

On the skew and curvature of implied and local volatilities

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Abstract

In this paper, we study the relationship between the short-end of the local and the implied volatility surfaces. Our results, based on Malliavin calculus techniques, recover the recent $\frac{1}{H+3/2}$ rule (where H denotes the Hurst parameter of the volatility process) for rough volatilities (see Bourgey, De Marco, Friz, and Pigato (2022)), that states that the short-time skew slope of the at-the-money implied volatility is $\frac{1}{H+3/2}$ the corresponding slope for local volatilities. Moreover, we see that the at-the-money short-end curvature of the implied volatility can be written in terms of the short-end skew and curvature of the local volatility and viceversa, and that this relationship depends on H .

Key words. Stochastic volatility; local volatility; rough volatility; Malliavin calculus

AMS subject classification. 60G44, 60H07, 91G20

1 Introduction

Local volatilities are a main tool in real market practice (see Dupire (1994)), since they are the simplest models that capture the empirical implied volatility surface. They are an example of mimicking process (see Gyöngy (1986)), in the sense that they are one-dimensional models that can reproduce the marginal distributions of asset prices S_t . In a local volatility model, the volatility process is a deterministic function $\sigma(t, S_t)$ of time and the underlying asset price. The values of this function can be computed via Dupire's formula (see again Dupire (1994)). The plot of this function σ , is called the local volatility surface.

One challenging problem in this context is the study of the relationship between implied and local volatilities. Even when both surfaces are similar, we can easily notice that short-end local volatility smiles are more pronounced than implied volatility smiles. In fact, some

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empirical studies (see Derman, Kani, and Zou (1996)) state that, for short and intermediate maturities, the ATM implied volatility skew is approximately half the skew of the local volatility (a property that is known as the *one-half rule*).

There have been many attempts to address this phenomena from the analytical point of view. Classical proofs of this property for stochastic volatility models can be found in the literature. For example, in Derman, Kani, and Zou (1996) or in Gatheral (2006), this property is deduced from the expression of implied volatilities as averaged local volatilities. In Lee (2001), the expansion of implied and local volatility allow to proof this property by a direct comparison. In Alòs and García-Lorite (2021), Malliavin calculus techniques give a representation of the short-limit at-the-money (ATM) implied volatility skew as an averaged local volatility skew, from where the one-half rule follows directly.

Nevertheless, recent studies (see Bourgey, De Marco, Friz, and Pigato (2022)) state that the one-half rule is not true for rough volatility models (where the volatility process is driven by a fBm with Hurst parameter $H < \frac{1}{2}$). More precisely, the ATM short-end implied volatility skew is $\frac{1}{H+\frac{3}{2}}$ the ATM short-end local volatility skew, a result that can be obtained via large deviations techniques.

Our aim in this paper is twofold. First of all, we see how Malliavin calculus leads to an easy proof of this $\frac{1}{H+\frac{3}{2}}$ rule. On the other hand, we study the relationship between the curvature of implied and local volatilities. In particular, we see how the ATM short-end implied volatility curvature can be written in terms of the ATM short-limit skew and curvature of the local volatility, and viceversa. Our results are valid for every $H \in (0, 1)$. That is, they hold for rough ($H < \frac{1}{2}$) volatilities, for classical stochastic volatility models ($H = \frac{1}{2}$), and for long-memory processes ($H > \frac{1}{2}$).

This paper is organized as follows. In Section 2 we present the main tools of Malliavin calculus needed in this work. Section 3 is devoted to introduce the framework and the notations. In Section 4 we analyze the relationship between the local and the implied volatility skews. The local and the implied curvatures is studied in Section 5.

2 Basic concepts of Malliavin calculus

In this section we recall the key tools of Malliavin calculus that we use in this paper. We refer to Alòs and García-Lorite (2021) for a deeper introduction to this topic and its applications to finance.

2.1 Basic definitions

If $Z = (Z_t)_{t \in [0, T]}$ is a standard Brownian motion, we denote by \mathcal{S} the set of random variables of the form

$$F = f(Z(h_1), \dots, Z(h_n)), \tag{1}$$

where $h_1, \dots, h_n \in L^2([0, T])$, $Z(h_i)$ denotes the Wiener integral of h_i , for $i = 1, \dots, n$, and $f \in C_b^\infty(\mathbb{R}^n)$ (i.e., f and all its partial derivatives are bounded). If $F \in \mathcal{S}$, the Malliavin

derivative of F with respect to Z , $D^Z F$, is defined as the stochastic process in $L^2(\Omega \times [0, T])$ given by

$$D_s^Z F = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(W(h_1), \dots, W(h_n))(s) h_j(s).$$

Moreover, for $m \geq 1$, we can define the iterated Malliavin derivative operator $D^{m,Z}$, as

$$D_{s_1, \dots, s_m}^{m,Z} F = D_{s_1}^Z \dots D_{s_m}^Z F, \quad s_1, \dots, s_m \in [0, T].$$

The operators $D^{m,Z}$ are closable in $L^2(\Omega)$ and we denote by $\mathbb{D}_Z^{n,2}$ the closure of \mathcal{S} with respect to the norm

$$\|F\|_{n,2} = \left(E|F|^p + \sum_{i=1}^n E \|D^{i,Z} F\|_{L^2([0,T]^i)}^2 \right)^{\frac{1}{2}}.$$

Notice that the Malliavin derivative operator satisfies the *chain rule*. That is, given $f \in \mathcal{C}_Z^{1,2}$, and $F \in \mathbb{D}_Z^{1,2}$, the random variable $f(F)$ belongs to $\mathbb{D}_Z^{1,2}$, and $D^Z f(F) = f'(F) D^W F$. We will also make use of the notation $\mathbb{L}^{n,2} = \mathbb{D}_Z^{n,p}(L^2([0, T]))$.

The adjoint of the derivative operator D^Z is the divergence operator δ^Z , which coincides with the Skorohod integral. Its domain, denoted by $\text{Dom } \delta$, is the set of processes $u \in L^2(\Omega \times [0, T])$ such that there exists a random variable $\delta^Z(u) \in L^2(\Omega)$ such that

$$E(\delta^Z(u)F) = E\left(\int_0^T (D_s^Z F)u_s ds\right), \quad \text{for every } F \in \mathcal{S}. \quad (2)$$

We use the notation $\delta^Z(u) = \int_0^T u_s dZ_s$. It is well known that δ is an extension of the Itô integral. That is, δ , applied to adapted and square integrable processes, coincides with the classical Itô integral. Moreover, the space $\mathbb{L}^{1,2}$ is included in the domain of δ .

From the above relationship between the operators D^Z and δ^Z , it is easy to see that, for an Itô process of the form

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t b_s dZ_s,$$

where a and b are adapted processes in $\mathbb{L}_Z^{1,2}$, its Malliavin derivative is given by

$$D_u^Z X_t = \int_0^t D_u^Z a_s ds + b_u \mathbf{1}_{[0,t]}(u) + \int_0^t D_u^Z b_s dZ_s. \quad (3)$$

Then, if we consider an equation of the form

$$X_t = X_0 + \int_0^t a(s, X_s) ds + \int_0^t b(s, X_s) dZ_s,$$

where $a(s, \cdot)$ and $b(s, \cdot)$ are differentiable functions with bounded derivatives, a direct application of (3) allows us to see that

$$D_u^Z X_t = \int_u^t \frac{\partial a}{\partial x}(s, X_s) D_u^Z X_s ds + b(u, X_u) + \int_u^t \frac{\partial b}{\partial x}(s, X_s) D_u^Z X_s dZ_s. \quad (4)$$

Notice that the above equality also holds if a and b are globally Lipschitz functions with polynomial growth (see Theorem 2.2.1 in Nualart (2006)), replacing $\frac{\partial a}{\partial x}$ and $\frac{\partial b}{\partial x}$ by adequate processes.

2.2 Malliavin calculus for local volatilities

Consider a local volatility model of the form

$$S_t = S_0 + \int_0^t \sigma(u, S_u) S_u dW_u, \quad (5)$$

where W is a Brownian motion, $\sigma(u, \cdot)$ is a bounded and twice differentiable function and where we take the interest rate $r = 0$ for the sake of simplicity. According to (4), its Malliavin derivative is given by

$$D_r^W S_t = \sigma(r, S_r) S_r + \int_r^t a(u, S_u) D_r^W S_u dW_u,$$

where $r < t$ and $a(u, S_u) : \partial_S \sigma(u, S_u) S_u + \sigma(u, S_u)$. This implies that

$$D_r^W S_t = \sigma(r, S_r) S_r \exp \left(-\frac{1}{2} \int_r^t a^2(u, S_u) du + \int_r^t a(u, S_u) dW_u \right) \quad (6)$$

Now, take $\theta < r$. Then

$$D_\theta^W D_r^W S_t \quad (7)$$

$$\begin{aligned} &= a(r, S_r) D_\theta^W S_r \exp \left(-\frac{1}{2} \int_r^t a^2(u, S_u) du + \int_r^t a(u, S_u) dW_u \right) \\ &+ \sigma(r, S_r) S_r \exp \left(-\frac{1}{2} \int_r^t a^2(u, S_u) du + \int_r^t a(u, S_u) dW_u \right) \\ &\times \left(-\frac{1}{2} \int_r^t D_\theta^W (a^2(u, S_u)) du + \int_r^t D_\theta^W (a(u, S_u)) dW_u \right) \end{aligned} \quad (8)$$

$$\begin{aligned} &= a(r, S_r) \sigma(\theta, S_\theta) S_\theta \exp \left(-\frac{1}{2} \int_\theta^r a^2(u, S_u) du + \int_\theta^r a(u, S_u) dW_u \right) \\ &+ \sigma(r, S_r) S_r \exp \left(-\frac{1}{2} \int_r^t a^2(u, S_u) du + \int_r^t a(u, S_u) dW_u \right) \\ &\times \left(-\frac{1}{2} \int_r^t D_\theta^W (a^2(u, S_u)) du + \int_r^t D_\theta^W (a(u, S_u)) dW_u \right). \end{aligned} \quad (9)$$

Notice that, as $\theta, r \rightarrow t$,

$$D_r^W S_t \rightarrow \sigma(t, S_t) S_t,$$

and

$$D_\theta^W D_r^W S_t \rightarrow a(t, S_t) \sigma(t, S_t) S_t = \partial \sigma(t, S_t) S_t^2 + \sigma^2(t, S_t) S_t.$$

3 Statement of the Model and notation

Consider a risk-neutral probability model for asset prices of the form

$$dS_t = \sigma_t S_t (\rho dW_+ + \sqrt{1 - \rho^2} dB_t), \quad (10)$$

where we assume the interest rate to be zero, $\rho \in [-1, 1]$, W and B are two independent Brownian motions, and where σ is a stochastic process adapted to the filtration generated by W . Notice that we do not assume σ to be a diffusion nor a Markov process. Then, (10) includes both the cases of classical stochastic volatility models (where σ is assumed to be a diffusion) and fractional volatilities (where σ is driven by a fractional Brownian motion). In particular, it includes the case of rough volatilities (fractional volatilities with Hurst parameter $H < \frac{1}{2}$).

Through this paper, we assume the following hypotheses.

Hypothesis 1 *The process $\sigma = (\sigma_t)_{t \in [0, T]}$ is positive and continuous a.s., and satisfies that for all $t \in [0, T]$,*

$$c_1 \leq \sigma_t \leq c_2,$$

for some positive constants c_1 and c_2 .

Hypothesis 2 *$\sigma \in \mathbb{L}_W^{3;2}$, and there exist $C > 0$ and $H \in (0, 1)$ such that for all $t \leq \tau \leq \theta \leq r \leq u \leq T$*

$$\begin{aligned} |(D_\theta^W \sigma_r^2)| &\leq C(r - \theta)^{H - \frac{1}{2}}, \\ |(D_\theta^W D_r^W \sigma_u^2)| &\leq C(u - r)^{H - \frac{1}{2}}(u - \theta)^{H - \frac{1}{2}}, \\ |(D_\tau^W D_\theta^W D_r^W \sigma_u^2)| &\leq C(u - r)^{H - \frac{1}{2}}(u - \theta)^{H - \frac{1}{2}}(u - \tau)^{H - \frac{1}{2}}. \end{aligned}$$

Hypothesis 3 *For every $t \in [0, T]$, the following quantities*

$$\begin{aligned} &\frac{1}{(T - t)^{\frac{3}{2} + H}} \int_t^T \left(\int_s^T D_s^W \sigma_u^2 du \right) ds, \\ &\frac{1}{(T - t)^{2 + 2H}} \int_t^T \left(\mathbb{E}_r \int_r^T D_r^W \sigma_u^2 du \right)^2 dr, \\ &\frac{1}{(T - t)^{2 + 2H}} \int_t^T \left(\int_s^T D_s^W \left(\sigma_r \int_s^T D_s^W \sigma_u^2 du \right) dr \right) ds. \end{aligned}$$

have finite limit as $T \rightarrow t$.

Remark 1 Notice that the above hypotheses have been chosen for the sake of simplicity, but they can be replaced by adequate integrability conditions.

In the next section, we compare the short-end of the implied skew slope for a model of the form (10) and the short-end of its corresponding local volatility.

4 The skew

Let us consider the following adaptation of Theorem 6.3 in Alòs, León and Vives (2007) (see also Theorem 7.5.2 in Alòs and García-Lorite (2021)).

Theorem 2 Under Hypothesis 1, 2 and 3, and for every fixed $t \in [0, T]$,

$$\lim_{T \rightarrow t} (T-t)^{\frac{1}{2}-H} \partial_k I(t, k^*) = \frac{\rho}{2\sigma_t^2} \lim_{T \rightarrow t} \frac{1}{(T-t)^{\frac{3}{2}+H}} \mathbb{E}_t \left(\int_t^T \left(\int_r^T D_r^W \sigma_u^2 du \right) dr \right). \quad (11)$$

Notice that the above theorem is valid not only for the model (10), but also for local volatility models of the form (5), where $\sigma_u = \sigma(u, S_u)$. Then, we are in a position to prove the main result of this section.

Theorem 3 Under Hypothesis 1, 2 and 3 for every fixed $t \in [0, T]$

$$\lim_{T \rightarrow t} (T-t)^{\frac{1}{2}-H} \partial_k I(t, k^*) = \frac{1}{\frac{3}{2}+H} \lim_{T \rightarrow t} (T-t)^{\frac{1}{2}-H} \partial_x \hat{\sigma}(T, X_T),$$

where $\hat{\sigma}$ denotes the local volatility function in terms of the log-price X .

Proof. Theorem 2 gives us that the limit

$$\lim_{T \rightarrow t} (T-t)^{\frac{1}{2}-H} \partial_k I(t, k^*)$$

exists and it is finite, which also implies that the limit in the right-hand-side in (11) is also finite. Moreover, as the vanilla implied volatilities are the same for the model (10) and for the corresponding local volatility model, the result in Theorem 2 is also true if we replace σ_u by the local volatility $\sigma(u, S_u)$. Notice that, because of the chain rule,

$$\begin{aligned} D_r^W \sigma^2(u, S_u) &= 2\sigma(u, S_u) D_r^W (\sigma(u, S_u)) \\ &= 2\sigma(u, S_u) \partial_S \sigma(u, S_u) D_r^W S_u, \end{aligned} \quad (12)$$

Then, a direct application of Theorem 2 and Equation (6) give us that

$$\lim_{T \rightarrow t} (T-t)^{\frac{1}{2}-H} \partial_k I(t, k^*) \quad (13)$$

$$= \frac{1}{2\sigma(t, S_t)^2} \lim_{T \rightarrow t} \frac{1}{(T-t)^{\frac{3}{2}+H}} \int_t^T \left(\int_r^T 2\sigma(u, S_u) \partial_S \sigma(u, S_u) \sigma(r, S_r) S_r du \right) dr. \quad (14)$$

Now, because of the continuity of the local volatility function σ and the asset price S we can write

$$\lim_{T \rightarrow t} (T-t)^{\frac{1}{2}-H} \partial_k I(t, k^*) = \lim_{T \rightarrow t} \frac{1}{(T-t)^{\frac{3}{2}+H}} \int_t^T \left(\int_r^T S_u \partial_S \sigma(u, S_u) du \right) dr. \quad (15)$$

Now, notice that

$$S_u \partial_S \sigma(u, S_u) = \partial_X \hat{\sigma}(u, X_u),$$

where $\hat{\sigma}$ denotes the local volatility function in terms of the log-price X . Then

$$\lim_{T \rightarrow t} (T-t)^{\frac{1}{2}-H} \partial_k I(t, k^*) \quad (16)$$

$$= \lim_{T \rightarrow t} \frac{1}{(T-t)^{\frac{3}{2}+H}} \int_t^T \left(\int_r^T \partial_x \hat{\sigma}(u, x_u) du \right) dr \quad (17)$$

$$= \lim_{T \rightarrow t} \frac{1}{(T-t)^{\frac{3}{2}+H}} \int_t^T (u-t) \partial_x \hat{\sigma}(u, X_u) du. \quad (18)$$

Now, a direct application of l'Hôpital rule gives us that

$$\lim_{T \rightarrow t} (T-t)^{\frac{1}{2}-H} \partial_k I(t, k^*) \quad (19)$$

$$= \lim_{T \rightarrow t} \frac{1}{(T-t)^{\frac{3}{2}+H}} \int_t^T \left(\int_r^T \partial_x \hat{\sigma}(u, X_u) du \right) dr \quad (20)$$

$$= \lim_{T \rightarrow t} \frac{1}{\left(\frac{3}{2}+H\right)(T-t)^{\frac{1}{2}+H}} (T-t) \partial_x \hat{\sigma}(T, X_T) du \quad (21)$$

$$= \frac{1}{\frac{3}{2}+H} \lim_{T \rightarrow t} (T-t)^{\frac{1}{2}-H} \partial_x \hat{\sigma}(T, X_T), \quad (22)$$

as we wanted to prove. ■

Remark 4 *In the case $H = \frac{1}{2}$, the above result recovers the classical one-half rule. For the case $H \neq \frac{1}{2}$, we recover the recent results by Bourgey, De Marco, Friz, and Pigato (2022).*

5 The curvature

Let us recall the following result, that is an adaptation of Theorem 4.6 in Alòs and León (2017) (see also Theorem 8.3.3 in Alòs and García-Lorite (2021)).

Theorem 5 Under Hypothesis 1, 2 and 3, and for every fixed $t \in [0, T]$,

$$\begin{aligned}
& \lim_{T \rightarrow t} (T-t)^{1-2H} \partial_{kk}^2 I(t, k^*) \\
&= \frac{1}{4\sigma_t^5} \lim_{T \rightarrow t} \frac{1}{(T-t)^{2+2H}} \mathbb{E}_t \left(\int_t^T \left(\mathbb{E}_r \int_r^T D_r^W \sigma_u^2 du \right)^2 dr \right) \\
&\quad - \frac{3\rho^2}{2\sigma_t^5} \lim_{T \rightarrow t} \frac{1}{(T-t)^{3+2H}} \mathbb{E}_t \left(\int_t^T \left(\int_r^T D_r^W \sigma_u^2 du \right) dr \right)^2 \\
&\quad + \frac{\rho^2}{\sigma_t^4} \lim_{T \rightarrow t} \frac{1}{(T-t)^{2+2H}} \mathbb{E}_t \left(\int_t^T \int_s^T D_s^W \left(\sigma_r \int_r^T D_r^W \sigma_u^2 du \right) dr ds \right).
\end{aligned} \tag{23}$$

Similar arguments as in the proof of Theorem 3 allow us to study the relationship between the short-end curvature of the implied and the local volatilities. More precisely, we get the following theorem

Theorem 6 Under Hypothesis 1, 2 and 3, and for every fixed $t \in [0, T]$,

$$\begin{aligned}
& \lim_{T \rightarrow t} (T-t)^{1-2H} \partial_{kk}^2 I(t, k^*) \\
&= \frac{1}{\sigma(t, S_t)} \lim_{u \rightarrow t} (u-t)^{1-2H} (\partial_x \hat{\sigma}(u, X_u))^2 \\
&\quad \times \left[\frac{3}{(H + \frac{3}{2})(H+1)} - \frac{6}{(H + \frac{3}{2})^2} + \frac{1}{2(H+1)} \right] \\
&\quad + \frac{1}{2(1+H)} \lim_{T \rightarrow t} (T-t)^{1-2H} \partial_{xx}^2 \hat{\sigma}(T, X_T),
\end{aligned} \tag{24}$$

Proof. Because of Theorem 5, we know that the limit

$$\lim_{T \rightarrow t} (T-t)^{1-2H} \partial_{kk}^2 I(t, k^*)$$

is finite. Moreover, as local volatilities replicate vanilla prices, the result in Theorem 5 is also true if we replace the spot volatility σ_u by the local volatility $\sigma(u, S_u)$. Then we can write

$$\begin{aligned}
& \lim_{T \rightarrow t} (T-t)^{1-2H} \partial_{kk}^2 I(t, k^*) \\
&= \frac{1}{4\sigma(t, S_t)^5} \lim_{T \rightarrow t} \frac{1}{(T-t)^{2+2H}} \mathbb{E}_t \left(\int_t^T \left(\mathbb{E}_r \int_r^T D_r^W \sigma^2(u, S_u) du \right)^2 dr \right) \\
&\quad - \frac{3}{2\sigma(t, S_t)^5} \lim_{T \rightarrow t} \frac{1}{(T-t)^{3+2H}} \mathbb{E}_t \left(\int_t^T \left(\int_r^T D_r^W \sigma^2(u, S_u) du \right) dr \right)^2 \\
&\quad + \frac{1}{\sigma(t, S_t)^4} \lim_{T \rightarrow t} \frac{1}{(T-t)^{2+2H}} \mathbb{E}_t \left(\int_t^T \int_s^T D_s^W \left(\sigma(r, S_r) \int_r^T D_r^W \sigma^2(u, S_u) du \right) dr ds \right) \\
&= T_1 + T_2 + T_3.
\end{aligned} \tag{25}$$

Now the proof is decomposed into several steps.

Step 1 Let us study the term T_1 . Let us study the term T_1 . As

$$D_r^W \sigma^2(u, S_u) = 2\sigma(u, S_u) \partial_S \sigma(u, S_u) D_r^W S_u, \quad (26)$$

and because of the continuity of σ , and S , we get

$$T_1 = \frac{1}{\sigma(t, S_t)} \lim_{T \rightarrow t} \frac{1}{(T-t)^{2+2H}} \mathbb{E}_t \left(\int_t^T \left(\mathbb{E}_r \int_r^T \partial_S \sigma(u, S_u) S_u du \right)^2 dr \right) \quad (27)$$

$$= \frac{1}{\sigma(t, S_t)} \lim_{T \rightarrow t} \frac{1}{(T-t)^{2+2H}} \mathbb{E}_t \left(\int_t^T \left(\mathbb{E}_r \int_r^T \partial_x \hat{\sigma}(u, X_u) du \right)^2 dr \right). \quad (28)$$

Because of Theorem 3 we know that

$$(u-t)^{\frac{1}{2}-H} \partial_x \hat{\sigma}(u, X_u)$$

tends to a finite limit. Then we can write

$$\begin{aligned} T_1 &= \frac{1}{\sigma(t, S_t)} \lim_{u \rightarrow t} (u-t)^{1-2H} (\partial_x \hat{\sigma}(u, X_u))^2 \lim_{T \rightarrow t} \frac{\int_t^T \left(\int_r^T (u-t)^{H-\frac{1}{2}} du \right)^2 dr}{(T-t)^{2+2H}} \\ &= \frac{1}{\sigma(t, S_t)} \lim_{u \rightarrow t} (u-t)^{1-2H} (\partial_x \hat{\sigma}(u, X_u))^2 \lim_{T \rightarrow t} \frac{\int_t^T [(T-t)^{H+\frac{1}{2}} - (r-t)^{H+\frac{1}{2}}]^2 dr}{(H+\frac{1}{2})^2 (T-t)^{2+2H}} \\ &= \frac{1}{\sigma(t, S_t) (H+\frac{1}{2})^2} \left(1 - \frac{2}{(H+\frac{3}{2})} + \frac{1}{2H+2} \right) \lim_{u \rightarrow t} (u-t)^{1-2H} (\partial_x \hat{\sigma}(u, X_u))^2. \end{aligned} \quad (29)$$

Now, notice that

$$1 - \frac{2}{(H+\frac{3}{2})} + \frac{1}{2H+2} = \frac{(H+\frac{1}{2})^2}{(H+\frac{3}{2})(H+1)},$$

and then

$$T_1 = \frac{1}{\sigma(t, S_t)} \frac{1}{(H+\frac{3}{2})(H+1)} \lim_{u \rightarrow t} (u-t)^{1-2H} (\partial_x \hat{\sigma}(u, X_u))^2. \quad (30)$$

Step 2 In a similar way,

$$\begin{aligned} T_2 &= -\frac{6}{\sigma(t, S_t)} \lim_{u \rightarrow t} (u-t)^{1-2H} (\partial_x \hat{\sigma}(u, X_u))^2 \lim_{T \rightarrow t} \frac{1}{(T-t)^{3+2H}} \left(\int_t^T \left(\int_r^T (u-t)^{H-\frac{1}{2}} du \right) dr \right)^2 \\ &= -\frac{6}{\sigma(t, S_t)} \frac{1}{(H+\frac{3}{2})^2} \lim_{u \rightarrow t} (u-t)^{1-2H} (\partial_x \hat{\sigma}(u, X_u))^2. \end{aligned} \quad (31)$$

Step 3 Let us now study the term T_3 . Similar arguments as before allow us to write

$$\begin{aligned}
T_3 &= \frac{1}{\sigma(t, S_t)^4} \lim_{T \rightarrow t} \frac{1}{(T-t)^{2+2H}} \mathbb{E}_t \left(\int_t^T \int_s^T D_s^W \left(\sigma(r, S_r) \int_r^T D_r^W \sigma^2(u, S_u) du \right) dr ds \right) \\
&= \frac{1}{\sigma(t, S_t)^4} \lim_{T \rightarrow t} \frac{1}{(T-t)^{2+2H}} \mathbb{E}_t \left(\int_t^T \int_s^T \left(D_s^W \sigma(r, S_r) \int_r^T D_r^W \sigma^2(u, S_u) du \right) dr ds \right) \\
&+ \frac{1}{\sigma(t, S_t)^4} \lim_{T \rightarrow t} \frac{1}{(T-t)^{2+2H}} \mathbb{E}_t \left(\int_t^T \int_s^T \left(\sigma(r, S_r) \int_r^T D_s^W D_r^W \sigma^2(u, S_u) du \right) dr ds \right) \\
&= T_3^1 + T_3^2. \tag{32}
\end{aligned}$$

As $D_s^W \sigma(r, S_r) = \partial_S \sigma(r, S_r) D_s^W S_r$, the continuity of σ and S allows us to write

$$\begin{aligned}
T_3^1 &= \frac{2}{\sigma(t, S_t)^4} \lim_{T \rightarrow t} \frac{1}{(T-t)^{2+2H}} \mathbb{E}_t \left(\int_t^T \int_s^T \left(\partial_S \sigma(r, S_r) S_r \int_r^T \partial_S \sigma(u, S_u) S_u du \right) dr ds \right) \\
&= \frac{2}{\sigma(t, S_t)^4} \lim_{T \rightarrow t} \frac{1}{(T-t)^{2+2H}} \mathbb{E}_t \left(\int_t^T \int_s^T \left(\partial_x \hat{\sigma}(r, X_r) \int_r^T \partial_x \hat{\sigma}(u, X_u) du \right) dr ds \right). \tag{33}
\end{aligned}$$

Then, as $(u-t)^{\frac{1}{2}-H} \partial_x \hat{\sigma}(u, X_u)$ has a finite limit, we get

$$\begin{aligned}
T_3^1 &= \frac{1}{\sigma(t, S_t)^4} \lim_{u \rightarrow t} (u-t)^{1-2H} (\partial_x \hat{\sigma}(u, X_u))^2 \\
&\quad \times \frac{1}{(T-t)^{2+2H}} \left(\int_t^T \left(\int_r^T (u-t)^{H-\frac{1}{2}} du \right)^2 dr \right) \\
&= \frac{1}{\sigma(t, S_t)^4} \frac{1}{(H + \frac{3}{2})(H+1)} \lim_{u \rightarrow t} (u-t)^{1-2H} (\partial_x \hat{\sigma}(u, X_u))^2. \tag{34}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
T_3^2 &= \frac{1}{\sigma(t, S_t)^4} \lim_{T \rightarrow t} \frac{1}{(T-t)^{2+2H}} \mathbb{E}_t \left(\int_t^T \int_s^T \left(\sigma(r, S_r) \int_r^T D_s^W D_r^W \sigma^2(u, S_u) du \right) dr ds \right) \\
&= \frac{1}{\sigma(t, S_t)^3} \lim_{T \rightarrow t} \frac{1}{(T-t)^{2+2H}} \mathbb{E}_t \left(\int_t^T \int_s^T \left(\int_r^T D_s^W D_r^W \sigma^2(u, S_u) du \right) dr ds \right) \tag{35}
\end{aligned}$$

Now, notice that

$$\begin{aligned}
D_\theta^W D_r^W \sigma^2(u, S_u) &= 2(\partial_S \sigma(u, S_u))^2 D_\theta^W S_u D_r^W S_u \\
&+ 2\sigma(u, S_u) \partial_{SS}^2 \sigma(u, S_u) D_\theta^W S_u D_r^W S_u \\
&+ 2\sigma(u, S_u) \partial_S \sigma(u, S_u) D_\theta^W D_r^W S_u \tag{36}
\end{aligned}$$

Then

$$\begin{aligned}
T_3^2 &= \frac{2}{\sigma(t, S_t)} \lim_{u \rightarrow t} (u-t)^{1-2H} (\partial_x \hat{\sigma}(u, X_u))^2 \\
&\quad \times \frac{1}{(T-t)^{2+2H}} \left(\int_t^T \int_s^T \int_r^T (u-t)^{2H-1} dudrds \right) \\
&+ 2 \lim_{T \rightarrow t} \frac{1}{(T-t)^{2+2H}} \mathbb{E}_t \left(\int_t^T \int_s^T \left(\int_r^T [\partial_{SS}^2 \sigma(u, S_u) S_u^2 + \partial_S \sigma(u, S_u) S_u] du \right) drds \right) \\
&+ \frac{2}{\sigma(t, S_t)} \lim_{u \rightarrow t} (u-t)^{1-2H} (\partial_x \hat{\sigma}(u, X_u))^2 \\
&\quad \times \frac{1}{(T-t)^{2+2H}} \left(\int_t^T \int_s^T (r-t)^{H-\frac{1}{2}} \left(\int_r^T (u-t)^{H-\frac{1}{2}} du \right) drds \right) \\
&= \frac{1}{\sigma(t, S_t)} \lim_{u \rightarrow t} (u-t)^{1-2H} (\partial_x \hat{\sigma}(u, X_u))^2 \frac{1}{2(H+1)} \left(1 + \frac{2}{(H+\frac{3}{2})} \right) \\
&+ 2 \lim_{T \rightarrow t} \frac{1}{(T-t)^{2+2H}} \mathbb{E}_t \left(\int_t^T \int_s^T \left(\int_r^T [\partial_{SS}^2 \sigma(u, S_u) S_u^2 + \partial_S \sigma(u, S_u) S_u] du \right) drds \right) \\
&= \frac{1}{2\sigma(t, S_t)(H+1)} \left(1 + \frac{2}{(H+\frac{3}{2})} \right) \lim_{u \rightarrow t} (u-t)^{1-2H} (\partial_x \hat{\sigma}(u, X_u))^2 \\
&+ \lim_{T \rightarrow t} \frac{1}{(T-t)^{2+2H}} \mathbb{E}_t \left(\int_t^T \int_s^T \int_r^T \partial_{xx}^2 \hat{\sigma}(u, X_u) (u-t)^2 dudrds \right). \tag{37}
\end{aligned}$$

Notice that, as all the other limits exist and are finite, the last term in the above equation is finite. Then, a direct application of l'Hôpital rule allows us to write

$$\begin{aligned}
T_3^2 &= \frac{1}{2\sigma(t, S_t)(H+1)} \left(1 + \frac{2}{(H+\frac{3}{2})} \right) \lim_{u \rightarrow t} (u-t)^{1-2H} (\partial_x \hat{\sigma}(u, X_u))^2 \\
&+ \frac{1}{2(1+H)} \lim_{T \rightarrow t} (T-t)^{1-2H} \partial_{xx}^2 \hat{\sigma}(T, X_T). \tag{38}
\end{aligned}$$

Now, (30), (31), (34), and (38) give us that

$$\begin{aligned}
&\lim_{T \rightarrow t} (T-t)^{1-2H} \partial_{kk}^2 I(t, k^*) \\
&= \frac{1}{\sigma(t, S_t)} \lim_{u \rightarrow t} (u-t)^{1-2H} (\partial_x \hat{\sigma}(u, X_u))^2 \\
&\quad \times \left[\frac{3}{(H+\frac{3}{2})(H+1)} - \frac{6}{(H+\frac{3}{2})^2} + \frac{1}{2(H+1)} \right] \\
&+ \frac{1}{2(1+H)} \lim_{T \rightarrow t} (T-t)^{1-2H} \partial_{xx}^2 \hat{\sigma}(T, X_T), \tag{39}
\end{aligned}$$

as we wanted to prove. ■

Remark 7 Notice that, if $H = \frac{1}{2}$, the above reduces to

$$\begin{aligned} & \lim_{T \rightarrow t} (T-t)^{1-2H} \partial_{kk}^2 I(t, k^*) \\ &= -\frac{1}{6\sigma(t, S_t)} \lim_{u \rightarrow t} (u-t)^{1-2H} (\partial_x \hat{\sigma}(u, X_u))^2 \\ &+ \frac{1}{3} \lim_{T \rightarrow t} (T-t)^{1-2H} \partial_{xx}^2 \hat{\sigma}(T, X_T), \end{aligned} \quad (40)$$

according to Equation (8.4.3) in Alòs and García-Lorite (2021). On the other hand, in the uncorrelated case $\rho = 0$ it reads

$$\lim_{T \rightarrow t} \partial_{kk}^2 I(t, k^*) = \frac{1}{2(H+1)} \lim_{T \rightarrow t} \partial_{xx}^2 \hat{\sigma}(T, X_T). \quad (41)$$

In particular, if $\rho = 0$ and $H = \frac{1}{2}$, we get

$$\lim_{T \rightarrow t} \partial_{kk}^2 I(t, k^*) = \frac{1}{3} \lim_{T \rightarrow t} \partial_{xx}^2 \hat{\sigma}(T, X_T), \quad (42)$$

according to the results in Hagan, Kumar, Lesniewski, and Woodward (2002).

Remark 8 As

$$\lim_{T \rightarrow t} (T-t)^{1-2H} (\partial_x \hat{\sigma}(T, X_T))^2 = 4 \lim_{T \rightarrow t} (T-t)^{1-2H} (\partial_k I(t, k^*))^2$$

the result in Theorem 6 can be written as

$$\begin{aligned} & \frac{1}{2(1+H)} \lim_{T \rightarrow t} (T-t)^{1-2H} \partial_{xx}^2 \hat{\sigma}(T, X_T) \\ &= \lim_{T \rightarrow t} (T-t)^{1-2H} \partial_{kk}^2 I(t, k^*) \\ &- \frac{4}{\sigma(t, S_t)} \lim_{T \rightarrow t} (T-t)^{1-2H} (\partial_k I(t, k^*))^2 \\ &\times \left[\frac{3}{(H + \frac{3}{2})(H+1)} - \frac{6}{(H + \frac{3}{2})^2} + \frac{1}{2(H+1)} \right]. \end{aligned} \quad (43)$$

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