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Variance Swaps

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Abstract

This document describes pricing of variance swaps by replication, as well as their market sensitivities. Additional information is provided regarding the pricing of variance swaps and their Greeks in OVML.

Keywords. Greeks, Sensitivity, Delta, Theta, Vega, Rho, Phi, Bump-and-Reprice, OVML.

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1 Payoff

Let S be an asset in a variance swap contract. We have a schedule of $N+1$ observation dates $\{t_0, t_1, \ldots, t_N\}$, and the corresponding number of log-returns is N. Typically the schedule is daily (each business day). We denote the fixing on date t_i by $S_i = S(t_i)$. Let A be a day count factor which is usually 252. Let v_k be a volatility strike which is often chosen to make the initial premium zero. Let $Notional$ denote the variance notional. Then the vega notional[1] is $Notional \cdot 2 \cdot v_k \cdot 100$, and the variance swap has the following payoff in the pay (notional) currency on the delivery date (usually two business days after the last observation):

$$Payoff = Notional \cdot \left(\frac{A}{N} \cdot \sum_{i=1}^{N} \log^2\left(\frac{S_i}{S_{i-1}}\right) - v_k^2 \right). \tag{1.1}$$

2 Pricing

The present value of the variance swap in the pay currency with variance $Notional = 1$ is

$$PV = Discount \cdot \left(\frac{A}{N} \cdot \mathbb{E}\left[\sum_{i=1}^{N} \log^2\left(\frac{S_i}{S_{i-1}}\right) \right] - v_k^2 \right),$$

where the discount is for the pay (notional) currency and the expectation is computed from the pay (notional) currency perspective. Suppose we already have M observations $\{S_0, S_1, \ldots, S_{M-1}\}$, the current spot is S and the next observation is on t_M. Then the total variance will depend on the realized variance v_r^2 and the future variance v_f^2 as follows:

$$PV = Discount \cdot \left(\frac{A}{N} \left(v_r^2 + v_f^2\right) - v_k^2 \right), \tag{2.1}$$

where

$$v_r^2 := \sum_{i=1}^{M-1} \log^2\left(\frac{S_i}{S_{i-1}}\right) + \log^2\left(\frac{S}{S_{M-1}}\right), \tag{2.2}$$

$$v_f^2 := \mathbb{E}\left[\log^2\left(\frac{S_M}{S}\right) + \sum_{i=M+1}^{N} \log^2\left(\frac{S_i}{S_{i-1}}\right) \right] \tag{2.3}$$

define the realized variance and future variance, respectively.

Note that the term $\log^2\left(\frac{S}{S_{M-1}}\right)$ in (2.2) is the source of the intraday delta and gamma for a variance swap[2]. If observations have not started yet ($M = 0$) then $v_r = 0$ and $v_f^2 = E\left[\sum_{i=1}^{N} \log^2\left(\frac{S_i}{S_{i-1}}\right)\right]$. We assume that there are no jumps in the dynamics for the spot, and that the interest rates

[1]The expression approximates the payoff when realized vol is one *vol point* (vega) above the strike: $(v^2 - v_k^2) \approx 2v_k$.

[2]Since $\mathbb{E}\left[\sum_{i=1}^{N} \log^2\left(S_i/S_{i-1}\right)\right] = v_r^2 + v_f^2 + 2\log\left(S/S_{M-1}\right)\mathbb{E}\left[\log\left(S_M/S\right)\right] \approx v_r^2 + v_f^2$.

are deterministic. Ignoring the discreteness of observations we can price the future variance by replication [Blo1, §A.1]:

$$v_f^2 = \int_0^F \frac{Put(K,\sigma(K))}{K^2} dK + \int_F^\infty \frac{Call(K,\sigma(K))}{K^2} dK, \tag{2.4}$$

where Put and $Call$ are undiscounted out of the money vanillas with expiry on the last observation date t_N, and F is the forward corresponding to the last observation date t_N (plus two business days). If the pay (notional) currency is foreign, then we may invert the currency pair and use the above formula[3] for the inverted currency pair $\widehat{S} = S^{-1}$.

$$\widehat{v}_f^2 = \frac{1}{F}\int_0^F \frac{Put(K,\sigma(K))}{K} dK + \frac{1}{F}\int_F^\infty \frac{Call(K,\sigma(K))}{K} dK. \qquad \text{(Inverted Pair)} \tag{2.5}$$

To improve the accuracy of integration in (2.4), the flat BS variance swap with ATMF $= \sigma(F)$ is used as a control variable (for which the left hand side of (2.4) is $v_f^2 = \sigma^2(F)\cdot t_N$):

$$v_f^2 = \sigma^2(F)\cdot t_N + \int_0^F \frac{Put(K,\sigma(K)) - Put(K,\sigma(F))}{K^2} dK + \int_F^\infty \frac{Call(K,\sigma(K)) - Call(K,\sigma(F))}{K^2} dK,$$

where again the vanillas in the integration are undiscounted. Furthermore, by making the change of variable $x = \log K$ in the above integrations, one can write:

$$v_f^2 = \sigma^2(F)\cdot t_N + \int_{-\infty}^{\log F} \frac{Put(e^x,\sigma(e^x)) - Put(e^x,\sigma(F))}{e^x} dx + \int_{\log F}^{+\infty} \frac{Call(e^x,\sigma(e^x)) - Call(e^x,\sigma(F))}{e^x} dx,$$

Similar formulas can be obtained for the inverted currency pair $\widehat{S}$, for example

$$\widehat{v}_f^2 = \sigma^2(F)\cdot t_N + \frac{1}{F}\int_{-\infty}^{\log F} (Put(e^x,\sigma(e^x)) - Put(e^x,\sigma(F)))dx + \frac{1}{F}\int_{\log F}^{+\infty} (Call(e^x,\sigma(e^x)) - Call(e^x,\sigma(F)))dx$$

for the variance swap paying in foreign currency. The integration in x is done numerically. The left endpoint is set to $\left[\log F - 10\cdot\sigma(F)\sqrt{t_N}\right]$, and the right endpoint[4] is set to $\left[\log F + 10\cdot\sigma(F)\sqrt{t_N}\right]$. We divide the left interval $[\log F - 10\cdot\sigma(F)\sqrt{t_N}, \log F]$ and right interval $[\log F, \log F + 10\cdot\sigma(F)\sqrt{t_N}]$ into $N_x = 200$ equally spaced nodes and apply trapezoidal rule integration which is sufficiently accurate. Note that the integrand at $x = \log F$ vanishes. Once we calculate the realized variance v_r^2 and the future variance v_f^2, they can be substituted into (2.1). The quoted fair volatility for the variance swap is

$$\sqrt{\frac{A}{N}\left(v_r^2 + v_f^2\right)}.$$

[3] So *e.g.* $\widehat{Put}(K,\sigma(K)) = \widehat{N}(0)\,\mathbb{E}\left[\widehat{N_t}^{-1}(K - S^{-1})^+\right] = F^{-1}N(0)\,\mathbb{E}\left[KN_t^{-1}(S-K^{-1})^+\right] = \frac{K}{F}Call(K^{-1},\sigma(K^{-1}))$, which gives $\int_0^{\widehat{F}} \widehat{Put}(K,\sigma(K))K^{-2}dK = \int_0^{\widehat{F}} \frac{K}{F}Call(K^{-1},\sigma(K^{-1}))K^{-2}dK = \int_F^\infty (Fu)^{-1}Call(u,\sigma(u))du$.

[4] Alternatively, we can use the points where the left cumulative density (digital put) is sufficiently small, and the right cumulative density (digital call) is sufficiently small.

3 Greeks

We will consider Greeks from the pay (notional) currency perspective. Before the initial observation S_0 the delta and gamma are zero (note that we use sticky-delta approach to delta and gamma, *i.e.* rolling the volsurface with the spot so that $vol(K,T;\lambda S) := vol(\lambda^{-1}K,T;S)$ where $vol(\cdot,\cdot;S)$ is the volsurface at the current spot and λS is the bumped-spot). After the initial observation S_0 varswap will have intraday delta and gamma. Suppose we have $M > 0$ observations $\{S_0, S_1, \ldots, S_{M-1}\}$, the previous fixing is S_{M-1}, the current spot is S and the next observation will be at t_M. Then straightforward differentiation of (2.1) gives

$$\begin{aligned} delta &= Discount \cdot 2 \cdot \frac{A}{N} \cdot \frac{1}{S}\eta, && (3.1)\\ gamma &= Discount \cdot 2 \cdot \frac{A}{N} \cdot \frac{1}{S^2}\left(1-\eta\right), && (3.2)\end{aligned}$$

where we have introduced the notation $\eta := \log\left(\frac{S}{S_{M-1}}\right)$.

The intraday delta is usually small since the current spot S is fairly close to the previous observation S_{M-1}. The intraday gamma is approximately $Discount \cdot 2 \cdot \frac{A}{N}\frac{1}{S^2}$ since the current spot S is fairly close to the previous observation S_{M-1}.

The sensitivities with respect to the domestic and foreign interest rates, respectively, are

$$\begin{aligned} rho &= -t_N \cdot PV,\\ phi &= 0.\end{aligned}$$

The vol sensitivities ATM vega (aega), RR vega (rega) and STR vega (sega) are done by bumping the volatility surface and re-pricing. ATM bump is 1 vol. RR bump is 0.1 vol. STR bump is 0.1 vol. Typically STR vega $\sim 2\cdot$ ATM vega since Fair Vol $\sim$ ATM $+2\cdot$ STR. RR vega is relatively small.

The decay theta is calculated by computing the PV for the next business day with today's market (spot, vol-surface, *etc.*) and subtracting today's PV.

If the pay (notional) currency is foreign then the above delta and gamma and interest rate sensitivities need to be modified accordingly:

$$\begin{aligned} delta &= PV/S + Discount \cdot 2 \cdot \frac{A}{N} \cdot \eta,\\ gamma &= Discount \cdot 2 \cdot \frac{A}{N} \cdot \frac{1}{S}\left(1+\eta\right),\\ rho &= 0,\\ phi &= -t_N \cdot PV.\end{aligned}$$

For example, writing $\hat{S} := S^{-1}$ and letting $g(\hat{S}) = \hat{S} f(\hat{S}^{-1})$ be the price with respect to the foreign currency, then $g'(\hat{S}) = f(\hat{S}^{-1}) - \hat{S}^{-1} f'(\hat{S}^{-1}) = g(\hat{S})/\hat{S} - delta(-\eta)/\hat{S} = PV/\hat{S} + S \cdot delta(\eta)$, where $S \cdot delta(\eta)$ refers to (3.1). Similarly, $g''(\hat{S}) = \frac{-1}{\hat{S}^2} f'(\hat{S}^{-1}) - \frac{-1}{\hat{S}^2} f'(\hat{S}^{-1}) - \frac{-1}{\hat{S}^3} f''(\hat{S}^{-1}) = S \cdot gamma(-\eta)$ where $S \cdot gamma(-\eta)$ refers to (3.2).

4 VSV

Although this document specifically treats FX variance swaps, the ideas are generally applicable to other assets. In particular, one defines the *variance swap par rate* as that volatility strike v_k which makes the expected value of (1.1) zero. The `VSV` function displays market bid-ask valuations for variance swap par rates of equity and equity index underlyings. Note that a screen `Forward Rates` is provided which gives the variance swap rate v_{12} of a forward starting variance swap, starting at t_1 and terminating at t_2, in terms of the variance swap rates (v_1, v_2) at (t_1, t_2) respectively. The relevant formula from which v_{12} is derived is based on the additivity of variances:

$$v_{12}^2(t_2 - t_1) = v_2^2 t_2 - v_1^2 t_1.$$

5 OVML

The `OVML` function displays many of the calculations discussed above, but in certain cases applies scalings depending on the notional amount, notional currency, premium currency, and user settings.

5.1 Settings: Position Delta and Percentage Delta

The `OVML` Settings menu has the following user options, which control the interpretation of the Premium and Delta values displayed in the UI:

- Pricing:Include premium in spot/forward hedge and delta (Yes if in Ccy1, No, and Interbank)
- Display:Premium Display (Pay or Receive premium, Buy or Sell position)
- Display:Delta Display (Percentage, Position)

5.2 Notional Units: Vega Notional and Variance Notional

The payoff of the variance swap at maturity is

$$VarianceNotional \cdot \left(\frac{A}{N} \cdot v_r^2 - v_k^2 \right),$$

where v_r is the realized variance, and v_k is the strike, both given in *vol-units*, which is to say when v_r, v_k are expressed in units of %. When v_r, v_k are converted to absolute units (so $v_k = 8$ vol-units becomes $v_k = 0.08$), a scaling factor of 100^2 will appear in the payoff. In terms of the *vega notional*, which already contains a factor of 100, one can write the payoff as

$$VegaNotional \cdot \left(\frac{A}{N} \cdot v_r^2 - v_k^2 \right) \frac{100}{2v_k},$$

where v_r, v_k are expressed in absolute units.

Given underlying FX in FOR:DOM = Ccy1:Ccy0, the `OVML` defaults for the Notional Currency and Premium currency are Ccy1 and Ccy0, respectively.

If the Premium currency is changed from Ccy0 to Ccy1, then clearly the premium is divided by S (which has units of [Ccy0/Ccy1]). Note that hovering over the premium field will display its value in both Ccy1 and Ccy0 currencies. Regarding the display of the Delta, Sticky Delta, and Hedge, these values do not change when manipulating the Premium currency, as they are fixed to the Notional currency.

The Notional is expressed in units of Ccy1 *per variance-point*, or Ccy0 *per variance-point*. Unlike the Premium currency which affects only the Premium value (Hedge and Delta Percentage being displayed always in FOR), the Notional currency affects each of the values Delta, Sticky Delta, and Hedge. Indeed, changing the Notional currency substantively changes the deal, as already evident from the different replication pricing formulas (2.4) and (2.5). In Table 5.1 we display the effect of changing the Notional currency from Foreign to Domestic for a particular EURUSD variance swap priced on 10-03-2017 with $v_k = 0.01$.

NOTIONAL	PREMIUM	DELTA	DELTA%	STICKY DELTA	HEDGE (FOR)
1234 DOM	69,663 DOM = 59,318 FOR	-47.45 DOM	3.2741% FOR/DOM	0 or 0%	-40.40
1234 FOR	82,365 DOM = 70,134 FOR	-54.92 FOR	4.4505%	7.01 or 0.5683%	-54.92

Table 5.1: Effect of Notional Currency in MC Stoch-Local Vol model. Spot = 1.1744 DOM/FOR.

5.3 Models: BS, MC-BS, MC-SLV

Black-Scholes (Smile Replication)

In `OVML`, the reference to the Black-Scholes model is actually a reference to an *analytic pricing model*, which in the current context is the *replication method* described in §2. This model is not a Black-Scholes model per se, since smiles in the market are represented in (2.4).

MC Black-Scholes

When selecting the MC Black-Scholes model, `OVML` displays Monte Carlo Data with which one can specify the number of simulation paths. Pricing is obtained using a Monte Carlo simulation of the Black-Scholes SDE, and taking an expected value of the var swap payoff over all paths. Unlike the (non MC) Black-Scholes model, this model assumes a constant volatility for the FX rate which is extracted from the vanilla ATM instrument of the same maturity. For example, pricing the variance swap with $v_k \approx 0$ should approximately reproduce the implied variance of the ATM instrument $(\text{ATM-Vol})^2$.

MC Stoch-Local Vol

As with the MC Black-Scholes model, when selecting the MC Stoch-Local Vol model, `OVML` displays Monte Carlo Data with which one can specify the number of simulation paths. Depending on the

Pricing:Auto-Calculate settings, pricing with this model may require that the user select the `Calc` button in the upper-left corner of the display. The stochastic local volatility model parameters, namely the correlation, vol-of-vol, and mixing fraction $(\rho(t), \nu(t), \lambda(t))$, each with a term-structure, are calibrated from the implied volatility surface. The user can view (or even modify) the calibrated parameters by selecting the Model:Parameters tab next to the model name MC Stoch-Local Vol.

Note that the Monte Carlo standard error is also displayed. By increasing the number of paths (up to 100,000) one may reduce the impact of simulation error.

5.4 Delta: Stickiness and Hedge

There are two stickiness assumptions that can be made when computing Delta: *sticky model* and *sticky moneyness* (also called *sticky delta*). The sticky moneyness delta, displayed as Sticky Delta in the `OVML` screen, is computed with the assumption that when the spot S is bumped to λS, the implied volatility function determined at the new spot $vol(K, T; \lambda S)$ is obtained as $vol(\lambda^{-1}K, T; S)$. In particular, the ATM vol before the bump will remain the ATM vol after the bump:

$$vol(\lambda S_0, T; \lambda S_0) = vol(S_0, T; S_0).$$

The other stickiness assumption, which may be called *model delta*, is obtained by re-using the calibrated model (whether calibrated LV surface or SLV leverage surface) without any adjustment. Needless to say, with this stickiness assumption the ATM vol will change accordingly with the bumped spot. When performing the bump-and-reprice methodology, the second Monte Carlo simulation for the bumped price uses a new initial condition for the SDE evolution, namely λS_0.

The units with respect to which Delta is displayed depends on the user's Display:Delta Display setting. When set to `Percentage`, the Delta is displayed as a percentage of the Variance Notional expressed in vol-units and in the foreign currency. When set to `Position`, the percentage Delta is multiplied by the Variance Notional expressed in vol-units, and possibly converted to domestic currency according to the Premium Currency setting. We recall Table 5.1, but re-expressed in generality with FX=S (or $\hat{S} = S^{-1}$), Notional=X, Premium=$f(S)$ (or $g(\hat{S})$), Delta=Δ (or Δ') and Sticky Delta=δ.

NOTIONAL	PREMIUM	DELTA	DELTA%	STICKY DELTA	HEDGE (FOR)
X DOM	$f(S)$ DOM $= f(S)/S$ FOR	$-X \cdot S \cdot \Delta\%$ DOM	$\Delta\%$ FOR/DOM	0 or 0%	$-X \cdot \Delta\%$
X FOR	$g(\hat{S})/\hat{S}$ DOM $= g(\hat{S})$ FOR	$-X \cdot \Delta'\%$ FOR	$\Delta'\%$	$X \cdot \delta\%$ or $\delta\%$	$-X \cdot \Delta'\%$

Table 5.2: Effect of Notional Currency in general. Spot = S (DOM/FOR).

Note that in a Black-Scholes model, the Sticky Delta should agree with Delta in theory, as the absence of a smile makes the distinction irrelevant. This is always the case in `OVML`, even in the MC-based models, as Delta will be displayed using the Sticky Delta value (rather than the model delta described above).

For variance and volatility swaps, more can be said. The presence of the terms $\log\left(\frac{S_i}{S_{i-1}}\right)$ in (1.1) shows the inconsequential effect of the change in spot $S \to \lambda S$, and hence the invariance of the payoff, in any so-called *scale-invariant* model. Such models, of which the Stochastic-Local Volatility model is one instance, have dynamics that respond to a (multiplicatively) bumped initial condition by shifting all ($\log S_i$) states; as a consequence the ratios (S_i/S_{i-1}) are unaffected[5], and therefore Sticky Delta *when priced in the domestic currency* is necessarily zero.

5.5 Miscellaneous Remarks

We conclude by making some additional observations regarding the Greek calculations:

- It was noted above that the Sticky Delta in the domestic currency C_0 is zero. On the other hand, when priced in the *foreign currency*, which is to say using $\widehat{S} = 1/S = C_0C_1$ as a currency pair, the Sticky Delta will be the Premium (FOR) divided by 10000.
  By specializing to the second row of Table 5.1 where Premium is 70134 FOR, we verify Sticky Delta is 7.01.

- More generally [Blo2], if the price in C_0 units is $f(S)$, then when one considers the price in C_1 units expressed as a function of $\hat{S} := S^{-1}$, one may write:

  $$g(\hat{S}) = f(S)/S = \hat{S} f(1/\hat{S}).$$

  Here, $S = C_1C_0$ (FOR:DOM), while $f(S)$ and $\Delta = f'(S)$ are the Premium and Delta in the first row of Table 5.2, respectively. Consequently the Delta in C_0 units appearing in the second row becomes [Blo2]:

  $$\Delta' \quad := \quad \frac{\partial g}{\partial \hat{S}} = f - S \cdot \Delta \qquad (\text{units} = C_1/[C_1/C_0] = C_0), \tag{5.1}$$

  If we specialize to Table 5.1 where $f(S) = 69,663$ DOM, and $\Delta = 47.45$ DOM, and also $\Delta' = 54.92$ FOR, then we can verify (5.1) for *Delta'*: $54.92 \cdot 1.174$ DOM $\approx 69663/10000$ DOM $+47.45$ DOM.

- The Monte Carlo pricers (MC Black-Scholes and Stoch-Local Vol) employ a bump-and-reprice methodology for Deltas and Gammas using a two-sided bump size of $\pm 0.005 \cdot \mathit{Spot}$.

- Hedge is always displayed in the FOR currency and is equal to the Delta Percentage multiplied by the Variance Notional.

- In the context of *intraday pricing*, the (S_M/S_{M-1}) term from (2.1) is *partially realized, i.e.*

  $$\mathbb{E}\left[\log^2(S_M/S_{M-1})\right] = \mathbb{E}\left[\left(\log(S_M/S) + \log(S/S_{M-1})\right)^2\right] \approx \mathbb{E}\left[\log^2(S_M/S)\right] + \left[\log^2(S/S_{M-1})\right],$$

  and thus contributes a fixed constant $\log^2(S/S_{M-1})$ from the realized variance v_r, to which the scaling argument presented above does not apply. In other words, one cannot expect the Sticky Delta in the domestic currency to be zero when the variance swap is priced intraday.

- Although there is no exact replication formula such as (2.4) applicable to volatility-swaps, the meaning of the elements of the `OVML` UI [Blo3], as described in Table 5.1, nonetheless apply to the `OVML` UI for volatility-swaps. One important exception is that the scaling for *vol-units* versus *var-units* introduces a factor of 100 rather than $10,000$.

[5] A more exact characterization would be those payoffs which can be rewritten as $f(S/S_0)$.

References

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