The short-time ATM skew and volatility swaps

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Abstract

The short-time ATM skew can be interpreted as the ratio of the difference between the volatility swap and the dual volatility swap to the ATM implied variance.

1 Introduction

Various papers have studied the short time behavior of the at-the-money (ATM) implied volatility level and skew (see for example Alòs et al. [2007] and Medvedev and Scaillet [2007]). Here we give another interpretation of the short-time ATM skew for diffusion models with stochastic volatility in terms of volatility swaps.

2 Assumptions and notations

We will work under the premise that the market implied volatility surface is generated by the following general stochastic volatility (SV) model

$$dS_u = \sigma_u S_u \left(\rho dW_u + \bar{\rho} dZ_u\right) \tag{1}$$

where dZ_u and dW_u are standard Brownian motions, $\bar{\rho} = \sqrt{1-\rho^2}$, and the instantaneous volatility σ_u is assumed to be a positive process that is independent of S_u and adapted to the filtration generated by W_u.

The SV process is assumed to be well-behaved in the sense that vanilla options prices are risk-neutral expectations of the payoff function:

$$C(S_t, t, K, T) = E_t\left[(S_T - K)_+\right] \tag{2}$$

The option price C can always be expressed in terms of the Black-Merton-Scholes (BS) price with an implied volatility parameter I:

$$C(S_t, K) = BS(S_t, K, I) \tag{3}$$

The BS price is given by the following well-known formula,

$$BS(S_t, K, I) = S_t N(d_+) - KN(d_-) \tag{4}$$

where $N(d_\pm)$ are normal distribution functions,

$$d_\pm = \frac{\log(S_t/K)}{I\sqrt{\tau}} \pm \frac{I\sqrt{\tau}}{2} \tag{5}$$

and $\tau = T - t$.

In what follows we will use the following notations:

- The llog spot price is defined as $s_t = \log S_t$ and the log strike as $k = \log K$
- The ATM implied volatility will be denoted as I_0 and the ATM log strike as k_0
- The zero vanna implied volatility I_-, the zero vanna log strike k_-, and their duals I_+ and k_+ are defined by

$$d_\pm(I_\pm, k_\pm) = 0 \tag{6}$$

- The realised volatility over the interval $[t, T]$ is

$$\sigma_{t,T} = \sqrt{\frac{1}{\tau}\int_t^T \sigma_u^2 \, du} \tag{7}$$

3 Three implied volatilities and the skew

As derived using straightforward Taylor expansion in Rolloos [2021] for the forwad starting case, and implicitly also in Rolloos [2018] for the spot starting case, there is a simple relationship between (dual) volatility swap prices and (dual) zero vanna implied volatilities:

$$E_t\left[\sigma_{t^*,T}\right] \approx I_-(t, t^*, T), \quad E_t\left[\frac{S_T}{S_{t^*}}\sigma_{t^*,T}\right] \approx I_+(t, t^*, T), \tag{8}$$

with $t \leq t^* < T$. Furthermore,

$$I_+ - I_- \approx \rho E_t\left[\sigma_{t^*,T}\int_{t^*}^T \sigma_u \, dW_u\right]. \tag{9}$$

It is reasonable to assume that as $\tau^* = T - t^* \to 0$ the (forward start) implied volatilities I_- and I_+ are almost equidistant from the (forward start) ATM implied volatility I_0. This is expressed as

$$I_- \approx I_0 - \frac{\partial I_0}{\partial k}dk, \quad I_+ \approx I_0 + \frac{\partial I_0}{\partial k}dk, \tag{10}$$

and thus

$$I_+ - I_- \approx 2\frac{\partial I_0}{\partial k}dk. \tag{11}$$

From the definition of $I_\pm$ it follows that

$$s_t - (k_0 - dk) = \frac{1}{2} I_-^2 \tau^*, \tag{12}$$

$$s_t - (k_0 + dk) = -\frac{1}{2} I_+^2 \tau^*. \tag{13}$$

Inserting (10) into the above and ignoring terms of order $(dk)^2$ we obtain

$$2dk = I_0^2 \tau^*. \tag{14}$$

Hence, by making use of (11)

$$I_0^2 \tau^* \frac{\partial I_0}{\partial k} \approx I_+ - I_- \tag{15}$$

Remembering[1] that $I_0 \to E_t[\sigma_{t^*}]$ as $\tau^* \to 0$, we can also write

$$\tau^* \frac{\partial I_0}{\partial k} \approx \frac{\rho}{(E_t[\sigma_{t^*}])^2} E_t \left[\sigma_{t^*,T} \int_{t^*}^T \sigma_u \, dW_u \right]. \tag{16}$$

4 Example: Bergomi model

Suppose the instantaneous variance is lognormal,

$$d\sigma_u^2 = \alpha \sigma_u^2 \, dW_u. \tag{17}$$

According to the integration by parts formula

$$E_t \left[\sigma_{t^*,T} \int_{t^*}^T \sigma_u \, dW_u \right] = E_t \left[\int_{t^*}^T \left(D_u^W \sigma_{t^*,T} \right) \sigma_u \, du, \right] \tag{18}$$

with D_u^W the Malliavin derivative with respect to W_u. Now

$$D_u^W \sigma_{t^*,T} = \frac{1}{2\tau^* \sigma_{t^*,T}} \int_u^T D_u^W \sigma_r^2 \, dr. \tag{19}$$

For the Bergomi model $D_u^W \sigma_r^2 = \alpha \sigma_r^2$, which for $\tau^* \to 0$ we can approximate by $\alpha \sigma_{t^*}^2$. Similarly we can approximate $\sigma_{t^*,T} \approx \sigma_{t^*}$. Hence,

$$E_t \left[\sigma_{t^*,T} \int_{t^*}^T \sigma_u \, dW_u \right] \approx \frac{1}{4} \alpha \tau^* E_t \left[\sigma_{t^*}^2 \right]. \tag{20}$$

By making use of (16) the short time (forward start) ATM skew for the Bergomi model is

$$\frac{\partial I_0}{\partial k} \approx \frac{1}{4} \rho \alpha \frac{E_t \left[\sigma_{t^*}^2 \right]}{(E_t[\sigma_{t^*}])^2}, \tag{21}$$

[1] For the forward start case this holds for forward start implied volatilities of cash forward start options. Forward start implied volatilities of asset forward start options do not have this property in general. In this note we consider cash forward start options and their implied volatilities.

Note that when $t = t^*$ we obtain the familiar expression for the Bergomi short-time ATM vanilla skew

$$\frac{\partial I_0}{\partial k} \approx \frac{\rho\alpha}{4}. \tag{22}$$

References

E. Alòs, J. A. Léon, and J. Vives. On the short-time behavior of the implied volatility skew in jump-diffusion models with stochastic volatility. *Finance and Stochastics*, 2007.

A. Medvedev and O. Scaillet. Approximation and calibration of short-term implied volatilities under jump-diffusion stochastic volatility. *Review of Financial Studies*, 2007.

F. Rolloos. It takes three to smile, 2018. Available at SSRN.com.

F. Rolloos. Managing forward volatility and skew risk, 2021. Available at SSRN.com.