5	9.9	9.9
9-Nov-15	EURUSD	1.077	9M	1.000-1.1500	20	19.6
6-Nov-15	EURUSD	1.0875	1Y	.97-1.19	35.5	35.7
25-Feb-15	EURUSD	1.105	1Y	1.0300-1.1700	9.5	9.4
29-Feb-16	EURUSD	1.092	1Y	1.000-1.1800	20.25	20.1
1-Mar-16	EURUSD	1.087	1Y	1.000-1.1800	21.25	21
2-Mar-16	EURUSD	1.0855	1Y	.9600-1.2000	40.25	39.9
17-Mar-16	EURUSD	1.122	1Y	1.0200-1.1750	14	14.1
5-Apr-16	EURUSD	1.1395	1Y	1.000-1.2000	24	24
5-Apr-16	USDJPY	110.5	1Y	100.00-120.00	37.75	37

	Barriers								
date	ccy	spot	expiry	strike		barrier	Call/Put	market	model
11-Nov-15	EURUSD	1.075	2M	1.0325	KO	1.0975	EUR P	0.455	0.46
10-Nov-15	EURUSD	1.0745	3M	1.03	RKO	0.97	EUR P	0.24	0.24
2-Mar-16	EURUSD	1.0845	3M	1.13	KO	1.0525	EUR C	0.655	0.66
10-Nov-15	USDJPY	123.2	4M	116	KO	127.5	USD P	0.38	0.37
2-Mar-16	EURUSD	1.086	4M	1.05	RKO	1	EUR P	0.16	0.16
7-Dec-15	EURUSD	1.0885	6M	1.035	KO	1.12	EUR P	0.78	0.77
7-Dec-15	EURUSD	1.084	6M	1.04	KI	1.04	EUR C	1.42	1.41
2-Feb-16	EURUSD	1.0905	6M	1.07	RKO	1.01	EUR P	0.24	0.23
5-Feb-16	EURUSD	1.12	6M	1.04	KO	1.15	EUR P	0.465	0.47
8-Feb-16	EURUSD	1.1125	6M	1.18	KO	1.075	EUR C	0.83	0.84
24-Feb-16	USDJPY	111.75	6M	107	KI	107	USD C	1.9	1.89
14-Apr-16	USDJPY	109.2	6M	103	KO	115	USD P	1.21	1.19
16-Nov-15	EURUSD	1.072	8M	1.15	KO	1.05	EUR C	0.615	0.63
8-Feb-16	EURUSD	1.1145	9M	1.02	KI	1.02	EUR C	0.965	0.95
17-Mar-16	USDJPY	112.15	10M	124	KO	105	USD C	0.48	0.48
5-Feb-16	EURUSD	1.1125	315 DAYS	1.01	KI	1.01	EUR C	1.015	1.011
8-Dec-15	EURUSD	1.0865	1Y	1.16	KI	1.16	EUR P	1.29	1.31
9-Feb-16	USDJPY	115.2	1Y	115.2	KO	130	USD C	0.46	0.46
9-Feb-16	USDJPY	115.2	1Y	130	KO	111	USD C	0.46	0.47
10-Feb-16	EURUSD	1.1295	1Y	1.05	KO	1.16	EUR P	0.81	0.8
1-Mar-16	EURUSD	1.087	1Y	1.05	RKO	0.95	EUR P	0.34	0.35
2-Mar-16	EURUSD	1.0875	1Y	1.21	KO	1	EUR C	1.165	1.18
3-Mar-16	USDJPY	113.95	1Y	100	KI	120	USD P	0.34	0.32
5-Apr-16	USDJPY	110.95	1Y	100	RKO	90	USD P	0.21	0.23
11-Apr-16	EURUSD	1.141	1Y	1.15	KI	1.05	EUR C	0.5	0.47
14-Apr-16	EURUSD	1.137	1Y	1.075	KO	1.18	EUR P	1.24	1.23

9. Summary and Conclusions

In this paper, we provided the first option pricing model that accurately reflects the prices of options in the market for underlying assets in all asset classes. We started by deriving the probability density function of all underlying assets- currencies, commodities, equities and interest rates- and showed that it is determined by any three option prices on the volatility smile. We then tested our model on an extremely large set of data on liquid assets in all asset classes and showed that the model closely matches the market. This demonstrates that the market assumes that all liquid financial products, i.e. stocks, currencies, commodities and interest rates behave according to the same 3 variable density function we found in this paper.

We then showed how to obtain the contingent density function (or the probability transfer density function) between 2 periods if the density functions from inception to these periods are known. Lastly, we used the contingent density function to calculate the price of different types of exotic options and compared the model to the prices in the market. Once again, the model matches market prices. We conclude that this new model reflects the market universally.

Appendix 1: Asymptotic Behavior of $\lambda(d_1)$, $\chi(d_1)$

In this Appendix, we calculate the asymptotic behavior of λ(d1) and χ(d1). We start with the model equations (43) and (44)

$$\zeta_c = df\ \mathrm{Fn}(d_1)\left(\frac{\lambda\sqrt{T}d_1^2}{\sigma_p} + \lambda d_1 T + \frac{\chi d_1}{2F}\left(\frac{1}{\sigma_c} + \frac{1}{\sigma_p}\right)\right) \quad \text{(A1)}$$

$$\zeta_p = df\ \mathrm{F\,n}(d_1)\left(\frac{\lambda\sqrt{Td_1^2}}{\sigma_p} - \lambda d_1 T - \frac{\chi d_1}{2F}\left(\frac{1}{\sigma_c} + \frac{1}{\sigma_p}\right)\right) \quad \text{(A2)}$$

We first calculate the asymptotic solution in the limit |log K/F| →∞.

In the BS formula (1) and (2), we use the asymptotic expression for N(x):

$$N(x) = 1 - n(x)/x, \quad x \to \infty \quad \text{(A3)}$$
$$N(x) = - n(x)/x, \quad x \to -\infty \quad \text{(A4)}$$

At $\pm\infty$ the contribution from the ATM volatility σ_0 is negligible compared to σ_c or σ_p since $\sigma_0 < \sigma_c$ and $\sigma_0 < \sigma_p$. We denote

$$v \equiv \sigma\sqrt{T} \quad \text{(A5)}$$

Using

$$K/F\ n(d_2) = n(d_1) \quad \text{(A6)}$$

we obtain (for simplicity we take *df*=1)

$$\zeta_c(K) = \mathrm{F\,n}(d_1)\left(\frac{1}{d_1 - v_c} - \frac{1}{d_1}\right) = \mathrm{Fn}(d_1)\frac{v_c}{d_1(d_1 - v_c)} \quad \text{(A7)}$$

$$\zeta_p(K) = \mathrm{F\,n}(d_1)\left(\frac{1}{d_1^p - v_p} - \frac{1}{d_1^p}\right) = \mathrm{Fn}(d_1)\frac{v_p}{d_1(d_1 + v_p)} \quad \text{(A8)}$$

Therefore

$$\frac{\zeta_c(K)+\zeta_p(K)}{Fn(d_1)} = \frac{1}{d_1}\left(\frac{v_c}{d_1 - v_c} + \frac{v_p}{d_1 + v_p}\right) = \frac{v_c + v_p}{(d_1 - v_c)(d_1 + v_p)} \tag{A4}$$

Comparing to the combination of (A1) and (A2)

$$\frac{\zeta_c(K)+\zeta_p(K)}{Fn(d_1)} = \lambda\sqrt{T}d_1^2\left(\frac{1}{\sigma_c} + \frac{1}{\sigma_p}\right) \tag{A10}$$

we obtain the asymptotic expression for λ:

$$\lambda T = \frac{1}{d_1^2}\frac{v_c v_p}{(d_1 - v_c)(d_1 + v_p)} \tag{A11}$$

Similarly

$$\frac{\zeta_c(K)-\zeta_p(K)}{Fn(d_1)} = \frac{1}{d_1}\left(\frac{v_c}{d_1 - v_c} - \frac{v_p}{d_1 + v_p}\right) = \frac{d_1(v_c - v_p) + 2v_c v_p}{d_1(d_1 - v_c)(d_1 + v_p)} \tag{A12}$$

And using (A11)

$$\frac{\zeta_c(K)-\zeta_p(K)}{Fn(d_1)} = \frac{d_1(v_c - v_p)}{d_1(d_1 - v_c)(d_1 + v_p)} + 2\lambda T d_1 \tag{A13}$$

Subtracting (A2) from (A1)

$$\frac{\zeta_c(K)-\zeta_p(K)}{n(d_1)} = -\chi d_1\left(\frac{1}{\sigma_c} + \frac{1}{\sigma_p}\right) + \lambda F\sqrt{T}\left(d_1^2\left(\frac{1}{\sigma_c} - \frac{1}{\sigma_p}\right) - 2\sqrt{T}d_1\right) \tag{A14}$$

and comparing to (A13)

$$\frac{d_1(v_c - v_p)}{d_1(d_1 - v_c)(d_1 + v_p)} + 2\lambda T d_1 = -\frac{\chi\sqrt{T}}{F}d_1\left(\frac{1}{v_c} + \frac{1}{v_p}\right) + \lambda T d_1\left(d_1\left(\frac{1}{v_c} - \frac{1}{v_p}\right) - 2\right) \tag{A15}$$

Thus,

$$\frac{\chi\sqrt{T}}{F}\left(\frac{1}{v_c}+\frac{1}{v_p}\right)=$$

$$-\frac{(v_c-v_p)}{d_1(d_1-v_c)(d_1+v_p)}+\lambda T\left(d_1\left(\frac{1}{v_c}-\frac{1}{v_p}\right)-4\right)=$$

$$\frac{1}{d_1^2}\frac{v_c v_p}{(d_1-v_c)(d_1+v_p)})(\left(d_1\left(\frac{1}{v_c}-\frac{1}{v_p}\right)-4\right)-\frac{d_1(v_c-v_p)}{v_c v_p}) \;= \tag{A16}$$

$$-2\frac{1}{d_1^2}\frac{2v_c v_p+d_1(v_c-v_p)}{(d_1-v_c)(d_1+v_p)}$$

Finally we have:

$$\lambda T=\frac{1}{d_1^2}\frac{v_c v_p}{(d_1-v_c)(d_1+v_p)} \tag{A17}$$

$$\chi=-\frac{2F}{\sqrt{T}}\frac{1}{d_1^2}\frac{2v_c v_p+d_1(v_c-v_p)}{(d_1-v_c)(d_1+v_p)}\times\frac{v_c v_p}{v_c+v_p} \tag{A5}$$

Equations (A17) and (A18) provide asymptotic expressions for λ and χ for $d_1 \to \infty$ with no assumptions on the behavior of the volatility smile. If the volatility stays finite at the limits, then $\lambda \sim 1/d_1^4$ and $\chi \sim 1/d_1^3$. Otherwise we need to use the results of Lee (2003) about finite moments of probability densities and their asymptotes. According to Lee, if there exists a supremum

$$\beta_p=\frac{\sigma_p^2 T}{\log(F/K_p)},\quad K_p\to 0 \tag{A20}$$

and

$$\beta_c=\frac{\sigma_c^2 T}{\log(F/K_c)},\qquad K_c\to \infty \tag{A21}$$

Then the moments $E[S^{1+p}]$ and $E[S^{-q}]$ are guaranteed to be finite, where

$$p=\frac{1}{2\beta_c}\left(1-\frac{\beta_c}{2}\right) \tag{A22}$$

$$q=\frac{1}{2\beta_p}\left(1-\frac{\beta_p}{2}\right) \tag{A23}$$

In order to have finite first moment $\beta_p, \beta_c \le 2$.

Since

$$v_c^2=\beta_c \log K_c/F,\qquad v_p^2=\beta_p \log F/K_p \tag{A24}$$

we obtain

$$d_1 = -\left(\frac{1}{\beta_p} + \frac{1}{2}\right) v_p = -\left(\frac{1}{\beta_c} - \frac{1}{2}\right) v_c \tag{A25}$$

$$d_1 + v_p = -\left(\frac{1}{\beta_p} - \frac{1}{2}\right) v_p, \quad d_1 - v_c = -\left(\frac{1}{\beta_p} + \frac{1}{2}\right) v_p \tag{A26}$$

Substituting all this into (A17) and (A18) we get the asymptotes of λ and χ

$$\lambda T d_1^2 = \frac{4\beta_c\beta_p}{(2-\beta_p)(2+\beta_c)} \tag{A27}$$

$$\frac{-\chi\sqrt{T}d_1}{2F} = \frac{4\beta_c\beta_p(\beta_c - \beta_p - \beta_c\beta_p)}{(2-\beta_p)(2+\beta_c)(\beta_c+\beta_p)} \tag{A28}$$

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Appendix 2: Additional Tables for F_λ, F_χ of λ and χ for different market data

Expiry	2 years															
σ ATM	25dRR	25dFly	d_1	0.25	0.5	0.75	1	1.25	1.5	1.75	2	2.25	2.5	2.75	3	3.25	3.5
10.00%	1.000%	0.100%	F_A	1.008	1.007	0.997	0.980	0.966	0.965	0.974	0.974	0.941	0.893	0.860	0.858	0.841	0.825
			F_B	0.998	0.999	1.000	0.999	0.998	1.002	1.007	0.997	0.964	0.940	0.913	0.886	0.860	0.835
			F_A/F_B	**1.0098**	**1.0082**	**0.9965**	**0.9809**	**0.9688**	**0.9632**	**0.9672**	**0.9770**	**0.9762**	**0.9502**	**0.9428**	**0.9689**	**0.9784**	**0.9880**
12.00%	2.000%	0.250%	F_A	1.042	1.024	0.990	0.950	0.919	0.902	0.889	0.861	0.835	0.821	0.817	0.808	0.755	0.652
			F_B	1.024	1.012	0.995	0.974	0.957	0.945	0.924	0.893	0.879	0.888	0.915	0.937	0.895	0.779
			F_A/F_B	**1.0169**	**1.0113**	**0.9950**	**0.9756**	**0.9599**	**0.9547**	**0.9615**	**0.9640**	**0.9505**	**0.9249**	**0.8922**	**0.8627**	**0.8439**	**0.8377**
15.00%	3.000%	0.750%	F_A	1.094	1.074	0.968	0.878	0.820	0.785	0.754	0.726	0.696	0.647	0.580	0.515	0.463	0.436
			F_B	1.145	1.076	0.967	0.888	0.848	0.821	0.791	0.766	0.734	0.681	0.607	0.537	0.478	0.434
			F_A/F_B	**0.9558**	**0.9977**	**1.0011**	**0.9882**	**0.9677**	**0.9569**	**0.9536**	**0.9479**	**0.9482**	**0.9501**	**0.9542**	**0.9599**	**0.9681**	**1.0035**
18.00%	4.000%	1.000%	F_A	1.046	1.066	0.971	0.883	0.826	0.786	0.750	0.722	0.693	0.644	0.579	0.525	0.506	0.474
			F_B	1.122	1.076	0.967	0.885	0.839	0.807	0.776	0.758	0.726	0.675	0.606	0.545	0.505	0.461
			F_A/F_B	**0.9330**	**0.9914**	**1.0041**	**0.9977**	**0.9838**	**0.9740**	**0.9667**	**0.9535**	**0.9545**	**0.9539**	**0.9558**	**0.9641**	**1.0029**	**1.0272**
25.00%	5.000%	1.200%	F_A	1.053	1.043	0.981	0.922	0.879	0.845	0.810	0.784	0.759	0.744	0.718	0.698	0.679	0.660
			F_B	1.076	1.042	0.982	0.929	0.891	0.863	0.839	0.835	0.814	0.794	0.753	0.725	0.693	0.662
			F_A/F_B	**0.9786**	**1.0008**	**0.9996**	**0.9929**	**0.9867**	**0.9787**	**0.9658**	**0.9399**	**0.9332**	**0.9374**	**0.9538**	**0.9630**	**0.9800**	**0.9973**
30.00%	6.000%	1.200%	F_A	1.029	1.020	0.991	0.963	0.943	0.923	0.897	0.882	0.863	0.843	0.824	0.806	0.788	0.770
			F_B	1.010	1.005	0.998	0.988	0.976	0.961	0.954	0.943	0.932	0.921	0.911	0.901	0.890	0.880
			F_A/F_B	**1.0192**	**1.0154**	**0.9933**	**0.9750**	**0.9669**	**0.9605**	**0.9402**	**0.9360**	**0.9255**	**0.9152**	**0.9050**	**0.8949**	**0.8849**	**0.8750**
45.00%	10.000%	2.000%	F_A	0.876	0.955	1.019	1.056	1.054	1.007	0.967	0.944	0.933	0.917	0.900	0.884	0.868	0.853
			F_B	0.863	0.943	1.024	1.074	1.056	0.994	0.932	0.875	0.822	0.772	0.725	0.681	0.640	0.601
			F_A/F_B	**1.0149**	**1.0126**	**0.9950**	**0.9837**	**0.9982**	**1.0132**	**1.0384**	**1.0785**	**1.1355**	**1.1875**	**1.2418**	**1.2986**	**1.3580**	**1.4201**

Expiry	3 years															
σ ATM	25dRR	25dFly	d_1	0.25	0.5	0.75	1	1.25	1.5	1.75	2	2.25	2.5	2.75	3	3.25	3.5
10.00%	1.000%	0.100%	F_A	1.019	1.011	0.995	0.977	0.964	0.962	0.970	0.969	0.933	0.886	0.855	0.818	0.783	0.749
			F_B	1.005	1.001	0.999	0.997	0.997	1.003	1.008	0.997	0.963	0.943	0.917	0.892	0.868	0.844
			F_A/F_B	**1.0138**	**1.0092**	**0.9960**	**0.9798**	**0.9666**	**0.9596**	**0.9629**	**0.9717**	**0.9688**	**0.9393**	**0.9317**	**0.9167**	**0.9019**	**0.8874**
12.00%	2.000%	0.250%	F_A	1.056	1.027	0.988	0.948	0.920	0.904	0.890	0.863	0.839	0.828	0.831	0.836	0.798	0.701
			F_B	1.030	1.012	0.995	0.978	0.965	0.954	0.934	0.907	0.901	0.919	0.963	1.010	0.990	0.885
			F_A/F_B	**1.0256**	**1.0152**	**0.9933**	**0.9700**	**0.9526**	**0.9472**	**0.9530**	**0.9513**	**0.9320**	**0.9016**	**0.8629**	**0.8278**	**0.8069**	**0.7924**
15.00%	3.000%	0.750%	F_A	1.080	1.065	0.972	0.889	0.836	0.800	0.766	0.737	0.709	0.662	0.596	0.537	0.509	0.471
			F_B	1.127	1.068	0.970	0.897	0.856	0.827	0.798	0.778	0.751	0.703	0.633	0.567	0.518	0.513
			F_A/F_B	**0.9576**	**0.9966**	**1.0016**	**0.9915**	**0.9762**	**0.9668**	**0.9603**	**0.9473**	**0.9443**	**0.9410**	**0.9412**	**0.9468**	**0.9820**	**0.9168**
18.00%	4.000%	1.000%	F_A	1.034	1.052	0.977	0.902	0.850	0.810	0.772	0.745	0.716	0.676	0.632	0.613	0.584	0.556
			F_B	1.093	1.060	0.974	0.901	0.853	0.821	0.792	0.781	0.752	0.711	0.656	0.617	0.574	0.535
			F_A/F_B	**0.9458**	**0.9926**	**1.0035**	**1.0021**	**0.9956**	**0.9867**	**0.9744**	**0.9536**	**0.9527**	**0.9506**	**0.9640**	**0.9942**	**1.0167**	**1.0397**
25.00%	5.000%	1.200%	F_A	1.015	1.024	0.990	0.953	0.927	0.901	0.872	0.852	0.851	0.841	0.831	0.821	0.811	0.802
			F_B	1.029	1.018	0.992	0.966	0.941	0.919	0.907	0.902	0.893	0.885	0.876	0.868	0.860	0.852
			F_A/F_B	**0.9862**	**1.0062**	**0.9973**	**0.9867**	**0.9851**	**0.9797**	**0.9618**	**0.9444**	**0.9531**	**0.9508**	**0.9484**	**0.9461**	**0.9438**	**0.9414**
30.00%	6.000%	1.200%	F_A	0.848	0.930	1.030	1.098	1.123	1.094	1.057	0.979	0.913	0.886	0.843	0.802	0.763	0.726
			F_B	0.867	0.939	1.026	1.090	1.105	1.076	1.050	0.989	0.977	0.943	0.909	0.877	0.846	0.817
			F_A/F_B	**0.9780**	**0.9899**	**1.0039**	**1.0072**	**1.0162**	**1.0167**	**1.0066**	**0.9890**	**0.9346**	**0.9401**	**0.9272**	**0.9146**	**0.9021**	**0.8897**
44.00%	10.000%	2.000%	F_A	0.713	0.877	1.051	1.133	1.120	1.060	1.026	0.998	0.968	0.947	0.922	0.898	0.874	0.851
			F_B	0.751	0.895	1.044	1.098	1.048	0.943	0.920	0.862	0.808	0.757	0.709	0.665	0.623	0.583
			F_A/F_B	**0.9495**	**0.9800**	**1.0074**	**1.0316**	**1.0691**	**1.1233**	**1.1156**	**1.1577**	**1.1986**	**1.2505**	**1.2996**	**1.3507**	**1.4037**	**1.4589**

Appendix 3: Dynamic Replication in the New Model

One of the most appealing features in the BS model is the dynamic replication of the option with the underlying asset. One can form a dynamic portfolio of one unit of the option and a varying amount of the underlying asset where the amount of the underlying asset in the portfolio is reset in each time interval so that the total risky (i.e. non deterministic) element of the portfolio vanishes until the next time interval. In the BS model, the underlying asset follows a simple stochastic process and which carries into the option price that depends on it. Thus, the amount of the underlying asset in the portfolio can be determined by setting the stochastic component in the portfolio to 0. This results in a dynamic delta hedging of the option. There is yet an assumption that the transaction cost of readjusting the portfolio is zero. In deriving our model so far, we used quasi-no arbitrage conditions in the smile and therefore did not deal with readjusting portfolios and hence had no transaction cost involved. Now we want to demonstrate how to implement the dynamic replication approaches to our formalism.

As we have seen, at time t_1 and underlying asset price s_1, the density function $g(s_1, t_1 \rightarrow s_2, t_2)$ of the underlying asset for time t_2 depends on three stochastic factors,. $\sigma_0(s_1, t_1, t_2)$, $25\Delta RR(s_1, t_1, t_2)$, $25\Delta fly(s_1, t_1, t_2)$ which are strongly correlated with each other and the underlying asset price. The correlation is a highly non trivial one because it also has to take into account the price history of the underlying asset. Since we do not know the stochastic processes of s, σ_0, 25ΔRR and 25Δfly, we will construct the dynamic replication using a more fundamental approach that assumes nothing about the behavior of the three variables and their correlation.

$\delta s(t) = \hat{f}_0(s,t,\sigma_0(s,t), RR(s,t), fly(s,t))$

$\delta\sigma_0(t) = \hat{f}_1(s,t,\sigma_0(s,t), RR(s,t), fly(s,t))$

$\delta RR(t) = \hat{f}_2(s,t,\sigma_0(s,t), RR(s,t), fly(s,t))$

$\delta Fly(t) = \hat{f}_3(s,t,\sigma_0(s,t), RR(s,t), fly(s,t))$ (A29)

Therefore

$$g\left(\frac{s_T}{s_0}, 2\delta t, \sigma_{0,2\delta t}, RR_{0,2\delta t}, Fly_{0,2\delta t}\right) = \int_0^\infty ds\ g\left(\frac{s}{s_0}, \delta t, \sigma_{0,\delta t}, RR_{0,\delta t}, Fly_{0,\delta t}\right)$$

$$\iiint d\sigma(s,\delta t)\ dRR(s,\delta t)\ dFly(s,\delta t)$$

$$\omega(\sigma(s,\delta t), RR(s,\delta t), Fly(s,\delta t), s, \delta t)\ g\big(s_T/s, \delta t, \sigma(s,t), RR(s,t), Fly(s,t)\big) =$$

$$= \int_0^\infty ds\ g\left(s/s_0, \delta t, \sigma_{0,\delta t}, RR_{0,\delta t}, Fly_{0,\delta t}\right) g\left(s_T/s, \delta t, \boldsymbol{\sigma}_{s,\delta t}, \mathbf{RR}_{s,\delta t}, \mathbf{Fly}_{s,\delta t}\right)$$

(A30)

Where $\omega(\sigma(s,\delta t), RR(s,\delta t), Fly(s,\delta t), s, \delta t)$ is the 'combined' density function and $\boldsymbol{\sigma}_{s,\delta t}$, $\mathbf{RR}_{s,\delta t}$, $\mathbf{Fly}_{s,\delta t}$ are considered the **effective** values of $\sigma(s,\delta t), RR(s,\delta t), Fly(s,\delta t)$ at (s, δt).

For the option price P, we can write

$$P(t+\delta t)=\int_0^{\infty} ds_T \int_0^{\infty} ds\ g\,(s/s_0, t, \sigma_t, RR_t, Fly_t)\ \iiint d\sigma(s,t)\, dRR(s,t)\, dFly(s,t)$$

$$\omega(\sigma(s,t), RR(s,t), Fly(s,t), s, t)\ g\big(s_T/s, \delta t,\ \sigma(s,t), RR(s,t), Fly(s,t)\big)$$

Payoff (s_T , t+δt)

$$= \int_0^{\infty} ds_T \int_0^{\infty} ds\ g\,(s/s_0, t, \sigma_t, RR_t, Fly_t)\, g(s_T/s, \delta t, \boldsymbol{\sigma}_{s,\delta t}, \mathbf{RR}_{s,\delta t}, \mathbf{Fly}_{s,\delta t})$$

Payoff(t+δt)

$$=\int_0^{\infty} ds_T\ g\,(s_T/s_0, t+\delta t, \sigma_{t+\delta t}, RR_{t+\delta t}, Fly_{t+\delta t})\ \text{Payoff}(s_T, t+\delta t) \qquad \text{(A31)}$$

Where $\boldsymbol{\sigma}_{s,\delta t}$, $\mathbf{RR}_{s,\delta t}$, $\mathbf{Fly}_{s,\delta t}$ are considered the **effective** values of $\sigma(s,t), RR(s,t), Fly(s,t)$ at (s,t).

Dynamic Replication of a given option with 3 options

Given an option with strike K and expiry time T, we divide T into small intervals δt=T/N. At t_i = i δt and spot s_i we want to calculate the replication strategy until time t_{i+1}. Since at each spot and time the price of the given option depends on 4 stochastic factors ($s(t), \sigma(s,t), RR(s,t), Fly(s,t)$), we need to replicate it with the spot price and three different options with the same expiry so their prices depend on the same three stochastic factors. We replicate the given option with three specific vanilla options, i.e. their strike is determined at the inception of the given option (t=0). The most natural replication at time t=0 is the way traders hedge their portfolio, i.e. hedge the spot, hedge the Vega with an ATM delta neutral straddle, hedge the dVega/dvol with a d_1 Vega neutral butterfly and hedge the dVega/dSpot with a d_1 Risk Reversal. If there is no transaction cost, then theoretically there is no preferred d1. However, in the real world, d1 can be determined by optimization. Hence at each interval i and underlying asset price s_1 (s_1, t_i) we construct the portfolio **Π** as follows.

$$\Pi(s_1, t_i) = P(K, s_1, t_i,T)) + D_i S_1 + V_i\, P_{straddle}(K_0, s_1, t_i,T) + R_i\, P_{RR}(K_{d1}, s_1, t_i,T) + F_i P_{butterfly}(K_{d1}, s_1, t_i,T) \qquad \text{(A40)}$$

where P(K) is the price of the given option with strike K which can be either a call or put and V_i, R_i, F_i are the dynamic hedging amounts that are selected so that

$$\Pi(s_1, t_i, D_i, V_i, R_i, F_i) = \Pi(s, t_{i+1}, D_i, V_i, R_i, F_i) \ \text{ for all s} \qquad \text{(A41)}$$

Before rearranging terms, we define the fixed strikes

$K_1 = K_{d1}$; $K_2 = K_{-d1}$ (A42)

We get

$\mathbf{\Pi}(s_1, t_i) = P(K, s_1, t_i,T)) + D_iS_1+W_{0i} (P_{call}(K_0, s_1, t_i,T)+ P_{put}(K_0, s_1, t_i,T))+$

$W_{1i} P_{call}(K_1, s_1, t_i,T)+ W_{2i} P_{put}(K_2, s_1, t_i,T)$ (A43)

W_{0i}, W_{1i}, W_{2i} are three numbers which represent the amounts of the options with strikes K_0 K_1 K_2 in the portfolio at time interval i respectively. $P(K_i)$ is the price of the option with strike K_i which can be either call or put. Notice that we could replace the delta by

$D_iS_1=D_i (P_{call}(K_0, s_1, t_i,T) - P_{put}(K_0, s_1, t_i,T))$ (A44)

The notional (weights) D_i, W_{0i}, W_{1i}, W_{2i} are determined such that the value of the portfolio remains the same at time t_{i+1} regardless of the underlying move from t_i to t_{i+1} , but practically we mean in the vicinity of s_1.

$\mathbf{\Pi}_i(s_1, t_i,T) \approx \mathbf{\Pi}_i(s, t_{i+1},T) = P(K, s, t_{i+1},T)) + D_iS + W_{0i} (P_{call}(K_0\ s, t_{i+1},T) + P_{put}(K_0\ s, t_{i+1},T)) + W_{1i} P_{call}(K_1\ s, t_{i+1},T)+$
$W_{2i} P_{put}(K_2, s, t_{i+1},T)$ (A45)

for all s not too far from s_1. The accuracy of the replication depends on how small the time interval δt is.

In order to calculate D_i and W_i's at (s_1, t_i), we need to obtain the implied local smile at s_1 for expiry T and the implied local smile at s_1 for expiry t_{i+1}. This means

$\sigma_0(s_1, t_i, t_{i+1})$, $25\Delta RR(s_1, t_i, t_{i+1})$, $25\Delta fly(s_1, t_i, t_{i+1})$ (A46)

$\sigma_0(s_1, t_i, T)$, $25\Delta RR(s_1, t_i, T)$, $25\Delta fly(s_1, t_i, T)$ (A47)

In addition, we need to calculate the implied local smile for all s at t_{i+1} at each s_2

$\sigma_0(s_2, t_{i+1}, T)$, $25\Delta RR(s_2, t_{i+1}, T)$, $25\Delta fly(s_2, t_{i+1}, T)$ (A48)

Using (143)-(145) we look for W_{0i}, W_{1i}, W_{2i} that minimize

$$\text{Min} \sum_{\{s_2\}} \left(\Pi_i(s_1, t_i, T) - \Pi_i(s_2, t_{i+1}, T)\right)^2 g(s_1, t_i \rightarrow s_2, t_{i+1})$$

where $\{S_2\}$ corresponds to all the underlying asset prices at time t_{i+1}. In practice, we can take N points around s_1 and sum over all of them.

Appendix 4: Comparison of the model to the market

Currencies Dec 31 2015

EUR/USD		spot	1.0861		
1 month	**31 days**	**Forward**	**1.08692**		
	ATM	25RR	25Fly	10RR	10Fly
market	9.725	-0.45	0.175	-0.8	0.5
model		-0.45	0.175	**-0.847**	**0.601**
	10d P	25d P	atm	25d call	10 d call
model vol	1.025	10.125	9.725	9.675	9.903
Market vol	10.625	10.125	9.725	9.675	9.825
3 month	**92 days**	**Forward**	**1.08867**		
	ATM	25RR	25Fly	10RR	10Fly
market	9.975	-1.05	0.2	-1.7	0.6
model	9.975	-1.05	0.2	**-1.774**	**0.662**
2	10d P	25d P	atm	25d call	10 d call
model vol	11.524	10.7	9.975	9.65	9.750
Market vol	11.425	10.7	9.975	9.65	9.725
1 Year	**365 days**	**Forward**	**1.10012**		
	ATM	25RR	25Fly	10RR	10Fly
market	10.1	-1.725	0.275	-2.875	0.9
model	10.1	-1.725	0.275	**-2.892**	**0.901**
	10d P	25d P	atm	25d call	10 d call
model vol	12.447	11.2375	10.1	9.5125	9.555
Market vol	12.4375	11.2375	10.1	9.5125	9.5625

USD/JPY		spot	120.32		
1 month	**34 days**	**Forward**	**120.243**		
	ATM	25RR	25Fly	10RR	10Fly
market	7.45	-0.95	0.35	-1.8	1.025
model	7.45	-0.95	0.35	**-1.73**	**1.058**
	10d P	25d P	atm	25d call	10 d call
model vol	9.374	8.275	7.45	7.325	7.642
Market vo	9.375	8.275	7.45	7.325	7.575
3 month	**92 days**	**Forward**	**120.022**		
	ATM	25RR	25Fly	10RR	10Fly
market	8.025	-0.7	0.375	-1.35	1.175
model	8.025	-0.7	0.375	**-1.38**	**1.205**
	10d P	25d P	atm	25d call	10 d call
model vol	9.921	8.75	8.025	8.05	8.540
Market vo	9.875	8.75	8.025	8.05	8.525
1 Year	**369 days**	**Forward**	**119.0774**		
	ATM	25RR	25Fly	10RR	10Fly
market	9.05	-0.275	0.65	-0.45	2.225
model	9.05	-0.275	0.65	**-0.422**	**2.209**
	10d P	25d P	atm	25d call	10 d call
model vol	11.470	9.8375	9.05	9.5625	11.048
Market vo	11.5	9.8375	9.05	9.5625	11.05

USD/GBP		spot	1.4736		
1 month	34 days	**Forward**	1.4737		
	ATM	25RR	25Fly	10RR	10Fly
market	8.1	-0.55	0.15	-0.95	0.45
model	8.1	-0.55	0.15	**-0.99**	**0.49**
	10d P	25d P	atm	25d call	10 d call
model vol	9.085	8.525	8.1	7.975	8.095
Market vo	9.025	8.525	8.1	7.975	8.075
3 month	92 days	**Forward**	1.4738		
	ATM	25RR	25Fly	10RR	10Fly
market	8.8	-1.2	0.25	-2.1	0.775
model	8.8	-1.2	0.25	**-2.06**	**0.784**
	10d P	25d P	atm	25d call	10 d call
model vol	10.614	9.65	8.8	8.45	8.554
Market vo	10.625	9.65	8.8	8.45	8.525
1 Year	369 days	**Forward**	1.4764		
	ATM	25RR	25Fly	10RR	10Fly
market	10.425	-2.625	0.475	-4.7	1.575
model	10.425	-2.625	0.475	**-4.58**	**1.52**
	10d P	25d P	atm	25d call	10 d call
model vol	14.235	12.213	10.425	9.5875	9.655
Market vo	14.35	12.213	10.425	9.5875	9.65

EUR/JPY		spot	130.6		
1 month	**34 days**	**Forward**	130.6123		
	ATM	25RR	25Fly	10RR	10Fly
market	8.25	-0.75	0.275	-1.325	0.775
model	8.25	-0.75	0.275	**-1.34**	**0.82**
	10d P	25d P	atm	25d call	10 d call
model vol	9.742	8.9	8.25	8.15	8.401
Market vo	9.6875	8.9	8.25	8.15	8.3625
3 month	**92 days**	**Forward**	130.5845		
	ATM	25RR	25Fly	10RR	10Fly
market	8.925	-1.05	0.35	-1.9	1.05
model	8.925	-1.05	0.35	**-1.78**	**1.088**
	10d P	25d P	atm	25d call	10 d call
model vol	10.903	9.8	8.925	8.75	9.123
Market vo	10.925	9.8	8.925	8.75	9.025
1 Year	**369 days**	**Forward**	130.5155		
	ATM	25RR	25Fly	10RR	10Fly
market	10.3	-1.8	0.6	-3.3	1.975
model	10.3	-1.8	0.6	**-3.21**	**1.962**
	10d P	25d P	atm	25d call	10 d call
model vol	13.867	11.8	10.3	10	10.657
Market vo	13.925	11.8	10.3	10	10.625

EUR/GBP		spot	0.7369		
1 month	**34 days**	**Forward**	0.737395		
	ATM	25RR	25Fly	10RR	10Fly
market	8.775	0.2	0.2	0.35	0.55
model	8.775	0.2	0.2	**0.38**	**0.66**
	10d P	25d P	atm	25d call	10 d call
model vol	9.245	8.875	8.775	9.075	9.625
Market vol	9.15	8.875	8.775	9.075	9.5
3 month	**92 days**	**Forward**	0.738533		
	ATM	25RR	25Fly	10RR	10Fly
market	9.15	0.325	0.25	0.55	0.775
model	9.15	0.325	0.25	**0.58**	**0.813**
	10d P	25d P	atm	25d call	10 d call
model vol	9.673	9.2375	9.15	9.5625	10.253
Market vol	9.65	9.2375	9.15	9.5625	10.2
1 Year	**369 days**	**Forward**	0.745168		
	ATM	25RR	25Fly	10RR	10Fly
market	10.525	0.6	0.425	1.15	1.4
model	10.525	0.6	0.425	**1.15**	**1.426**
	10d P	25d P	atm	25d call	10 d call
model vol	11.376	10.65	10.525	11.25	12.526
Market vol	11.35	10.65	10.525	11.25	12.5

EUR/CHF		spot	1.0889		
1 month	**34 days**	**Forward**	1.08828		
	ATM	25RR	25Fly	10RR	10Fly
market	6.2	-0.4	0.35	-0.7	0.975
model	6.2	-0.4	0.35	**-0.68**	**1.06**
	10d P	25d P	atm	25d call	10 d call
model vol	7.600	6.75	6.2	6.35	6.920
Market vol	7.525	6.75	6.2	6.35	6.825
3 month	**92 days**	**Forward**	1.0872		
	ATM	25RR	25Fly	10RR	10Fly
market	6.775	-0.725	0.525	-1.325	1.6
model	6.775	-0.725	0.525	**-1.295**	**1.671**
	10d P	25d P	atm	25d call	10 d call
model vol	9.11	7.6625	6.775	6.9375	7.799
Market vol	9.0375	7.6625	6.775	6.9375	7.7125
1 Year	**369 days**	**Forward**	1.08144		
	ATM	25RR	25Fly	10RR	10Fly
market	7.7	-2.05	0.75	-3.85	2.525
model	7.7	-2.05	0.75	**-3.69**	**2.555**
	10d P	25d P	atm	25d call	10 d call
model vol	9.11	9.475	7.7	7.425	8.410
Market vol	12.15	9.475	7.7	7.425	8.36

USD/KRW	spot	1175.9			
1 month	**34 days**	**Forward**	1176.15		
	ATM	25RR	25Fly	10RR	10Fly
market	9.85	0.8	0.2	1.45	0.6
model	9.85	0.8	0.2	**1.46**	**0.68**
	10d P	25d P	atm	25d call	10 d call
model vol	9.804	9.65	9.85	10.45	11.262
Market vol	9.725	9.65	9.85	10.45	11.175
3 month	**92 days**	**Forward**	1177.9		
	ATM	25RR	25Fly	10RR	10Fly
market	10.6	1.45	0.325	2.7	0.9
model	10.6	1.45	0.325	**2.56**	**0.96**
	10d P	25d P	atm	25d call	10 d call
model vol	10.280	10.2	10.6	11.65	12.840
Market vol	10.15	10.2	10.6	11.65	12.85
1 Year	**369 days**	**Forward**	1179.38		
	ATM	25RR	25Fly	10RR	10Fly
market	11.3	2.8	0.55	5	1.6
model	11.3	2.8	0.55	**4.89**	**1.62**
	10d P	25d P	atm	25d call	10 d call
model vol	10.475	10.45	11.3	13.25	15.365
Market vol	10.4	10.45	11.3	13.25	15.39

EUR/PLN		spot	4.2635		
1 month	**34 days**	**Forward**	4.26975		
	ATM	25RR	25Fly	10RR	10Fly
market	6.75	0.475	0.2	0.8	0.625
model	6.75	0.475	0.2	**0.86**	**0.71**
	10d P	25d P	atm	25d call	10 d call
model vol	7.030	6.7125	6.75	7.1875	7.890
Market vol	6.975	6.7125	6.75	7.1875	7.775
3 month	**92 days**	**Forward**	4.2813		
	ATM	25RR	25Fly	10RR	10Fly
market	6.75	0.85	0.275	1.6	0.875
model	6.75	0.85	0.275	**1.51**	**0.91**
	10d P	25d P	atm	25d call	10 d call
model vol	6.905	6.6	6.75	7.45	8.415
Market vol	6.825	6.6	6.75	7.45	8.425
1 Year	**369 days**	**Forward**	4.3339		
	ATM	25RR	25Fly	10RR	10Fly
market	7.15	1.65	0.45	3.15	1.35
model	7.15	1.65	0.45	**2.98**	**1.391**
	10d P	25d P	atm	25d call	10 d call
model vol	7.051	6.775	7.15	8.425	10.031
Market vol	6.925	6.775	7.15	8.425	10.075

Commodities: Dec 31 2015

BRENT Crude oil

Expiry	26-Jan-16			Expiry	24-Mar-16		
Forward	ATM	25dRR	25fFly	Forward	ATM	25dRR	25fFly
37.6569	44.65	-3.71	1.43	39.37	43.26	-2.89	1.22

Strike	Exchange	model	Strike	Exchange	model	Strike	Exchange	model
20	17.68	17.667	20	19.44	19.437	66	0.07	0.075
21	16.68	16.671	21	18.45	18.451	67	0.07	0.069
22	15.68	15.677	22	17.46	17.467	68	0.06	0.064
23	14.68	14.684	23	16.48	16.487	69	0.05	0.059
24	13.69	13.692	24	15.51	15.512	70	0.05	0.055
25	12.69	12.702	25	14.54	14.544	71	0.04	0.051
26	11.71	11.714	26	13.58	13.585	72	0.04	0.048
26.5	11.21	11.221	26.5	13.11	13.109	73	0.04	0.045
27	10.72	10.729	27	12.64	12.637	74	0.03	0.042
27.5	10.23	10.238	28	11.71	11.705	75	0.03	0.039
28	9.74	9.749	29	10.80	10.791	76	0.03	0.037
28.5	9.26	9.261	30	9.91	9.900	77	0.03	0.035
29	8.77	8.776	31	9.04	9.034	78	0.02	0.033
29.5	8.29	8.294	31.5	8.62	8.613	79	0.02	0.031
30	7.82	7.816	32	8.20	8.199	80	0.02	0.030
30.5	7.35	7.342	32.5	7.80	7.794	81	0.02	0.029
31	6.88	6.875	33	7.40	7.398	82	0.02	0.027
31.5	6.43	6.413	33.5	7.01	7.012	83	0.02	0.026
32	5.97	5.960	34	6.63	6.635	84	0.02	0.025
32.5	5.53	5.516	34.5	6.26	6.268	85	0.02	0.024
33	5.09	5.082	35	5.91	5.912	86	0.02	0.023
33.5	4.67	4.659	35.5	5.56	5.567	87	0.02	0.022
34	4.25	4.250	36	5.23	5.234	88	0.02	0.021
34.5	3.85	3.856	37	4.60	4.602	89	0.01	0.020
35	3.47	3.478	37.5	4.30	4.305	90	0.01	0.019
35.5	3.11	3.118	38	4.02	4.019			
36	2.77	2.778	38.5	3.75	3.747			
36.5	2.45	2.459	39	3.49	3.487			
37	2.16	2.163	39.5	3.24	3.240			
37.5	1.89	1.890	40	3.01	3.006			
38	1.64	1.641	40.5	2.79	2.785			
38.5	1.42	1.418	41	2.58	2.577			
39	1.22	1.219	41.5	2.39	2.384			
39.5	1.05	1.045	42	2.20	2.203			
40	0.9	0.893	42.5	2.03	2.035			
40.5	0.76	0.761	43	1.88	1.879			
41	0.65	0.648	43.5	1.73	1.735			
41.5	0.55	0.552	44	1.60	1.601			
42	0.47	0.469	44.5	1.47	1.477			
42.5	0.4	0.399	45	1.36	1.362			
43	0.34	0.340	45.5	1.25	1.256			
43.5	0.29	0.291	46	1.16	1.158			
44	0.25	0.249	46.5	1.06	1.068			
44.5	0.21	0.213	47	0.98	0.984			
45	0.18	0.183	47.5	0.90	0.907			
45.5	0.16	0.158	48	0.83	0.836			
46	0.14	0.137	48.5	0.77	0.771			
46.5	0.12	0.120	49	0.71	0.711			
47	0.11	0.105	49.5	0.65	0.656			
47.5	0.09	0.092	50	0.61	0.605			
48	0.08	0.081	50.5	0.56	0.559			
48.5	0.07	0.072	51	0.52	0.516			
49	0.06	0.065	51.5	0.48	0.477			
49.5	0.06	0.058	52	0.44	0.441			
50	0.05	0.053	52.5	0.41	0.407			
50.5	0.05	0.048	53	0.38	0.377			
51	0.04	0.044	53.5	0.35	0.349			
51.5	0.04	0.040	54	0.33	0.324			
52	0.04	0.037	54.5	0.31	0.300			
52.5	0.03	0.034	55	0.29	0.279			
53	0.03	0.032	55.5	0.27	0.259			
53.5	0.03	0.030	56	0.25	0.241			
54	0.03	0.028	56.5	0.23	0.225			
54.5	0.02	0.026	57	0.22	0.210			
55	0.02	0.024	57.5	0.20	0.196			
55.5	0.02	0.023	58	0.19	0.183			
56	0.02	0.021	58.5	0.18	0.172			
56.5	0.02	0.020	59	0.17	0.161			
57	0.02	0.019	59.5	0.16	0.151			
57.5	0.02	0.018	60	0.15	0.142			
58	0.02	0.017	61	0.13	0.126			
58.5	0.02	0.016	62	0.12	0.112			
59	0.02	0.015	63	0.10	0.101			
59.5	0.01	0.014	64	0.09	0.091			
60	0.01	0.014	65	0.08	0.082			

Expiry	**22-Dec-16**		
Forward	ATM	25dRR	25fFly
45.49	33.51	-1.7	0.99

Strike	**Exchange**	**model**
5	40.52	40.498
10	35.54	35.517
15	30.56	30.572
17.5	28.09	28.120
20	25.65	25.686
25	20.93	20.923
27.5	18.65	18.635
30	16.47	16.440
35	12.42	12.419
37.5	10.62	10.629
39	9.62	9.634
40	8.99	9.005
41	8.39	8.403
42	7.82	7.830
43	7.28	7.285
44	6.77	6.768
45	6.29	6.280
46	5.83	5.819
46.5	5.61	5.599
47	5.40	5.386
48	5.00	4.981
49	4.62	4.604
50	4.27	4.255
51	3.94	3.932
52	3.64	3.635
53	3.36	3.360
54	3.10	3.107
55	2.86	2.875
56	2.65	2.660
57	2.45	2.463
58	2.27	2.281
59	2.1	2.113
60	1.95	1.959
61	1.81	1.816
62	1.68	1.685
63	1.56	1.564
64	1.45	1.453
65	1.35	1.350
66	1.26	1.255
67	1.17	1.168
68	1.09	1.088
69	1.01	1.014
70	0.94	0.946
71	0.88	0.883
72	0.82	0.825
73	0.76	0.771
74	0.71	0.722
75	0.67	0.676
76	0.62	0.634
77	0.59	0.595
78	0.55	0.560
79	0.52	0.526
80	0.49	0.496
81	0.47	0.467
82	0.44	0.441
83	0.42	0.417
84	0.4	0.395
85	0.38	0.374
86	0.37	0.355
87	0.35	0.337
88	0.34	0.320
89	0.32	0.305
90	0.31	0.291
91	0.29	0.277
92	0.28	0.265
93	0.27	0.253
94	0.26	0.242
95	0.25	0.232
96	0.24	0.223
97	0.23	0.214
98	0.22	0.206
99	0.21	0.198
100	0.2	0.191

Strike	**Exchange**	**model**
103	0.18	0.172
104	0.17	0.166
105	0.17	0.161
106	0.16	0.156
107	0.15	0.152
108	0.15	0.147
110	0.14	0.139
115	0.11	0.123
120	0.10	0.110
125	0.08	0.100
130	0.07	0.092
135	0.06	0.085
140	0.06	0.078
145	0.05	0.073
150	0.05	0.068
175	0.04	0.051
200	0.03	0.040

Expiry	**21-Dec-18**		
Forward	ATM	25dRR	25fFly
53.205	24.15	-4.35	2.51

Strike	**Exchange**	**model**
48	12.22	12.341
49	11.64	11.767
50	11.08	11.202
51	10.54	10.645
52	10.01	10.095
53	9.50	9.551
54	9.00	9.014
55	8.53	8.485
56	8.07	7.964
57	7.62	7.458
58	7.20	6.972
59	6.79	6.527
60	6.40	6.136
61	6.03	5.789
62	5.68	5.480
63	5.34	5.201
64	5.02	4.946
65	4.73	4.712
66	4.45	4.495
67	4.20	4.292
68	3.96	4.102
69	3.74	3.923
70	3.55	3.753
71	3.37	3.593
72	3.20	3.441
73	3.06	3.297
74	2.92	3.160
75	2.80	3.031
76	2.70	2.908
77	2.60	2.793
78	2.51	2.683
79	2.43	2.580
80	2.35	2.482
81	2.28	2.390
82	2.22	2.302
83	2.15	2.218
84	2.10	2.137
85	2.04	2.061
86	1.99	1.988
87	1.94	1.919
88	1.90	1.853
89	1.85	1.791
90	1.81	1.732
91	1.77	1.676
92	1.73	1.624
93	1.69	1.574
94	1.66	1.526
95	1.62	1.480
96	1.59	1.437
97	1.56	1.395
98	1.52	1.355
99	1.49	1.317
100	1.46	1.280
101	1.43	1.246
102	1.41	1.213
103	1.38	1.181
104	1.35	1.151

GOLD Exchange: COMEX

Expiry	29-Jan-16			Expiry	31-Mar-16		
Forward	**ATM**	**25dRR**	**25fFly**	**Forward**	**ATM**	**25dRR**	**25fFly**
1060.15	**11.63**	**-0.7**	**0.3**	**1060.35**	**14.53**	**-1.26**	**0.6**

Strike	Exchange	model	Strike	Exchange	model	Strike	Exchange	model
675	385.2	385.146	675	385.8	385.48	1190	2.9	2.85
700	360.2	360.146	700	360.8	360.54	1195	2.7	2.62
725	335.2	335.146	725	335.8	335.62	1200	2.5	2.41
750	310.2	310.146	750	310.8	310.73	1205	2.3	2.22
775	285.2	285.147	775	285.8	285.87	1210	2.2	2.05
800	260.2	260.148	800	260.9	261.06	1215	2	1.90
825	235.2	235.152	825	236.1	236.28	1220	1.9	1.77
850	210.2	210.160	850	211.4	211.57	1225	1.7	1.64
860	200.2	200.166	860	201.5	201.71	1230	1.6	1.53
870	190.2	190.174	870	191.7	191.86	1235	1.5	1.43
875	185.2	185.179	875	186.8	186.95	1240	1.4	1.34
880	180.2	180.185	880	181.9	182.04	1245	1.3	1.26
890	170.2	170.201	890	172.1	172.25	1250	1.2	1.18
900	160.2	160.222	900	162.4	162.50	1255	1.1	1.11
910	150.2	150.249	910	152.7	152.81	1260	1	1.05
920	140.2	140.284	920	143.1	143.19	1265	1	0.99
925	135.3	135.304	925	138.4	138.41	1270	0.9	0.94
930	130.3	130.327	930	133.7	133.65	1275	0.9	0.89
940	120.3	120.382	940	124.3	124.22	1280	0.8	0.85
950	110.4	110.449	950	115.1	114.93	1285	0.8	0.80
955	105.4	105.489	955	110.5	110.34	1290	0.7	0.77
960	100.5	100.535	960	106	105.79	1295	0.7	0.73
965	95.5	95.589	965	101.5	101.30	1300	0.7	0.70
970	90.6	90.654	970	97	96.85	1305	0.6	0.67
975	85.7	85.731	975	92.6	92.46	1310	0.6	0.64
980	80.8	80.826	980	88.2	88.13	1315	0.6	0.61
985	75.9	75.943	985	83.9	83.86	1320	0.5	0.59
990	71.1	71.089	990	79.6	79.66	1325	0.5	0.56
995	66.3	66.273	995	75.5	75.53	1330	0.5	0.54
1000	61.5	61.504	1000	71.5	71.48	1335	0.5	0.52
1005	56.8	56.794	1005	67.5	67.52	1340	0.4	0.50
1010	52.2	52.156	1010	63.6	63.64	1345	0.4	0.48
1015	47.6	47.607	1015	59.8	59.86	1350	0.4	0.46
1020	43.2	43.164	1020	56.1	56.17			
1025	38.9	38.848	1025	52.5	52.59			
1030	34.7	34.681	1030	49.1	49.13			
1035	30.7	30.689	1035	45.8	45.78			
1040	26.9	26.899	1040	42.5	42.55			
1045	23.3	23.338	1045	39.4	39.46			
1050	20.1	20.033	1050	36.5	36.50			
1055	17.1	17.008	1055	33.6	33.68			
1060	14.3	14.281	1060	30.9	31.00			
1065	11.9	11.865	1065	28.5	28.48			
1070	9.7	9.761	1070	26.2	26.11			
1075	7.9	7.961	1075	24	23.90			
1080	6.4	6.445	1080	22	21.85			
1085	5.1	5.187	1085	20	19.95			
1090	4.1	4.156	1090	18.2	18.19			
1095	3.3	3.319	1095	16.6	16.57			
1100	2.6	2.646	1100	15	15.09			
1105	2.1	2.109	1105	13.7	13.73			
1110	1.7	1.681	1110	12.4	12.49			
1115	1.4	1.344	1115	11.3	11.36			
1120	1.1	1.077	1120	10.3	10.33			
1125	0.9	0.868	1125	9.3	9.39			
1130	0.7	0.703	1130	8.5	8.54			
1135	0.6	0.574	1135	7.7	7.76			
1140	0.5	0.472	1140	7	7.06			
1145	0.5	0.391	1145	6.4	6.42			
1150	0.4	0.326	1150	5.8	5.85			
1155	0.3	0.275	1155	5.3	5.32			
1160	0.3	0.233	1160	4.8	4.85			
1165	0.2	0.198	1165	4.4	4.43			
1170	0.2	0.169	1170	4.1	4.04			
1175	0.2	0.146	1175	3.7	3.70			
1180	0.2	0.126	1180	3.4	3.38			

Expiry	30-Jun-16					
Forward	ATM	25dRR	25fFly			
1061.58	15.57	-1.47	0.72			
Strike	Exchange	model		Strike	Exchange	model
675	387.4	387.60		1190	10.4	10.31
700	362.5	362.86		1195	9.8	9.73
725	337.7	338.14		1200	9.2	9.18
750	313	313.48		1205	8.7	8.66
775	288.4	288.87		1210	8.2	8.18
800	264	264.36		1215	7.8	7.73
825	239.8	239.99		1220	7.3	7.30
850	215.9	215.86		1225	6.9	6.91
860	206.5	206.31		1230	6.5	6.53
870	197.1	196.82		1235	6.2	6.19
875	192.4	192.10		1240	5.9	5.86
880	187.7	187.41		1245	5.6	5.56
890	178.5	178.09		1250	5.4	5.27
900	169.3	168.87		1255	5.1	5.00
910	160.2	159.76		1260	4.9	4.75
920	151.2	150.79		1265	4.7	4.51
925	146.8	146.35		1270	4.5	4.29
930	142.4	141.96		1275	4.3	4.09
940	133.7	133.29		1280	4.1	3.89
950	125.1	124.79		1285	3.9	3.71
955	120.8	120.61		1290	3.7	3.54
960	116.6	116.48		1295	3.5	3.38
965	112.5	112.40		1300	3.4	3.23
970	108.4	108.38		1305	3.2	3.09
975	104.4	104.41		1310	3.1	2.96
980	100.5	100.50		1315	3	2.84
985	96.6	96.65		1320	2.8	2.72
990	92.8	92.86		1325	2.7	2.61
995	89	89.14		1330	2.6	2.51
1000	85.3	85.48		1335	2.5	2.42
1005	81.7	81.89		1340	2.4	2.33
1010	78.1	78.38		1345	2.3	2.24
1015	74.6	74.93		1350	2.2	2.16
1020	71.2	71.56		1355	2.1	2.09
1025	68	68.27		1360	2	2.02
1030	64.8	65.07		1365	1.9	1.95
1035	61.7	61.94		1370	1.9	1.89
1040	58.7	58.90		1375	1.8	1.83
1045	55.7	55.96		1380	1.7	1.77
1050	52.9	53.10		1385	1.7	1.72
1055	50.1	50.33		1390	1.6	1.67
1060	47.4	47.67		1395	1.5	1.62
1065	45	45.10		1400	1.5	1.58
1070	42.6	42.63		1405	1.4	1.53
1075	40.3	40.26		1410	1.4	1.49
1080	38.1	38.01		1420	1.3	1.41
1085	35.9	35.87		1425	1.2	1.38
1090	33.9	33.83		1430	1.2	1.35
1095	32	31.90		1440	1.1	1.28
1100	30.2	30.07				
1105	28.5	28.34				
1110	26.8	26.71				
1115	25.3	25.16				
1120	23.8	23.70				
1125	22.4	22.33				
1130	21.1	21.03				
1135	19.8	19.81				
1140	18.6	18.66				
1145	17.6	17.58				
1150	16.6	16.56				
1155	15.6	15.60				
1160	14.7	14.70				
1165	13.9	13.85				
1170	13.1	13.05				
1175	12.3	12.30				
1180	11.6	11.59				
1185	11	10.93				

Expiry	30-Dec-16					
Forward	ATM	25dRR	25fFly			
1065.5	16.93	-0.91	0.34			
Strike	Exchange	model		Strike	Exchange	model
675	391.5	392.23		1195	27.8	27.81
700	367.0	367.66		1200	26.7	26.74
725	342.8	343.24		1205	25.6	25.71
750	318.8	319.02		1210	24.6	24.72
775	295.1	295.07		1215	23.7	23.77
800	271.7	271.47		1220	22.8	22.85
825	248.8	248.33		1225	21.9	21.96
850	226.3	225.76		1230	21.0	21.11
860	217.4	216.91		1235	20.2	20.28
870	208.7	208.19		1240	19.4	19.49
875	204.3	203.87		1245	18.6	18.73
880	200.0	199.59		1250	17.9	18.00
890	191.5	191.12		1255	17.2	17.29
900	183.0	182.80		1260	16.5	16.61
910	174.8	174.62		1265	15.9	15.96
920	166.8	166.60		1270	15.2	15.34
925	162.8	162.66		1275	14.6	14.73
930	158.9	158.75		1280	14.0	14.16
940	151.2	151.08		1285	13.5	13.60
950	143.7	143.58		1290	13.0	13.06
960	136.3	136.27		1295	12.5	12.55
965	132.7	132.69		1300	12.0	12.06
970	129.1	129.16		1305	11.5	11.59
975	125.6	125.68		1310	11.1	11.13
980	122.1	122.25		1315	10.7	10.70
985	118.6	118.88		1320	10.3	10.28
990	115.2	115.55		1325	9.9	9.87
995	111.9	112.28		1330	9.5	9.49
1000	108.7	109.07		1335	9.1	9.12
1005	105.6	105.90		1340	8.8	8.76
1010	102.5	102.80		1350	8.1	8.10
1015	99.5	99.75		1360	7.5	7.48
1020	96.5	96.76		1370	7.0	6.92
1025	93.6	93.82		1375	6.7	6.65
1030	90.7	90.94		1380	6.4	6.40
1035	87.9	88.12		1390	6.0	5.92
1040	85.1	85.36		1400	5.5	5.48
1045	82.4	82.65		1410	5.1	5.08
1050	79.8	80.01		1420	4.7	4.71
1055	77.2	77.42		1425	4.5	4.53
1060	74.6	74.89		1430	4.4	4.37
1065	72.1	72.42		1450	3.7	3.77
1070	69.8	70.01		1475	3.1	3.15
1075	67.5	67.66		1500	2.5	2.65
1080	65.3	65.36		1525	2.1	2.24
1085	63.1	63.13		1550	1.7	1.91
1090	61.0	60.95		1575	1.5	1.64
1095	59.0	58.84		1600	1.2	1.42
1100	57.0	56.78		1625	1.0	1.24
1105	55.0	54.79				
1110	53.1	52.85				
1115	51.2	50.96				
1120	49.4	49.14				
1125	47.7	47.36				
1130	45.9	45.65				
1135	44.3	43.98				
1140	42.7	42.37				
1145	41.1	40.81				
1150	39.6	39.30				
1155	38.1	37.84				
1160	36.7	36.43				
1165	35.3	35.07				
1170	33.9	33.75				
1175	32.6	32.47				
1180	31.4	31.24				
1185	30.2	30.06				
1190	29.0	28.91				

COPPER Exchange: COMEX

Expiry	31-Jan-16		
Forward	**ATM**	**25dRR**	**25fFly**
2.12992	**21.75**	**-2.33**	**0.48**

Strike	Exchange	model
0.25	1.8800	1.87993
0.5	1.6300	1.62993
0.75	1.3800	1.37993
1	1.1300	1.12993
1.25	0.8800	0.87995
1.3	0.8300	0.82998
1.35	0.7800	0.78001
1.4	0.7300	0.73007
1.45	0.6800	0.68014
1.5	0.6300	0.63024
1.55	0.5800	0.58038
1.6	0.5300	0.53057
1.65	0.4805	0.48082
1.7	0.4305	0.43114
1.75	0.3810	0.38157
1.8	0.3320	0.33219
1.82	0.3125	0.31253
1.83	0.3030	0.30273
1.84	0.2930	0.29296
1.85	0.2835	0.28321
1.86	0.2735	0.27350
1.87	0.2640	0.26382
1.88	0.2545	0.25419
1.89	0.2450	0.24460
1.9	0.2355	0.23507
1.91	0.2260	0.22560
1.92	0.2165	0.21619
1.93	0.2075	0.20687
1.94	0.1980	0.19763
1.95	0.1890	0.18848
1.96	0.1800	0.17943
1.97	0.1710	0.17050
1.98	0.1620	0.16170
1.99	0.1530	0.15303
2	0.1445	0.14451
2.01	0.1360	0.13615
2.02	0.1280	0.12797
2.03	0.1195	0.11998
2.04	0.1115	0.11219
2.05	0.1040	0.10461
2.06	0.0965	0.09727
2.07	0.0895	0.09017
2.08	0.0830	0.08333
2.09	0.0765	0.07676
2.1	0.0700	0.07048
2.11	0.0645	0.06449
2.12	0.0585	0.05881
2.13	0.0535	0.05345
2.14	0.0485	0.04840
2.15	0.0440	0.04368
2.16	0.0395	0.03929
2.17	0.0355	0.03522
2.18	0.0320	0.03147
2.19	0.0285	0.02804
2.2	0.0255	0.02491
2.21	0.0225	0.02207
2.22	0.0200	0.01951
2.23	0.0175	0.01722
2.24	0.0155	0.01516
2.25	0.0135	0.01333
2.26	0.0115	0.01171
2.27	0.0100	0.01027
2.28	0.0085	0.00900
2.29	0.0075	0.00789
2.3	0.0065	0.00691
2.31	0.0055	0.00605
2.32	0.0045	0.00530
2.33	0.0040	0.00465
2.34	0.0035	0.00407
2.35	0.0030	0.00358
2.36	0.0025	0.00314
2.37	0.0020	0.00276
2.38	0.0015	0.00243
2.39	0.0015	0.00215
2.4	0.0010	0.00190
2.41	0.0010	0.00168
2.42	0.0005	0.00149
2.43	0.0005	0.00132
2.44	0.0005	0.00118

Expiry	31-Mar-16		
Forward	**ATM**	**25dRR**	**25fFly**
2.14054	**24.44**	**-2.58**	**0.57**

Strike	Exchange	model
0.25	1.8915	1.89054
0.5	1.6415	1.64054
0.75	1.3915	1.39057
1	1.1415	1.14076
1.05	1.0915	1.09087
1.1	1.0415	1.04101
1.15	0.9915	0.99119
1.2	0.9415	0.94142
1.25	0.8915	0.89169
1.3	0.8415	0.84199
1.35	0.7915	0.79235
1.4	0.7415	0.74276
1.45	0.692	0.69326
1.5	0.6425	0.64386
1.55	0.5935	0.59465
1.6	0.5445	0.54571
1.65	0.4965	0.49717
1.7	0.449	0.44923
1.75	0.4025	0.40216
1.8	0.357	0.35625
1.85	0.3135	0.31189
1.9	0.2715	0.26945
1.95	0.2305	0.22933
2	0.192	0.19195
2.05	0.157	0.15772
2.1	0.126	0.12701
2.15	0.0995	0.10012
2.2	0.077	0.07734
2.25	0.0585	0.05876
2.3	0.044	0.04406
2.35	0.032	0.03269
2.4	0.024	0.02407
2.45	0.018	0.01763
2.5	0.0135	0.01289
2.55	0.01	0.00943
2.6	0.0075	0.00693
2.65	0.0055	0.00514
2.7	0.004	0.00385
2.75	0.003	0.00292
2.8	0.002	0.00225
2.85	0.0015	0.00176
2.9	0.001	0.00139
2.95	0.001	0.00112
3	0.0005	0.00091
3.05	0.0005	0.00075
3.1	0.0005	0.00062
3.15	0.0005	0.00052
3.2	0.0005	0.00044
3.25	0.0005	0.00038
3.3	0.0005	0.00032
3.35	0.0005	0.00028
3.4	0.0005	0.00025
3.45	0.0005	0.00021
3.5	0.0005	0.00019
3.55	0.0005	0.00016
3.6	0.0005	0.00014
3.65	0.0005	0.00013
3.7	0.0005	0.00011
3.75	0.0005	0.00010
3.8	0.0005	0.00009
3.85	0.0005	0.00008
3.9	0.0005	0.00007
3.95	0.0005	0.00006
4	0.0005	0.00006
4.05	0.0005	0.00005
4.1	0.0005	0.00005
4.15	0.0005	0.00004
4.2	0.0005	0.00004
4.25	0.0005	0.00003
4.5	0.0005	0.00002

Expiry	30-Jun-16		
Forward	**ATM**	**25dRR**	**25fFly**
2.143	**24.66**	**-2.7**	**0.6**

Strike	Exchange	model
0.25	1.8965	1.89300
0.5	1.6465	1.64304
0.75	1.3965	1.39334
1	1.1465	1.14456
1.05	1.0965	1.09495
1.1	1.0465	1.04538
1.15	0.9965	0.99586
1.2	0.9465	0.94641
1.25	0.8965	0.89704
1.3	0.8465	0.84779
1.35	0.7965	0.79871
1.4	0.7465	0.74986
1.45	0.697	0.70132
1.5	0.648	0.65318
1.55	0.5995	0.60561
1.6	0.552	0.55875
1.65	0.5055	0.51280
1.7	0.46	0.46794
1.75	0.416	0.42441
1.8	0.3745	0.38239
1.85	0.3345	0.34210
1.9	0.297	0.30375
1.95	0.262	0.26752
2	0.2295	0.23356
2.05	0.1995	0.20203
2.1	0.1725	0.17306
2.15	0.148	0.14674
2.2	0.1265	0.12319
2.25	0.1075	0.10266
2.3	0.0905	0.08504
2.35	0.076	0.07012
2.4	0.063	0.05760
2.45	0.0525	0.04716
2.5	0.043	0.03851
2.55	0.0355	0.03139
2.6	0.029	0.02555
2.65	0.0235	0.02079
2.7	0.019	0.01693
2.75	0.015	0.01380
2.8	0.012	0.01128
2.85	0.0095	0.00926
2.9	0.0075	0.00764
2.95	0.006	0.00633
3	0.005	0.00528
3.05	0.004	0.00443
3.1	0.003	0.00375
3.15	0.0025	0.00320
3.2	0.002	0.00275
3.25	0.0015	0.00237
3.3	0.001	0.00206
3.35	0.001	0.00180
3.4	0.0005	0.00159

Stocks & Indices

GOOGLE			
Expiry	**15-Jan-16**		
Forward	ATM	25dRR	25fFly
778.879	19.8	-4	0.45

strike	bid	ask	model	strike	bid	ask	model	strike	bid	ask	model
260	517.00	520.5	518.88	600	177.8	180.7	178.99	767.5	18.9	20.4	19.68
270	507.00	510.7	508.88	602.5	174.9	178.2	176.50	770	17.4	18	17.99
275	502.1	505.7	503.88	605	172.8	175.7	174.01	772.5	15.9	17.1	16.37
280	497.1	500.7	498.88	607.5	169.9	173.2	171.51	775	14.3	15	14.83
285	492.2	495.4	493.88	610	167.8	170.7	169.02	777.5	12.9	13.6	13.37
290	487.1	490.4	488.88	612.5	165	168.2	166.53	780	11.8	12.2	11.98
295	482.2	485.4	483.88	615	162.8	165.7	164.04	782.5	10.3	11.1	10.69
300	477.1	480.4	478.88	617.5	159.8	163.2	161.55	785	9.3	10	9.47
305	472	475.4	473.88	620	157.8	160.7	159.06	787.5	8.3	9	8.35
310	467.1	470.4	468.88	622.5	154.8	158.2	156.57	790	7.3	8	7.31
315	462.1	465.4	463.88	625	152.8	155.8	154.08	792.5	6.2	7	6.37
320	457	460.4	458.88	627.5	149.8	153.3	151.59	795	5.5	6	5.51
325	452.2	455.4	453.88	630	147.9	150.7	149.10	797.5	4.7	5.3	4.75
330	447	450.4	448.88	632.5	144.8	148.2	146.61	800	4.1	4.4	4.11
335	442.2	445.4	443.88	635	142.8	145.7	144.12	802.5	3.5	3.8	3.52
340	437.6	440.4	438.88	637.5	139.9	143.4	141.63	805	2.95	3.3	2.96
345	432.2	435.4	433.88	640	137.8	140.8	139.14	807.5	2.45	2.75	2.51
350	427.3	430.5	428.88	642.5	134.9	138.3	136.66	810	2.1	2.35	2.13
355	422.3	425.5	423.88	645	132.9	135.8	134.17	812.5	1.8	1.95	1.81
360	417.3	420.5	418.88	647.5	129.8	133.3	131.69	815	1.45	1.65	1.53
365	412.2	415.5	413.88	650	127.9	130.8	129.20	817.5	1.3	1.4	1.30
370	407	410.5	408.88	652.5	125	128.3	126.72	820	1	1.2	1.11
375	402	405.5	403.88	655	122.6	125.8	124.24	822.5	0.9	1	0.95
380	397.6	400.5	398.88	657.5	119.9	123.4	121.76	825	0.7	0.85	0.81
385	392.3	395.5	393.88	660	117.9	120.8	119.28	827.5	0.6	0.7	0.69
390	387.2	390.5	388.88	662.5	114.9	118.3	116.80	830	0.45	0.6	0.59
395	382.2	385.5	383.88	665	112.9	116	114.32	832.5	0.35	0.55	0.51
400	377.6	380.5	378.88	667.5	110	113.4	111.84	835	0.3	0.5	0.44
405	372.3	375.5	373.88	670	108	110.9	109.36	837.5	0.3	0.45	0.38
410	367.1	370.5	368.88	672.5	105.1	108.4	106.89	840	0.2	0.4	0.32
415	362.2	365.5	363.88	675	103	105.9	104.41	842.5	0.15	0.35	0.28
420	357.7	360.5	358.88	677.5	100.1	103.4	101.94	845	0.1	0.45	0.24
425	352.3	355.5	353.88	680	97.6	101	99.47	847.5	0.1	0.6	0.21
430	347.2	350.5	348.88	682.5	95.2	98.4	97.00	850	0.05	0.3	0.18
435	342.2	345.5	343.88	685	92.6	96.1	94.53				
440	337.4	340.5	338.88	687.5	90.2	93.5	92.06				
445	332.3	335.5	333.88	690	88.1	91	89.60				
450	327.8	330.5	328.88	692.5	85.2	88.6	87.14				
455	322.4	325.5	323.88	695	82.7	86	84.68				
460	317.7	320.5	318.88	697.5	80.3	83.6	82.22				
465	312.8	315.5	313.88	700	78.2	81.1	79.77				
470	307.8	310.5	308.88	702.5	75.4	78.7	77.32				
475	302.5	305.5	303.88	705	73.1	76.2	74.88				
480	297.7	300.5	298.88	707.5	70.3	73.8	72.44				
485	292.8	295.8	293.88	710	68.4	71.3	70.01				
490	287.8	290.8	288.88	712.5	65.4	68.9	67.59				
495	282.8	285.8	283.88	715	63.5	66.5	65.18				
500	277.5	280.5	278.88	717.5	61.3	64.2	62.77				
505	272.7	275.5	273.88	720	58.3	61.6	60.38				
510	267.8	270.5	268.89	722.5	55.8	59.2	58.00				
515	262.8	265.5	263.89	725	54	56.9	55.63				
520	257.6	260.5	258.89	727.5	51.7	54.5	53.28				
525	252.8	255.5	253.89	730	49.3	52.2	50.95				
530	247.3	250.7	248.90	732.5	46.9	49.9	48.63				
535	242.3	245.6	243.90	735	44.7	47.6	46.34				
540	237.3	240.6	238.90	737.5	42.3	45.3	44.07				
545	232.5	235.6	233.91	740	39.9	43.1	41.82				
550	227.9	230.6	228.91	742.5	38	40.9	39.60				
555	222.3	225.6	223.92	745	35.5	38.7	37.42				
560	217.8	220.6	218.92	747.5	33.9	36.6	35.26				
565	212.3	215.6	213.93	750	31.8	34.5	33.15				
570	207.8	210.6	208.94	752.5	29.2	32.5	31.07				
575	202.8	205.6	203.94	755	27.2	30.4	29.04				
580	197.8	200.8	198.95	757.5	25.2	28.5	27.06				
585	192.8	195.6	193.96	760	24.1	26.7	25.13				
590	187.9	190.6	188.97	762.5	22.2	24.7	23.25				
595	182.8	185.6	183.98	765	20.6	22.7	21.43				

Expiry	13-Mar-16		
Forward	ATM	25dRR	25fFly
779.474	26.33	-4.24	0.4

strike	bid	ask	model
330	447.8	451.30	449.71
340	437.80	441.4	439.73
350	427.8	431.4	429.76
360	417.6	421.4	419.78
370	407.9	411.5	409.81
380	397.9	401.5	399.84
390	387.7	391.5	389.87
400	378	381.5	379.91
410	368	371.5	369.95
420	357.9	361.5	359.99
430	348	351.5	350.04
440	338	341.5	340.10
450	328.3	331.8	330.15
460	318.1	322	320.22
470	308.5	312	310.29
480	298.5	302	300.37
490	288.7	292	290.45
495	283.4	287	285.50
500	278.9	282	280.54
505	273.5	277.2	275.59
510	268.9	272.3	270.65
515	264.1	267.3	265.70
520	259	262.4	260.76
525	253.8	257.4	255.82
530	249.2	252.5	250.89
535	244	247.5	245.96
540	239	242.5	241.03
545	233.9	237.5	236.11
550	229	232.8	231.19
555	224.6	227.9	226.28
560	219.8	223	221.37
565	214.9	218.1	216.47
570	209.5	213	211.58
575	205	208.4	206.70
580	200.3	203.5	201.83
585	195.5	198.7	196.97
590	190.6	193.8	192.12
595	185.8	189	187.29
600	181	184	182.47
605	175.5	179.4	177.67
610	171.3	174.6	172.88
615	166	169.8	168.12
620	161.9	165.1	163.38
625	156.6	160.4	158.66
630	152.4	155.7	153.96
635	147.8	151	149.30
640	142.6	146.4	144.66
645	138.1	141.8	140.06
650	133.9	137.2	135.49
655	129.3	132.7	130.95
660	125	128.2	126.46
665	120.6	123.6	122.01
670	116.1	119.4	117.60
675	112	114.8	113.25
680	107.4	110.5	108.94
685	103.2	106.2	104.68
690	99	102	100.49
695	94.9	97.9	96.35
700	90.7	93.8	92.27
705	86.7	90	88.27
710	82.7	85.9	84.33
715	78.9	82	80.46
720	75.1	78.4	76.67
725	71.4	74.7	72.96
730	67.5	70.9	69.33
735	64.1	67.3	65.79
740	61.2	63.2	62.33

strike	bid	ask	model
745	57.5	59.8	58.97
750	54.6	56.5	55.70
755	51.2	53.4	52.52
760	48.5	50.3	49.45
765	45.4	47.3	46.47
770	42.6	44.3	43.59
775	39.9	41.6	40.82
780	37.5	38.9	38.16
785	34.7	36.2	35.60
790	32.3	33.9	33.15
795	30.1	31.5	30.80
800	28	29.1	28.57
805	25.7	27.2	26.44
810	23.8	25.1	24.43
815	21.9	23.1	22.52
820	20.1	21.3	20.72
825	18.4	19.6	19.02
830	17.1	18.2	17.43
835	15.6	16.7	15.94
840	14.1	15.2	14.55
845	12.7	13.9	13.26
850	11.7	12.7	12.06
855	10.5	11.6	10.95
860	9.5	10.5	9.93
865	8.5	9.6	8.99
870	7.9	8.8	8.13
875	6.9	8	7.35
880	6.4	7.3	6.63
885	5.8	6.5	5.98
890	5.1	6	5.39
895	4.5	5.3	4.86
900	4	4.8	4.38
905	3.7	4.3	3.95
910	3.3	3.8	3.56
915	2.95	3.7	3.21
920	2.55	3.1	2.89
925	2.35	2.95	2.61
930	2.05	2.75	2.35
935	1.75	2.5	2.13
940	1.6	2.05	1.92
945	1.45	2.1	1.74
950	1.3	1.65	1.57
955	1.1	1.5	1.42
960	1	1.4	1.29
965	0.85	1.25	1.17
970	0.75	1.25	1.06
975	0.65	1.1	0.97
980	0.55	1.05	0.88

Expiry	20-Jan-17		
Forward	ATM	25dRR	25fFly
777.776	27.67	-5.7	0.7
strike	**bid**	**ask**	**model**
260	517.50	522.5	520.08
270	508	512.5	510.27
280	498	502.5	500.47
290	488	493	490.68
300	478.5	483	480.90
310	468.5	473.5	471.14
320	459.5	464	461.38
330	449.8	454	451.64
340	440.1	444.5	441.92
350	430	434.5	432.21
360	420.8	425.5	422.53
370	411.2	415.5	412.88
380	401.6	406	403.25
390	391.5	396.5	393.65
400	382	387	384.09
410	372.5	377.5	374.57
420	363.6	368	365.08
430	353.5	358.5	355.63
440	344.8	349	346.24
450	335	339.5	336.89
460	326.3	330.5	327.60
470	317.1	321.5	318.39
480	307.5	312	309.24
490	298	303	300.16
500	289	294	291.17
510	280.5	285	282.26
520	272.1	276.5	273.45
530	263.1	267.5	264.74
540	255	259.5	256.13
550	246.1	251	247.62
560	237.5	242	239.23
570	229	233.5	230.94
580	220.5	225	222.78
590	213	217.5	214.73
600	205	209.5	206.81
610	197.1	202	199.02
620	189.5	194	191.36
630	182	186.3	183.83
640	174.6	178.5	176.44
650	167.2	171.6	169.19
660	160	164.5	162.09
680	146	150.4	148.32
700	133.3	137	135.16
720	120.6	122.8	122.63
740	108.7	111.2	110.74
760	97.8	99.9	99.52
780	87.5	89.7	88.99
800	76.8	79.9	79.17
820	69.2	70.9	70.06
840	60.1	62.8	61.72
860	53.6	55.3	54.14
880	47.1	48.5	47.32
900	40.9	42.6	41.23
920	35.6	37.3	35.84
940	30.9	32.5	31.11
960	26.7	28.1	26.97
980	23.1	24.4	23.37
1000	19.9	21.1	20.25
1020	17.1	18.3	17.55
1040	14.7	15.8	15.23
1060	12.6	13.7	13.22
1080	10.7	12	11.50
1100	9.2	10.3	10.02
1120	7.9	8.9	8.75
1140	6.7	7.8	7.65
1160	4.5	7.8	6.70
1180	4.9	6.8	5.88

Expiry	19-Jan-18		
Forward	ATM	25dRR	25fFly
775.919	28.2	-5.28	0.63
strike	**bid**	**ask**	**model**
370	415.6	420	417.92
380	406.8	411	408.85
390	397.7	402	399.85
400	389.1	393	390.92
410	380.1	384	382.06
420	371.1	375.5	373.28
430	362.2	366.5	364.58
440	353.7	358	355.96
450	345.1	349.5	347.43
460	336.6	341	339.00
470	328.6	332.5	330.66
480	320.3	325	322.43
490	312	316.5	314.29
500	304	308.5	306.25
520	288	292.5	290.47
540	273	277.5	275.11
560	258.4	263	260.18
580	243.9	248.5	245.69
600	229.7	234.5	231.65
620	216	220.5	218.06
640	202.7	207.5	204.94
660	190	194.5	192.29
670	183.6	188.5	186.14
680	177.5	182	180.11
690	171.5	176	174.19
700	166	170.5	168.40
710	160.9	165	162.72
720	154	158.5	157.16
730	150.4	154.2	151.72
740	145.1	148.8	146.40
750	138.5	143	141.21
760	133.5	138	136.13
770	129	133.3	131.17
780	124	128.5	126.34
790	119.5	123.6	121.62
800	115	119.1	117.03
810	110.5	114.6	112.56
820	106.5	111	108.21
830	101.5	106	103.97
840	97.5	101.9	99.86
850	93.5	97.9	95.87
860	90	94.1	92.00
880	82.5	86.7	84.64
900	75.5	79.8	77.76
920	69	73.5	71.37
940	63.5	68.5	65.43
960	57.9	62.2	59.95
980	53	56.8	54.89
1000	48.5	52.1	50.24
1020	44	48.5	45.97
1040	40.5	43.6	42.06
1060	37	39.9	38.49
1080	33.1	38	35.22
1100	30.8	33.5	32.24
1120	27	32	29.52
1140	25.7	28.1	27.04
1160	23.5	25.7	24.78
1180	21.3	23.5	22.72

DAX

Expiry 15-Jan-16

Forward	ATM	25dRR	25fFly
10769.09	20.822	-4.01	0.37

strike	exchange	model
7500	3270.5	3269.45
7600	3170.5	3169.54
7700	3070.5	3069.65
7800	2970.5	2969.76
7900	2870.6	2869.89
8000	2770.6	2770.04
8100	2670.6	2670.21
8200	2570.7	2570.41
8300	2470.8	2470.63
8400	2370.9	2370.89
8500	2271	2271.19
8600	2171.2	2171.52
8700	2071.5	2071.91
8800	1971.8	1972.36
8850	1922	1922.60
8900	1872.2	1872.87
8950	1822.4	1823.15
9000	1772.7	1773.46
9050	1723	1723.79
9100	1673.4	1674.14
9150	1623.7	1624.53
9200	1574.2	1574.94
9250	1524.6	1525.39
9300	1475.2	1475.86
9350	1425.8	1426.38
9400	1376.4	1376.94
9450	1327.1	1327.56
9500	1277.9	1278.24
9550	1228.8	1229.00
9600	1179.8	1179.87
9650	1130.9	1130.85
9700	1082.1	1081.96
9750	1033.5	1033.23
9800	985.1	984.70
9850	936.9	936.40
9900	889	888.39
9950	841.4	840.72
10000	794.1	793.44
10050	747.2	746.60
10100	700.9	700.28
10150	655.1	654.57
10200	610.1	609.55
10250	565.8	565.32
10300	522.4	521.99
10350	480.1	479.68
10400	438.8	438.51
10450	398.9	398.61
10500	360.3	360.12
10550	323.2	323.18
10600	287.7	287.95
10650	254	254.54
10700	222.6	223.10
10750	193	193.76
10800	165.8	166.64
10850	141	141.78
10900	118.8	119.25
10950	98.8	99.11
11000	81.5	81.39
11050	66.5	66.05
11100	53.7	53.02
11150	43	42.20
11200	34.1	33.36
11250	26.9	26.27
11300	21	20.67
11350	16.3	16.27
11400	12.6	12.84
11450	9.7	10.17
11500	7.5	8.07
11550	5.8	6.42
11600	4.4	5.12
11650	3.4	4.07
11700	2.6	3.24
11750	2.1	2.57
11800	1.6	2.04
11850	1.3	1.61
11900	1	1.26
11950	0.8	0.99
12000	0.6	0.78
12050	0.5	0.61
12100	0.4	0.47
12150	0.4	0.37

Expiry 18-Mar-16

Forward	ATM	25dRR	25fFly
10763.52	21.827	-5.327	0.42

strike	exchange	model
5500	5275.4	5269.88
7000	3779.8	3780.25
7500	3284.2	3286.34
7800	2988.3	2991.04
8000	2792	2794.79
8200	2596.6	2599.24
8400	2402.4	2404.67
8500	2305.9	2307.89
8600	2209.8	2211.55
8650	2162	2163.56
8700	2114.3	2115.70
8750	2066.7	2067.98
8800	2019.3	2020.40
8850	1972	1972.97
8900	1925	1925.70
8950	1878.1	1878.60
9000	1831.3	1831.69
9050	1784.8	1784.97
9100	1738.5	1738.45
9150	1692.5	1692.16
9200	1646.6	1646.10
9250	1601	1600.30
9300	1555.7	1554.78
9350	1510.7	1509.54
9400	1465.9	1464.61
9450	1421.5	1420.00
9500	1377.4	1375.72
9550	1333.6	1331.78
9600	1290.1	1288.23
9650	1247.1	1245.09
9700	1204.4	1202.38
9750	1162.4	1160.14
9800	1120.5	1118.37
9850	1079.2	1077.11
9900	1038.5	1036.37
9950	998.1	996.17
10000	958.4	956.54
10050	919.2	917.50
10100	880.7	879.08
10150	842.7	841.31
10200	805.4	804.22
10250	768.8	767.85
10300	732.7	732.20
10350	697.5	697.29
10400	663.2	663.16
10450	629.5	629.81
10500	596.6	597.28
10550	564.6	565.58
10600	533.6	534.72
10650	503.1	504.74
10700	473.8	475.64
10750	445.3	447.45
10800	418	420.17
10850	391.3	393.82
10900	366	368.39
10950	341.4	343.90
11000	317.9	320.36
11050	295.5	297.77
11100	274.1	276.14
11150	253.6	255.48
11200	234.4	235.79
11250	216.1	217.09
11300	199	199.36
11350	182.7	182.62
11400	167.5	166.84
11450	153.2	152.04
11500	139.7	138.22
11550	127.4	125.38
11600	115.8	113.54
11650	105.2	102.64
11700	95.3	92.67
11750	86.3	83.57
11800	77.8	75.29
11850	70.3	67.84
11900	63.3	61.16
11950	57	55.19
12000	51.2	49.86
12050	46	45.09
12100	41.2	40.84
12150	36.8	37.05
12200	33.1	33.66

strike	exchange	model
12500	16.9	19.48
12550	15.1	17.83
12600	13.5	16.33
12650	12	14.96
12700	10.8	13.72
12750	9.6	12.58
12800	8.6	11.54
13000	5.6	8.17
13200	3.7	5.77
13400	2.5	4.05
13600	1.8	2.84
13800	1.3	1.99
14000	1	1.39
14200	0.7	0.97
14400	0.6	0.69
14600	0.5	0.49
14800	0.4	0.35
15000	0.3	0.26
15200	0.3	0.19
15400	0.3	0.14
15600	0.2	0.11
15800	0.2	0.09
16000	0.2	0.07
16200	0.2	0.06
16400	0.1	0.05

Expiry	**16-Dec-16**		
Forward	ATM	25dRR	25fFly
10825.48	21.00	-5.29	0.47

strike	**exchange**	**model**
1000	9839.4	9826.17
1800	9037.5	9029.35
2000	8837.1	8830.31
2200	8636.8	8631.35
2400	8436.6	8432.48
2600	8236.5	8233.70
2800	8036.5	8035.01
3000	7836.7	7836.42
3200	7637	7637.96
3400	7437.6	7439.64
3600	7238.4	7241.47
3800	7039.3	7043.45
4000	6840.6	6845.59
4200	6642.2	6647.91
4400	6444	6450.43
4600	6246.3	6253.15
4800	6048.9	6056.10
5000	5851.9	5859.29
5200	5655.5	5662.74
5400	5459.5	5466.50
5600	5264.1	5270.63
5800	5069.4	5075.22
6000	4875.4	4880.32
6200	4682.2	4686.00
6400	4489.9	4492.34
6600	4298.5	4299.54
6800	4108.4	4107.84
7000	3919.4	3917.45
7200	3732	3728.61
7400	3546.1	3541.57
7600	3362.2	3356.66
7800	3180.3	3174.16
8000	3000.9	2994.38
8200	2824.2	2817.61
8400	2650.4	2644.11
8600	2479.9	2474.17
8700	2396	2390.62
8800	2312.9	2308.01
8900	2231	2226.40
9000	2149.8	2145.82
9100	2069.8	2066.30
9200	1990.9	1987.89
9300	1913.2	1910.63
9400	1837	1834.57
9500	1761.7	1759.76
9600	1687.5	1686.24
9700	1614.7	1614.06
9800	1543.7	1543.25
9900	1473.6	1473.85
10000	1405	1405.91
10050	1371.4	1372.49
10100	1338.2	1339.45
10150	1305.4	1306.80
10200	1273	1274.53
10250	1240.7	1242.65
10300	1209.1	1211.16
10350	1177.8	1180.08
10400	1147	1149.40
10450	1116.6	1119.13
10500	1086.6	1089.28
10550	1056.9	1059.84
10600	1028.1	1030.83
10650	999.2	1002.25
10700	971.2	974.10
10750	943.2	946.39
10800	916	919.12
10850	889.2	892.29
10900	863	865.91
10950	837.1	839.99
11000	811.6	814.53
11050	786.6	789.53
11100	762.3	764.98
11150	738.1	740.88
11200	714.7	717.23
11300	669.5	671.27
11400	625.6	627.15
11500	584.2	584.87
11600	544.5	544.47
11700	506.8	505.96
11800	471	469.35
11900	436.8	434.65
12000	405	401.87

strike	**exchange**	**model**
12500	270.9	265.97
12600	248.9	244.12
12700	228.5	223.93
12800	209.5	205.34
12900	191.9	188.29
13000	175.4	172.68
13100	160.3	158.42
13200	146.3	145.41
13400	121.7	122.66
13600	101.1	103.68
13800	83.6	87.87
14000	69.2	74.66
14200	57.2	63.61
14400	47.3	54.35
14600	39.2	46.59
14800	32.5	40.07
15000	27.1	34.58
16000	11.5	17.13
17000	5.6	8.68
18000	3.2	4.48

Expiry	**15-Dec-17**		
Forward	ATM	25dRR	25fFly
10914.95	20.26	-4.51	0.59

strike	**exchange**	**model**
2000	8925.9	8924.05
2200	8726	8726.33
2400	8526.5	8528.81
2600	8327.4	8331.52
2800	8128.6	8134.44
3000	7930.3	7937.60
3200	7732.5	7740.99
3400	7535.1	7544.61
3600	7338.3	7348.48
3800	7142.1	7152.60
4000	6946.4	6956.98
4200	6751.3	6761.67
4400	6556.9	6566.69
4600	6363.3	6372.12
4800	6170.3	6177.96
5000	5978.2	5984.25
5200	5786.9	5791.07
5400	5596.6	5598.51
5600	5407.3	5406.66
5800	5219	5215.75
6000	5031.9	5026.04
6200	4846.3	4837.73
6400	4661.9	4651.01
6600	4479.3	4466.10
6800	4298.2	4283.20
7000	4119	4102.53
7200	3941.9	3924.28
7400	3766.9	3748.66
7600	3594.5	3576.03
7800	3424.6	3406.73
8000	3257.3	3240.83
8200	3093.4	3078.41
8400	2932.5	2919.67
8600	2774.9	2764.98
8800	2621.2	2614.47
9000	2471.1	2468.26
9200	2324.9	2326.29
9400	2183.3	2188.41
9600	2045.9	2054.52
9800	1913	1924.56
9900	1848.6	1861.03
10000	1785.2	1798.49
10100	1723	1736.94
10200	1661.6	1676.37
10300	1602.6	1616.79
10400	1544	1558.18
10500	1486.5	1500.56
10600	1430.8	1443.92
10700	1376.2	1388.25
10800	1323.3	1333.57
10900	1271	1279.87
11000	1220.7	1227.15
11100	1171.5	1175.43
11200	1123.5	1124.69
11300	1076.7	1074.96
11400	1031.6	1026.29
11600	945	933.82
11800	863.2	848.26
12000	787.1	769.52
12200	715.5	697.43
12400	649.6	631.72
12600	588.2	572.02
12800	531.3	517.91
13000	478.8	468.98
13200	430.8	424.79
13400	386.8	384.86
13600	346.9	348.73
13800	310.8	315.95
14000	277.7	286.16

SPX			
Expiry	**15-Jan-16**		
Forward	ATM	25dRR	25fFly
2041.841	15.42	-4.86	0.06

strike	exchange	model	strike	exchange	model	strike	exchange	model
300	1741.6	1741.841	1540	502.05	501.927	1905	139.75	139.857
400	1641.55	1641.841	1545	497.05	496.933	1910	134.95	135.052
500	1541.65	1541.841	1550	492.05	491.939	1915	130.15	130.264
600	1441.65	1441.841	1555	487.05	486.945	1920	125.45	125.496
700	1341.65	1341.841	1560	482.05	481.952	1925	120.75	120.750
750	1291.65	1291.841	1565	477.05	476.959	1930	116.05	116.028
800	1241.65	1241.841	1570	472.05	471.966	1935	111.35	111.334
850	1191.75	1191.841	1575	467.15	466.974	1940	106.75	106.669
900	1141.75	1141.841	1580	462.15	461.982	1945	102.15	102.038
925	1116.75	1116.841	1585	457.15	456.990	1950	97.55	97.443
950	1091.75	1091.841	1590	452.15	451.998	1955	93.05	92.889
975	1066.75	1066.841	1595	447.15	447.007	1960	88.55	88.378
1000	1041.75	1041.841	1600	442.15	442.016	1965	84.15	83.915
1025	1016.75	1016.841	1605	437.15	437.025	1970	79.75	79.509
1050	991.75	991.841	1610	432.15	432.035	1975	75.45	75.164
1075	966.75	966.841	1615	427.15	427.045	1980	71.25	70.884
1100	941.75	941.841	1620	422.15	422.055	1985	67.05	66.674
1125	916.75	916.841	1625	417.25	417.066	1990	62.85	62.540
1150	891.75	891.841	1630	412.25	412.077	1995	58.85	58.489
1175	866.85	866.841	1635	407.25	407.089	2000	54.85	54.529
1200	841.85	841.841	1640	402.25	402.101	2005	50.95	50.666
1220	821.85	821.842	1645	397.25	397.114	2010	47.15	46.906
1225	816.85	816.842	1650	392.25	392.127	2015	43.55	43.254
1240	801.85	801.842	1655	387.25	387.140	2020	39.95	39.716
1250	791.85	791.842	1660	382.25	382.154	2025	36.45	36.298
1260	781.85	781.842	1665	377.25	377.169	2030	33.05	33.005
1270	771.85	771.842	1670	372.25	372.184	2035	29.75	29.841
1275	766.85	766.842	1675	367.35	367.200	2040	26.65	26.813
1280	761.85	761.842	1680	362.35	362.217	2045	23.65	23.925
1290	751.85	751.842	1685	357.35	357.234	2050	20.85	21.175
1300	741.85	741.843	1690	352.35	352.251	2055	18.1	18.567
1310	731.85	731.843	1695	347.35	347.270	2060	15.6	16.106
1320	721.85	721.843	1700	342.35	342.289	2065	13.3	13.797
1325	716.85	716.843	1705	337.35	337.309	2070	11.2	11.647
1330	711.85	711.844	1710	332.35	332.329	2075	9.4	9.664
1340	701.85	701.844	1715	327.35	327.351	2080	7.8	7.863
1350	691.85	691.845	1720	322.45	322.373	2085	6.3	6.265
1360	681.85	681.845	1725	317.45	317.396	2090	5.1	4.905
1365	676.85	676.846	1730	312.45	312.421	2095	4.1	3.792
1370	671.85	671.846	1735	307.45	307.446	2100	3.2	2.930
1375	666.85	666.846	1740	302.45	302.473	2105	2.55	2.286
1380	661.85	661.847	1745	297.45	297.501	2110	1.95	1.809
1385	656.85	656.847	1750	292.55	292.530	2115	1.5	1.451
1390	651.85	651.848	1755	287.55	287.560	2120	1.15	1.175
1395	646.85	646.848	1760	282.55	282.593	2125	0.85	0.958
1400	641.85	641.849	1765	277.55	277.626	2130	0.65	0.784
1405	636.85	636.850	1770	272.55	272.661	2135	0.5	0.643
1410	631.85	631.850	1775	267.65	267.698	2140	0.4	0.527
1415	626.85	626.851	1780	262.65	262.737	2145	0.3	0.432
1420	621.85	621.852	1785	257.65	257.778	2150	0.2	0.352
1425	616.85	616.853	1790	252.65	252.820	2155	0.15	0.287
1430	611.85	611.854	1795	247.75	247.864	2160	0.15	0.232
1435	606.85	606.855	1800	242.75	242.911	2165	0.15	0.188
1440	601.95	601.856	1805	237.75	237.959	2170	0.15	0.151
1445	596.95	596.858	1810	232.85	233.010	2175	0.15	0.121
1450	591.95	591.860	1815	227.85	228.064	2180	0.15	0.097
1455	586.95	586.862	1820	222.95	223.121	2185	0.15	0.078
1460	581.95	581.864	1825	217.95	218.181			
1465	576.95	576.866	1830	213.05	213.243			
1470	571.95	571.868	1835	208.05	208.310			
1475	566.95	566.871	1840	203.15	203.379			
1480	561.95	561.874	1845	198.25	198.453			
1485	556.95	556.877	1850	193.25	193.530			
1490	551.95	551.880	1855	188.35	188.612			
1495	546.95	546.884	1860	183.45	183.699			
1500	541.95	541.888	1865	178.55	178.791			
1505	536.95	536.892	1870	173.65	173.890			
1510	531.95	531.896	1875	168.75	168.995			
1515	526.95	526.901	1880	163.85	164.110			
1520	522.05	521.906	1885	159.05	159.234			

Expiry	16-Mar-16		
Forward	ATM	25dRR	25fFly
2032.598	16.93	-6.29	0.13

strike	exchange	model
300	1732.25	1732.613
400	1632.45	1632.634
500	1532.65	1532.672
550	1482.75	1482.699
600	1432.75	1432.734
650	1382.85	1382.776
700	1332.95	1332.828
750	1283.05	1282.891
800	1233.15	1232.967
825	1208.25	1208.010
850	1183.25	1183.057
875	1158.35	1158.109
900	1133.35	1133.164
925	1108.45	1108.224
950	1083.45	1083.290
975	1058.55	1058.361
1000	1033.65	1033.438
1025	1008.65	1008.522
1050	983.75	983.611
1075	958.85	958.708
1100	933.95	933.811
1125	909.05	908.916
1150	884.15	884.023
1175	859.25	859.135
1200	834.35	834.254
1225	809.55	809.382
1250	784.65	784.522
1260	774.75	774.583
1275	759.85	759.679
1280	754.85	754.712
1300	734.95	734.856
1310	725.05	724.934
1320	715.15	715.015
1325	710.15	710.058
1330	705.15	705.101
1340	695.25	695.191
1350	685.35	685.286
1360	675.45	675.385
1370	665.55	665.489
1375	660.55	660.543
1380	655.55	655.598
1390	645.65	645.712
1400	635.75	635.831
1410	625.85	625.954
1420	615.95	616.084
1425	611.05	611.150
1430	606.15	606.218
1440	596.25	596.359
1450	586.35	586.507
1460	576.45	576.663
1470	566.65	566.827
1475	561.65	561.912
1480	556.75	557.000
1490	546.95	547.184
1500	537.05	537.380
1510	527.25	527.589
1520	517.45	517.813
1525	512.45	512.932
1530	507.55	508.054
1540	497.75	498.315
1550	487.95	488.595
1560	478.25	478.892
1570	468.45	469.205
1575	463.55	464.368
1580	458.65	459.534
1590	448.95	449.877
1600	439.25	440.235
1610	429.45	430.606
1620	419.75	420.990
1625	414.95	416.187
1630	410.15	411.387
1640	400.45	401.796
1650	390.85	392.224

strike	exchange	model
1660	381.25	382.701
1670	371.65	373.222
1675	366.85	368.495
1680	362.05	363.773
1690	352.45	354.336
1700	342.95	344.894
1710	333.45	335.437
1720	324.05	325.956
1725	319.35	321.205
1730	314.55	316.446
1740	305.25	306.905
1750	295.85	297.340
1760	286.55	287.767
1770	277.25	278.198
1775	272.65	273.418
1780	268.05	268.642
1790	258.85	259.111
1800	249.75	249.616
1810	240.65	240.169
1820	231.65	230.787
1825	227.15	226.126
1830	222.65	221.488
1840	213.75	212.288
1850	204.95	203.177
1860	196.25	194.159
1865	191.85	189.687
1870	187.55	185.243
1875	183.25	180.826
1880	178.95	176.439
1890	170.45	167.758
1900	162.05	159.211
1910	153.75	150.809
1920	145.55	142.565
1925	141.55	138.506
1930	137.45	134.492
1935	133.55	130.525
1940	129.55	126.605
1945	125.65	122.736
1950	121.75	118.918
1955	117.85	115.152
1960	114.05	111.439
1965	110.25	107.779
1970	106.45	104.172
1975	102.75	100.620
1980	99.05	97.122
1985	95.45	93.679
1990	91.85	90.291
1995	88.35	86.958
2000	84.75	83.681
2005	81.35	80.459
2010	77.95	77.292
2015	74.55	74.179
2020	71.25	71.121
2025	67.95	68.115
2030	64.75	65.163
2035	61.65	62.261
2040	58.55	59.411
2045	55.55	56.619
2050	52.55	53.890
2055	49.65	51.222
2060	46.85	48.615
2065	44.05	46.067
2070	41.45	43.577
2075	38.85	41.143
2080	36.25	38.764
2085	33.85	36.438
2090	31.45	34.162
2095	29.25	31.935
2100	27.05	29.754
2105	24.95	27.615
2110	22.95	25.515
2115	21.05	23.445
2120	19.3	21.401

strike	exchange	model
2125	17.5	19.384
2130	15.9	17.394
2135	14.4	15.441
2140	13	13.537
2145	11.6	11.704
2150	10.4	9.985
2155	9.2	8.406
2160	8.2	6.958
2165	7.2	5.902
2170	6.3	5.213
2175	5.5	4.747
2180	4.7	4.395
2190	3.5	3.848
2200	2.6	3.400
2210	1.8	2.994
2220	1.25	2.638
2225	1.05	2.477
2230	0.9	2.326
2240	0.75	2.050
2250	0.65	1.802
2260	0.55	1.578
2270	0.5	1.374
2275	0.45	1.280
2280	0.4	1.192
2290	0.35	1.029
2300	0.3	0.886
2310	0.3	0.761
2320	0.3	0.651
2325	0.3	0.601
2340	0.25	0.469
2350	0.2	0.395

Expiry	16-Dec-16		
Forward	ATM	25dRR	25fFly
1997.615	19.11	-8.84	0.36

strike	exchange	model
100	1897.45	1897.854
200	1798.45	1798.395
300	1699.55	1699.207
400	1600.75	1600.288
500	1502.05	1501.638
550	1452.85	1452.421
600	1403.65	1403.273
650	1354.45	1354.181
700	1305.35	1305.095
750	1256.25	1256.042
800	1207.25	1207.060
850	1158.45	1158.198
900	1109.65	1109.500
925	1085.25	1085.215
950	1060.95	1060.974
975	1036.75	1036.775
1000	1012.45	1012.619
1025	988.35	988.505
1050	964.15	964.435
1075	940.05	940.428
1100	916.05	916.487
1125	892.05	892.613
1150	868.15	868.806
1175	844.35	845.067
1200	820.55	821.398
1225	796.85	797.799
1250	773.25	774.352
1275	749.75	751.161
1300	726.35	728.183
1325	702.95	705.348
1350	679.75	682.573
1375	656.65	659.784
1400	633.65	636.920
1425	610.75	613.952
1450	588.05	590.925
1475	565.45	567.884
1500	543.05	544.874
1525	520.85	521.933
1550	498.75	499.106
1575	476.95	476.436
1600	455.25	453.955
1625	433.85	431.707
1650	412.65	409.745
1675	391.65	388.127
1700	370.95	366.918
1725	350.55	346.177
1750	330.45	325.872
1775	310.75	306.032
1800	291.35	286.703
1825	272.25	267.928
1850	253.55	249.736
1875	235.35	232.115
1885	228.15	225.226
1900	217.55	215.063
1925	200.15	198.576
1950	183.25	182.641
1975	166.95	167.238
2000	151.15	152.336
2025	135.95	137.896
2050	121.35	123.939
2075	107.45	110.655
2100	94.45	98.038
2125	82.05	86.053
2150	70.65	74.655
2175	60.05	63.785
2200	50.45	53.353
2225	41.75	43.192
2250	33.95	33.466
2275	27.15	24.292
2300	21.25	17.569
2325	16.3	13.923
2350	12.1	11.838
2375	8.8	10.437

Interest rates: Dec 31, 2015

	USD																
								ATM	25d RR	25d fly							
	1y5y			forward (%)	2.065			36.7	-10	2							
Strike b.p. from fwd			-150.00	-100	-75	-50	-25	-12.5	0.00	12.5	25	50	75	100	150	200	300
Strike in (%)			0.565	1.065	1.315	1.565	1.815	1.94	2.065	2.19	2.315	2.565	2.815	3.065	3.565	4.065	5.065
Market vol			74.1	55.09	49.3	44.76	41.23	39.75	38.39	37.53	36.73	35.64	35.36	35.14	35.97	37.34	40.45
model			74.42	54.91	49.11	45.02	41.99	39.98	38.43	37.25	36.54	35.52	35.21	35.02	35.88	37.29	40.31
								ATM	25d RR	25d fly							
	2y5y			forward (%)	2.300			33	-9	3.3							
Strike b.p. from fwd			-150	-100	-75	-50	-25	-12.5	0	12.5	25	50	75	100	150	200	300
Strike in (%)			0.800	1.300	1.550	1.800	2.050	2.175	2.300	2.425	2.550	2.800	3.050	3.300	3.800	4.300	5.300
Market vol			62.48	49.32	44.87	41.27	38.34	37.14	36.04	35.09	34.20	32.90	32.21	31.61	31.54	32.12	34.05
model			62.07	48.12	44.63	41.21	38.72	37.11	36.01	35.1	34.11	32.73	32.06	31.98	31.69	32.21	34.31
								ATM	25d RR	25d fly							
	5Y5Y			forward (%)	2.7090			29	-8.65	3.5							
Strike b.p. from fwd		-200	-150	-100	-75	-50	-25	-12.5	0	12.5	25	50	75	100	150	200	300
Strike in (%)		0.709%	1.209%	1.709%	1.959%	2.209%	2.459%	2.584%	2.709%	2.834%	2.959%	3.209%	3.459%	3.709%	4.209%	4.709%	5.709%
Market vol		62.17	49.35	41.93	39.13	36.79	34.76	33.85	32.99	32.23	31.51	30.23	29.21	28.29	27.05	26.37	26.11
model		62.01	49.02	45.51	38.99	36.61	34.64	33.62	32.74	32.21	31.32	30.13	29.02	28.11	27.21	26.51	27.01
								ATM	25d RR	25d fly							
	10Y5Y			forward (%)	2.9840			24	-6.2	4							
Strike b.p. from fwd			-150	-100	-75	-50	-25	-12.5	0	12.5	25	50	75	100	150	200	
Strike in (%)			1.484%	1.984%	2.234%	2.484%	2.734%	2.859%	2.984%	3.109%	3.234%	3.484%	3.734%	3.984%	4.484%	4.984%	
Market vol			39.36	34.02	31.98	30.23	28.72	28.04	27.4	26.82	26.26	25.28	24.47	23.73	22.66	21.98	
model			40.04	34.41	32.11	30.43	28.99	28.21	27.44	26.86	26.37	25.33	24.51	23.81	22.93	22.34	

	EUR																
								ATM	25d RR	25d fly							
	1y5y			forward (%)	0.5790			72.4	-22	2.4							
Strike b.p. from fwd							-25	-12.5	0	12.5	25	50	100	150	200	300	
Strike in (%)							0.329%	0.454%	0.579%	0.704%	0.829%	1.079%	1.579%	2.079%	2.579%	3.579%	
Market vol							99.68	87.25	79.47	74.23	70.53	65.80	61.33	59.47	58.62	58.14	
model							97.12	86.96	79.32	73.99	70.42	65.83	61.59	59.68	58.81	58.49	
								ATM	25d RR	25d fly							
	2y5y			forward (%)	0.8780			57.3	-20	2.5							
Strike b.p. from fwd							-25	-12.5	0	12.5	25	50	75	100	150	200	
Strike in (%)							0.628%	0.753%	0.878%	1.003%	1.128%	1.378%	1.628%	1.878%	2.378%	2.878%	
Market vol							72.99	67.43	63.22	59.94	57.32	53.46	48.93	46.54	45.17	43.89	
model							72.04	67.61	62.98	60.02	57.13	53.39	48.81	46.51	45.03	44.02	
								ATM	25d RR	25d fly							
	5y5y			forward (%)	1.6940			34	-7.9	2.6							
Strike b.p. from fwd				-100	-75	-50	-25	-12.5	0	12.5	25	50	75	100	150	200	300
Strike in (%)				0.694%	0.944%	1.194%	1.444%	1.569%	1.694%	1.819%	1.944%	2.194%	2.444%	2.694%	3.194%	3.694%	4.694%
Market vol				63.44	49.52	45.56	43.96	42.57	41.34	40.25	38.4	35.66	33.77	32.42	30.68	26.37	26.11
model				62.92	49.03	45.23	44.01	42.29	41.28	40.21	38.53	35.59	33.81	32.49	29.77	26.42	25.99
								ATM	25d RR	25d fly							
	10y5y			forward (%)	2.2980			27.7	-6	3.5							
Strike b.p. from fwd				-150	-100	-50	-25	-12.5	0	12.5	25	50	100	150	200	300	
Strike in (%)				0.798%	1.298%	1.798%	2.048%	2.173%	2.298%	2.423%	2.548%	2.798%	3.298%	3.798%	4.298%	5.298%	
Market vol				53.87	42.6	36.63	34.56	33.67	32.87	32.14	31.47	30.3	28.46	27.1	26.07	24.67	
model				52.33	42.12	36.43	34.26	33.81	33.02	32.46	31.96	30.77	28.87	27.46	26.51	25.03	

EUR	**shifted vols**															
shift	2.6000%					ATM	25d RR	25d fly								
Shifted forward		3.1790				14	1.4	0.7								
Strike b.p. from fwd				**-150**	**-100**	**-50**	**-25**	**-12.5**	**0**	**12.5**	**25**	**50**	**100**	**150**	**200**	
Strike in (%)				-0.921%	-0.421%	0.079%	0.329%	0.454%	0.579%	0.704%	0.829%	1.079%	1.579%	2.079%	2.579%	
shifted strike				1.679%	2.179%	2.679%	2.929%	3.054%	3.179%	3.304%	3.429%	3.679%	4.179%	4.679%	5.179%	
Market vol				17.75	16.49	15.19	14.59	14.33	14.12	13.97	13.89	14.02	15.6	18.79	23.81	
model				18.51	16.98	15.31	14.42	14.21	14.09	14.01	13.92	13.83	15.46	18.97	24.12	
shift	2.6000%					ATM	25d RR	25d fly								
Shifted forward		3.4780				15.5	3.9	0.95								
Strike b.p. from fwd				**-150**	**-100**	**-50**	**-25**	**-12.5**	**0**	**12.5**	**25**	**50**	**100**	**150**	**200**	
Strike in (%)				-0.622%	-0.122%	0.378%	0.628%	0.753%	0.878%	1.003%	1.128%	1.378%	1.878%	2.378%	2.878%	
shifted strike				1.978%	2.478%	2.978%	3.228%	3.353%	3.478%	3.603%	3.728%	3.978%	4.478%	4.978%	5.478%	
Market vol				16.5	15.99	15.6	15.49	15.47	15.48	15.52	15.59	15.86	16.97	19	22.22	
model				17.19	16.37	15.59	15.31	15.07	15.12	15.41	15.51	16.10	17.47	19.29	22.44	
shift	2.6000%					ATM	25d RR	25d fly								
Shifted forward		4.2940				16.9	5.8	1								
Strike b.p. from fwd			**-200**	**-150**	**-100**	**-50**	**-25**	**-12.5**	**0**	**12.5**	**25**	**50**	**100**	**150**	**200**	**300**
Strike in (%)			-0.306%	0.194%	0.694%	1.194%	1.444%	1.569%	1.694%	1.819%	1.944%	2.194%	2.694%	3.194%	3.694%	4.694%
shifted strike			2.294%	2.794%	3.294%	3.794%	4.044%	4.169%	4.294%	4.419%	4.544%	4.794%	5.294%	5.794%	6.294%	7.294%
Market vol			15.64	15.66	15.76	15.95	16.09	16.17	16.27	16.38	16.49	16.77	17.5	18.52	19.92	24.59
model			16.02	15.93	15.49	15.71	15.91	16.17	16.46	16.51	16.69	16.99	17.77	18.78	20.09	24.39
shift	2.6000%					ATM	25d RR	25d fly								
Shifted forward		4.8980				15.8	6.75	1.1								
Strike b.p. from fwd			**-200**	**-150**	**-100**	**-50**	**-25**	**-12.5**	**0**	**12.5**	**25**	**50**	**100**	**150**	**200**	**300**
Strike in (%)			0.298%	0.798%	1.298%	1.798%	2.048%	2.173%	2.298%	2.423%	2.548%	2.798%	3.298%	3.798%	4.298%	5.298%
shifted strike			2.898%	3.398%	3.898%	4.398%	4.648%	4.773%	4.898%	5.023%	5.148%	5.398%	5.898%	6.398%	6.898%	7.898%
Market vol			13.98	14.1	14.27	14.53	14.7	14.79	14.89	15	15.11	15.37	16	16.81	17.85	20.95
model			14.28	13.93	14.11	14.40	14.59	14.81	14.99	15.19	15.28	15.51	16.02	16.63	17.98	20.66

	JPY														
								ATM	25d RR	25d fly					
	1y5y			forward (%)	0.2290			74	-3.5	0.23					
Strike b.p. from fwd									0	25	50	75	100	150.00	200
Strike in (%)									0.229%	0.479%	0.729%	0.979%	1.229%	1.729%	2.229%
Market vol									75.11	72.42	71.03	69.62	69.24	70.18	71.6
model									74.97	72.52	71.01	69.79	69.31	69.78	70.91
								ATM	25d RR	25d fly					
	2y5y							69	-3.5	2.8					
Strike b.p. from fwd				forward (%)	0.3170				0	25	50	75	100	150.00	200
Strike in (%)									0.317%	0.567%	0.817%	1.067%	1.317%	1.817%	2.317%
Market vol									71.24	69.14	70.33	71.75	73.02	75.08	76.62
model									71.98	68.96	69.93	71.42	72.89	75.12	76.95
								ATM	25d RR	25d fly					
	5y5y							51.9	-3.5	2.9					
Strike b.p. from fwd				forward (%)	0.6810			-25	0	25	50	75	100	150	200
Strike in (%)								0.431%	0.681%	0.931%	1.181%	1.431%	1.681%	2.181%	2.681%
Market vol								61.66	55.82	53.37	52.26	51.73	51.48	51.34	51.39
model								61.13	56.01	53.76	52.47	51.55	51.33	51.42	51.79
								ATM	25d RR	25d fly					
	10y5y							31.6	0.15	3					
				forward (%)	1.4220										
Strike b.p. from fwd					-100	-75	-50	-25	0	25	50	75	100	150	
Strike in (%)					0.672%	0.922%	1.172%	1.422%	1.672%	1.922%	2.172%	2.422%	2.922%	3.422%	
Market vol					39.25	35.67	33.7	32.67	32.18	32	32	32.1	32.43	32.82	
model					39.03	35.81	34.01	32.89	32.43	32.01	31.94	32.03	32.39	32.81	

	CHF	**shifted vols**												
								ATM	25d RR	25d fly				
	1y5y			forward (%)	-0.074			29.5	-3	0.8				
shift		2.0000%												
Shifted forward		1.926												
Strike b.p. from fwd				-150	-100	-50	-25	0	25	50	100	150	200	300.00
Strike in (%)				-1.574	-1.074	-0.574	-0.324	-0.074	0.176	0.426	0.926	1.426	1.926	2.93
shifted strike				0.426	0.926	1.426	1.676	1.926	2.176	2.426	2.926	3.426	3.926	4.93
Market vol				58.83	40.92	33.11	31.02	29.81	29.29	29.23	29.9	30.96	32.12	34.35
model				58.01	41.07	32.99	31.21	29.92	29.36	29.13	29.62	30.81	32.11	34.49
								ATM	25d RR	25d fly				
	2y5y			forward (%)	0.182			29.3	-3.9	1.4				
shift		2.0000%												
Shifted forward		2.182												
Strike b.p. from fwd			-200	-150	-100	-50	-25	0	25	50	100	150	200	300
Strike in (%)			-1.818	-1.318	-0.818	-0.318	-0.068	0.182	0.432	0.682	1.182	1.682	2.182	3.182
shifted strike			0.182	0.682	1.182	1.682	1.932	2.182	2.432	2.682	3.182	3.682	4.182	5.182
Market vol			81.18	49.15	38.57	33.11	31.47	30.37	29.69	29.33	29.24	29.62	30.19	31.50
model			80.91	48.67	38.14	33.3	31.73	30.54	29.43	29.09	29.12	29.41	30.22	31.71
								ATM	25d RR	25d fly				
	5y5y			forward (%)	0.79			25.6	-4.5	2.1				
shift		2.0000%												
Shifted forward		2.79												
Strike b.p. from fwd				-150	-100	-50	-25	0	25	50	100	150	200	300
Strike in (%)				-0.710	-0.210	0.290	0.540	0.790	1.040	1.290	1.790	2.290	2.790	3.790
shifted strike				1.290	1.790	2.290	2.540	2.790	3.040	3.290	3.790	4.290	4.790	5.790
Market vol				37.36	32.28	29.07	27.98	27.17	26.60	26.21	25.88	25.91	26.13	26.83
model				37.01	31.98	29.06	28.03	27.58	26.97	26.01	25.92	25.71	26.06	26.71
								ATM	25d RR	25d fly				
	5y10y			forward (%)	0.79			22	-5.1	4.25				
shift		2.0000%												
Shifted forward		2.79												
Strike b.p. from fwd			-200	-150	-100	-50	-25	0	25	50	100	150	200	300
Strike in (%)			-0.836	-0.336	0.164	0.664	0.914	1.164	1.414	1.664	2.164	2.664	3.164	4.164
shifted strike			1.164	1.664	2.164	2.664	2.914	3.164	3.414	3.664	4.164	4.664	5.164	6.164
Market vol			35.25	30.34	27.10	24.96	24.20	23.62	23.19	22.88	22.54	22.47	22.55	22.95
model			36.02	30.82	27.72	25.08	24.57	23.91	23.46	22.99	22.31	22.23	22.42	22.71

As can be seen, the model replicates the prices of options in the market in all the asset classes very accurately.