

# First Baruch Volatility Workshop

## Session 1: The volatility surface: Statics and dynamics

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### Outline of Session 1

- Introduction to R and iPython notebook
- What is the volatility surface?
- Financial time series stylized facts
- Stochastic volatility
- The Bergomi-Guyon expansion
- Term structure of at-the-money skew
- The skew-stickiness ratio
- The volatility envelope

## **What is R? (<http://cran.r-project.org> (<http://cran.r-project.org>))**

From Wikipedia:

- In computing, R is a programming language and software environment for statistical computing and graphics. It is an implementation of the S programming language with lexical scoping semantics inspired by Scheme.
- R was created by Ross Ihaka and Robert Gentleman at the University of Auckland, New Zealand, and is now developed by the R Development Core Team. It is named partly after the first names of the first two R authors (Robert Gentleman and Ross Ihaka), and partly as a play on the name of S. The R language has become a de facto standard among statisticians for the development of statistical software.
- R is widely used for statistical software development and data analysis. R is part of the GNU project, and its source code is freely available under the GNU General Public License, and pre-compiled binary versions are provided for various operating systems. R uses a command line interface, though several graphical user interfaces are available.

## **RStudio (<http://www.rstudio.com> (<http://www.rstudio.com>))**

From RStudio.org:

- RStudio is an integrated development environment (IDE) for R. RStudio combines an intuitive user interface with powerful coding tools to help you get the most out of R.
- RStudio brings together everything you need to be productive with R in a single, customizable environment. Its intuitive interface and powerful coding tools help you get work done faster.
- RStudio is available for all major platforms including Windows, Mac OS X, and Linux. It can even run alongside R on a server, enabling multiple users to access the RStudio IDE using a web browser.
- Like R, RStudio is available under a free software license that guarantees the freedom to share and change the software, and to make sure it remains free software for all its users.

## The IPython Notebook (<http://ipython.org/notebook.html>)

From ipython.org:

The IPython Notebook is a web-based interactive computational environment where you can combine code execution, text, mathematics, plots and rich media into a single document:

The IPython notebook with embedded text, code, math and figures. These notebooks are normal files that can be shared with colleagues, converted to other formats such as HTML or PDF, etc. You can share any publicly available notebook by using the IPython Notebook Viewer service which will render it as a static web page. This makes it easy to give your colleagues a document they can read immediately without having to install anything.

[http://nbviewer.ipython.org/github/dboyliao/cookbook-code/blob/master/notebooks/chapter07\\_stats/08\\_r.ipynb](http://nbviewer.ipython.org/github/dboyliao/cookbook-code/blob/master/notebooks/chapter07_stats/08_r.ipynb) ([http://nbviewer.ipython.org/github/dboyliao/cookbook-code/blob/master/notebooks/chapter07\\_stats/08\\_r.ipynb](http://nbviewer.ipython.org/github/dboyliao/cookbook-code/blob/master/notebooks/chapter07_stats/08_r.ipynb)) has instructions on using R with iPython notebook.

## Set up R environment

In [1]:

```
%load_ext rpy2.ipython
```

In [2]:

```
%%R
```

```
download.file(url="http://mfe.baruch.cuny.edu/wp-content/uploads/2015/04/VW1.zip", destfile="VW1.zip")
unzip(zipfile="VW1.zip")
```

```
trying URL 'http://mfe.baruch.cuny.edu/wp-content/uploads/2015/04/VW1.zip'
Content type 'application/zip' length 92874 bytes (90 KB)
opened URL
```

```
=====
downloaded 90 KB
```

## The Black-Scholes formula

Code from BlackScholes.R:

In [3]:

```
##R

BSFormula <- function(S0, K, T, r, sigma)
{
  x <- log(S0/K)+r*T
  sig <- sigma*sqrt(T)
  d1 <- x/sig+sig/2
  d2 <- d1 - sig
  pv <- exp(-r*T)
  return( S0*pnorm(d1) - pv*K*pnorm(d2))
}
```

Note that this function can take a vector of strikes and volatilities.

In [4]:

```
##R

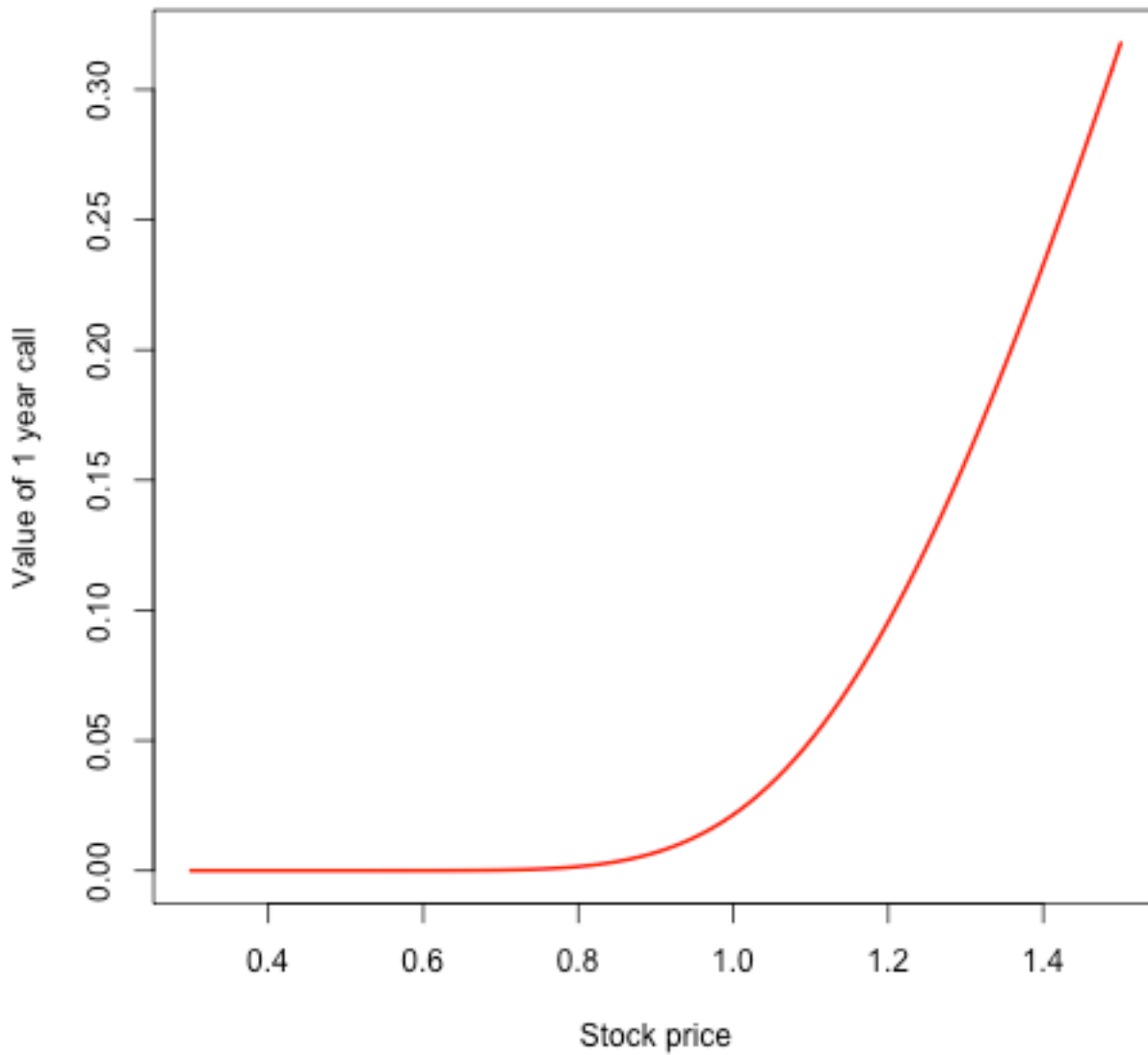
BSFormula(S0=1, K=c(.9,1.0,1.1), T=1, r=0, sigma=0.2)

[1] 0.13589108 0.07965567 0.04292011
```

In [5]:

```
##R

curve(BSFormula(S0=x, K=1.2, T=1, r=0, sigma=0.2),from=0.3,to=1.5,col="red",lwd=
2,xlab="Stock price",ylab="Value of 1 year call")
```



## Implied volatility

In what follows, we always assume zero rates and dividends.

Denote the market price of an option with log-strike  $k$  and expiration  $t$  by  $C(k, t)$ . Then the implied volatility  $\sigma_{BS}(k, t)$  is the number that solves:

$$C_{BS}(k, t, \sigma_{BS}(k, t)) = C(k, t)$$

where  $C_{BS}(\cdot)$  denotes the Black-Scholes formula for a call option:

$$C_{BS}(k, t, \sigma) = F N(d_+) - K N(d_-) = F \{N(d_+) - e^k N(d_-)\}$$

$$\text{with } d_{\pm} = \frac{-k}{\sigma \sqrt{t}} \pm \frac{\sigma \sqrt{t}}{2}.$$

Here is some more code from BlackScholes.R

In [6]:

```
##R

BSImpliedVolCall <- function(S0, K, T, r, C)
{
  nK <- length(K)
  sigmaL <- rep(1e-10,nK)
  CL <- BSFormula(S0, K, T, r, sigmaL)
  sigmaH <- rep(10,nK)
  CH <- BSFormula(S0, K, T, r, sigmaH)
  while (mean(sigmaH - sigmaL) > 1e-10)
  {
    sigma <- (sigmaL + sigmaH)/2
    CM <- BSFormula(S0, K, T, r, sigma)
    CL <- CL + (CM < C)*(CM-CL)
    sigmaL <- sigmaL + (CM < C)*(sigma-sigmaL)
    CH <- CH + (CM >= C)*(CM-CH)
    sigmaH <- sigmaH + (CM >= C)*(sigma-sigmaH)
  }
  return(sigma)
}
```

First, we generate some option prices:

In [7]:

```
##R

vols <- c(0.23,0.20,0.18)
K <- c(0.9,1.0,1.1)
(optVals <- BSFormula(S0=1,K,T=1,r=0,sigma=vols))

[1] 0.14589696 0.07965567 0.03557678
```

Now get implied volatilities from these option prices:

In [8]:

```
##R

(impVols <- BSImpliedVolCall(S0=1, K, T=1, r=0, C=optVals))

[1] 0.23 0.20 0.18
```

# The volatility surface

In practice, implied volatility depends on strike and expiration.

- Fixing one expiration, the *volatility smile* is the graph of implied volatility as a function of (log-)strike.
- Plotting implied volatility by strike and expiration gives the *volatility surface*.

## Smiles as of 15-Sep-2005

We see that the graph of implied volatility vs log-strike looks like a skewed smile. Some people say "smirk".

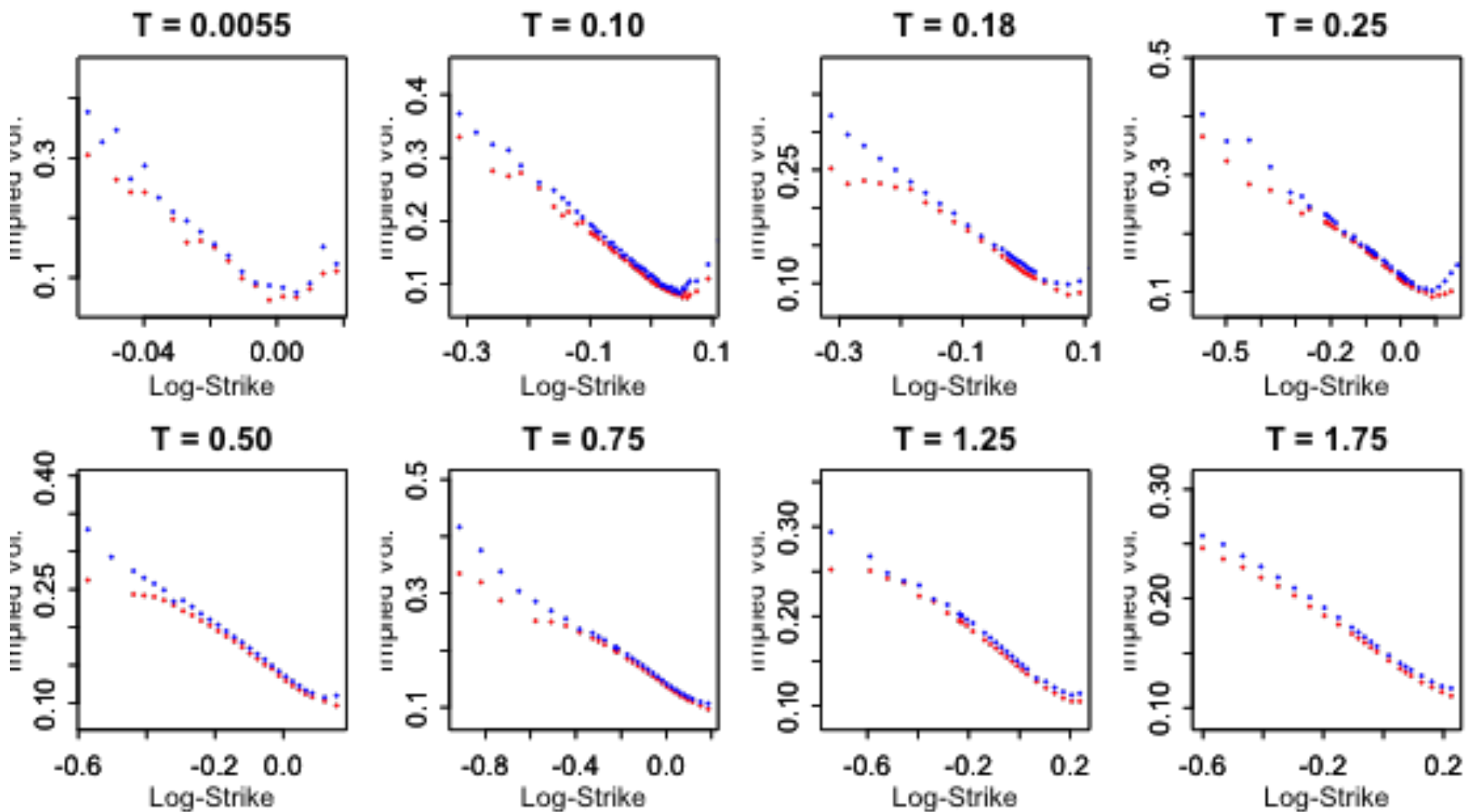


Figure 1: SPX volatility smiles as of September 15, 2005.

## 3D plot

Interpolating by time to expiration, we obtain the following picture of the SPX volatility surface as of the close on September 15, 2005:

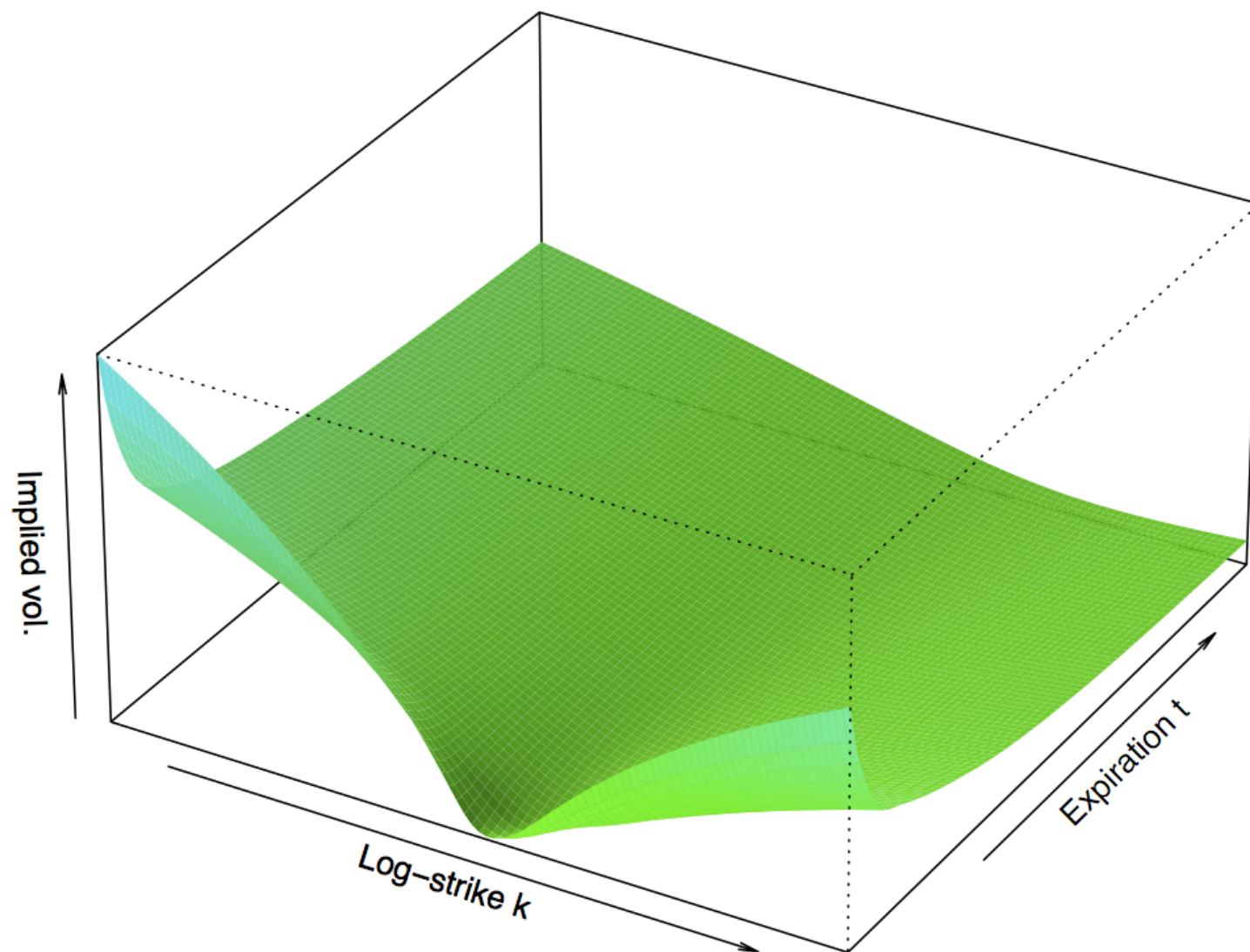


Figure 2: Figure 3.2 from *The Volatility Surface*.  $k := \log K/F$  is the log-strike and  $t$  is time to expiry.



## Why is the surface not flat?

Suppose

$$\frac{dS}{S} = \sigma dZ$$

with constant  $\sigma$ . The solution to this SDE is ( $Z \sim N(0, 1)$ )

$$S_t = S_0 e^{\sigma \sqrt{t} Z - \sigma^2 t/2}.$$

$S_t$  is then lognormally distributed and implied volatility, independent of strike or time to expiration.

- If  $\sigma$  were constant, the implied volatility surface would be flat.

In the real world, returns are not lognormal nor are they independent, as we will now see.

## Financial time series: stylized facts

From Cont (2001)<sup>[4]</sup>:

the seemingly random variations of asset prices do share some quite non-trivial statistical properties. Such properties, common across a wide range of instruments, markets and time periods are called *stylized empirical facts*.

Cont (2001)<sup>[4]</sup> lists the following stylized facts

- **Absence of autocorrelation:** (linear) autocorrelations of asset returns are often insignificant, except for very small intraday time scales.
- **Heavy tails:** the (unconditional) distribution of returns seems to display a power-law or Pareto-like tail, with a tail index which is finite, higher than two and less than five for most data sets studied. In particular this excludes stable laws with infinite variance and the normal distribution.
- **Gain/loss asymmetry:** one observes large drawdowns in stock prices and stock index values but not equally large upward movements.
- **Aggregational Gaussianity:** as one increases the time scale  $\Delta t$  over which returns are calculated, their distribution looks more and more like a normal distribution. In particular, the shape of the distribution is not the same at different time scales.

- **Intermittency:** returns display, at any time scale, a high degree of variability. This is quantified by the presence of irregular bursts in time series of a wide variety of volatility estimators.
- **Volatility clustering:** different measures of volatility display a positive autocorrelation over several days, which quantifies the fact that high-volatility events tend to cluster in time.
- **Conditional heavy tails:** even after correcting returns for volatility clustering (e.g. via GARCH-type models), the residual time series still exhibit heavy tails. However, the tails are less heavy than in the unconditional distribution of returns.
- **Slow decay of autocorrelation in absolute returns:** the autocorrelation function of absolute returns decays slowly as a function of the time lag, roughly as a power law with an exponent  $\beta \in [0.2, 0.4]$ . This is sometimes interpreted as a sign of long-range dependence.
- **Leverage effect:** most measures of volatility of an asset are negatively correlated with the returns of that asset.
- **Volatility/volume correlation:** trading volume is correlated with all measures of volatility.
- **Asymmetry in time scales:** coarse-grained measures of volatility predict fine-scale volatility better than the other way round.

## Power laws in daily SPX returns

Using R and the quantmod package, we analyze log-returns of SPX since 1950.

In [9]:

```
##R  
library(quantmod)  
library(boot)  
  
options("getSymbols.warning4.0"=FALSE)  
  
getSymbols("^GSPC",from="1927-01-01") #Creates the time series object GSPC
```

Loading required package: xts

Loading required package: zoo

Attaching package: 'zoo'

The following objects are masked from 'package:base':

as.Date, as.Date.numeric

Loading required package: TTR

Version 0.4-0 included new data defaults. See ?getSymbols.

[1] "GSPC"

Do the same for VIX and create joint dataset of VIX and SPX

In [10]:

```
##R
getSymbols("^VIX",from="1927-01-01") #Creates the time series object VIX

mm <- specifyModel(Cl(GSPC)~Cl(VIX))
spxVixData <-modelData(mm) #quantmod function automatically aligns data from two
series

vix <- spxVixData[, "Cl.VIX" ]
spx <- spxVixData[, "Cl.GSPC" ]

print(head(spxVixData))

print(tail(spxVixData))
```

	Cl.GSPC	Cl.VIX
1990-01-02	359.69	17.24
1990-01-03	358.76	18.19
1990-01-04	355.67	19.22
1990-01-05	352.20	20.11
1990-01-08	353.79	20.26
1990-01-09	349.62	22.20
	Cl.GSPC	Cl.VIX
2015-05-22	2126.06	12.13
2015-05-26	2104.20	14.06
2015-05-27	2123.48	13.27
2015-05-28	2120.79	13.31
2015-05-29	2107.39	13.84
2015-06-01	2111.73	13.97

In [11]:

```
%%R  
  
ret.spx <- log(Cl(GSPC)/lag(Cl(GSPC)))  
ret.spx <- ret.spx[!is.na(ret.spx)] # Remove missing values  
ret.spx <- ret.spx-mean(ret.spx)  
breaks <- seq(-.235,.115,.002)  
hist.spx <- hist(ret.spx,breaks=breaks,freq=F)
```

**Histogram of ret.spx**

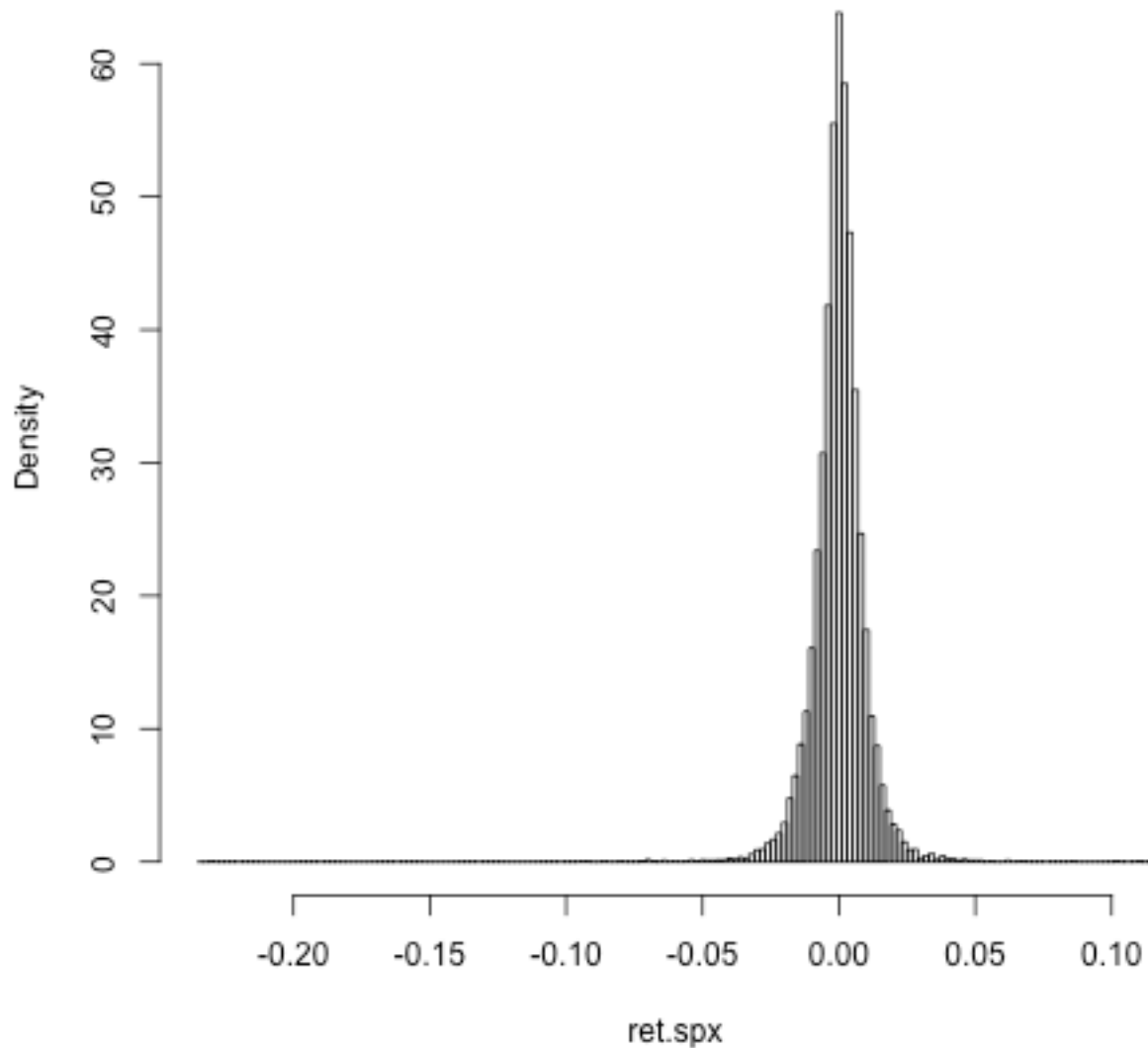


Figure 3: Histogram of SPX log returns. It doesn't look like a normal distribution!

## The Student-t distribution

$$p(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)} \frac{1}{\left(1 + \frac{x^2}{\nu}\right)^{\frac{\nu+1}{2}}} \sim \frac{1}{x^{\nu+1}} \text{ as } x \rightarrow \infty$$

So the tail-exponent  $\zeta = \nu$ , the degrees of freedom.

Obviously, this distribution has mean zero. Its variance is

$$\int x^2 p(x) dx = \frac{\nu}{\nu - 2}$$

With  $\nu = 3$ , the density simplifies to

$$p(x) = \frac{2}{\sqrt{3}\pi} \frac{1}{\left(1 + \frac{x^2}{3}\right)^2}$$

## SPX daily log-returns: Student-t fit

In [12]:

```
##R

plot(hist.spx,xlim=c(-.05,.05),freq=F,main=NA,xlab="Log return")
sig <- as.numeric(sd(ret.spx))
curve(dt(x*sqrt(3)/sig,df=3)*sqrt(3)/sig,from=-.05,to=.05,col="red", add=T)
curve(dnorm(x,mean=0,sd=sig),from=-.05,to=.05,col="blue", add=T)
```

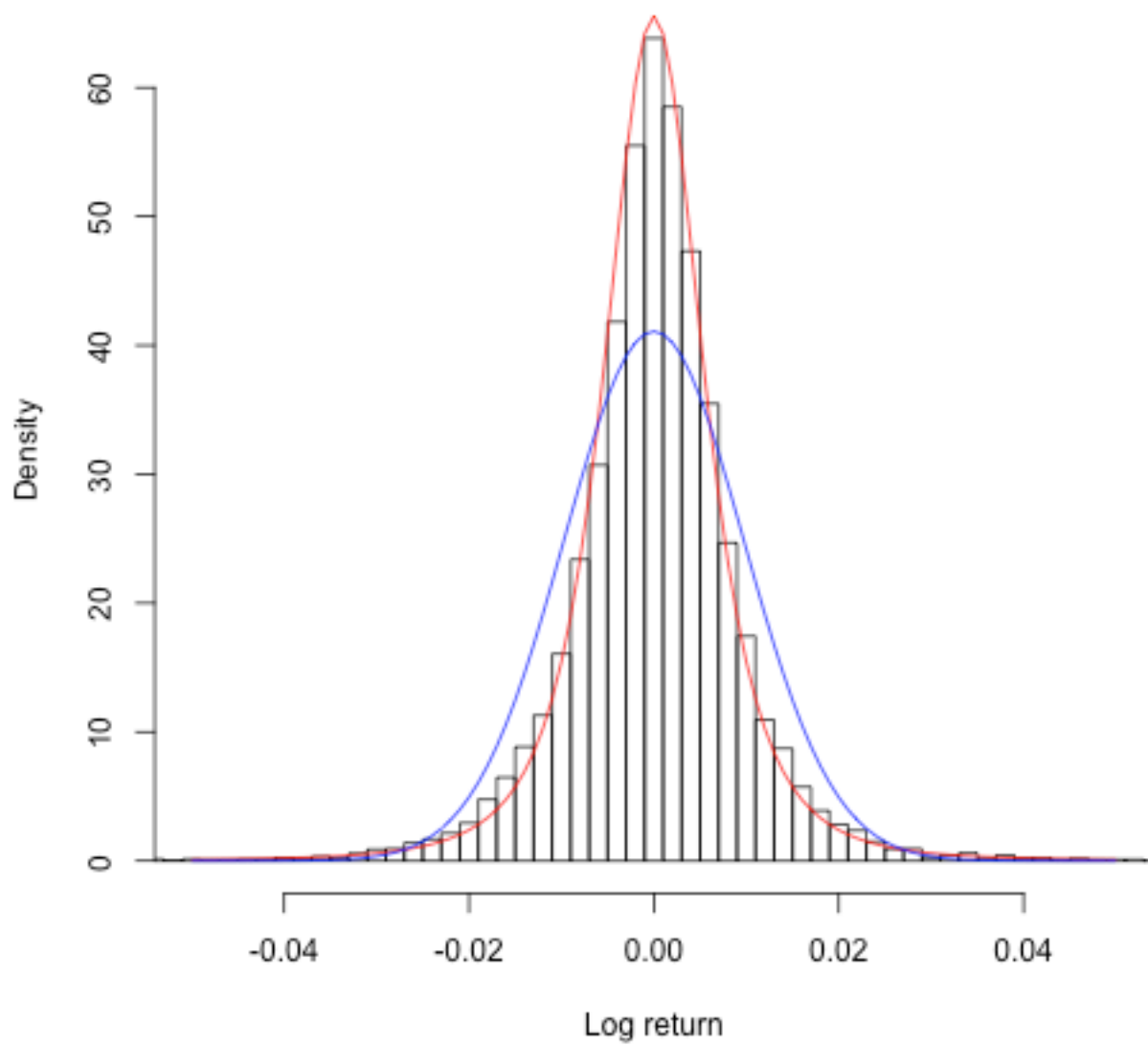


Figure 4: Student-t with  $\nu = 3$  fits almost perfectly! Normal fit is in blue. The cubic law of returns is confirmed.

## SPX weekly log-returns

In [13]:

```
%%R
```

```
pxw <- Cl(to.weekly(GSPC)) # Built-in quantmod function to get close  
retw <- Delt(log(pxw))[-1]  
retw <- retw-mean(retw)  
c(min(retw),max(retw))  
breaks <- seq(-.0345,.0345,.001)  
spx.histw <- hist(retw,breaks=breaks,freq=F,plot=F)  
sigw <- as.numeric(sd(retw))  
scale <- function(nu){sqrt(nu/(nu-2))}  
  
plot(spx.histw,xlim=c(-.035,.035),freq=F,main=NA,xlab="Weekly log return")  
curve(dt(x*scale(4)/sigw,df=4)*scale(4)/sigw,from=-.05,to=.05,col="red", add=T)  
curve(dnorm(x,mean=0,sd=sigw),from=-.05,to=.05,col="blue", add=T)
```

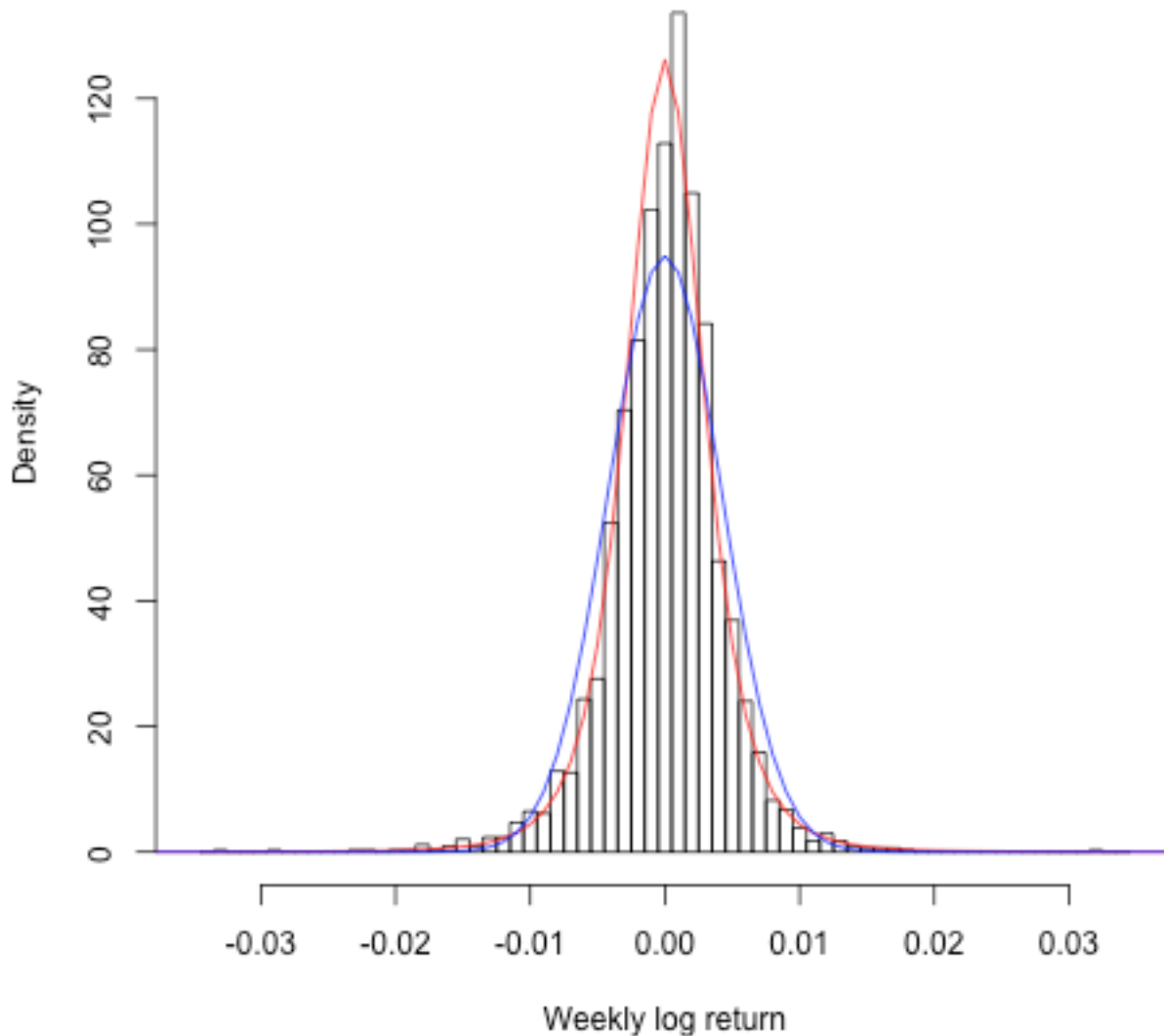


Figure 5: Student-t with  $\nu = 4$  in red. Normal fit is in blue. Aggregational Gaussianity?

## SPX monthly log-returns



In [14]:

```
##R

pxm <- Cl(to.monthly(GSPC)) # Built-in quantmod function to get close
retm <- Delt(log(pxm))[-1]
retm <- retm-mean(retm)
c(min(retm),max(retm))
breaks <- seq(-.046,.038,.004)
spx.histm <- hist(retm,breaks=breaks,freq=F,plot=F)
sigm <- as.numeric(sd(retm))
scale <- function(nu){sqrt(nu/(nu-2))}

plot(spx.histm,xlim=c(-.045,.045),freq=F,main=NA,xlab="monthly log return")
curve(dt(x*scale(5)/sigm,df=5)*scale(5)/sigm,from=-.05,to=.05,col="red", add=T)
curve(dnorm(x,mean=0,sd=sigm),from=-.05,to=.05,col="blue", add=T)
```

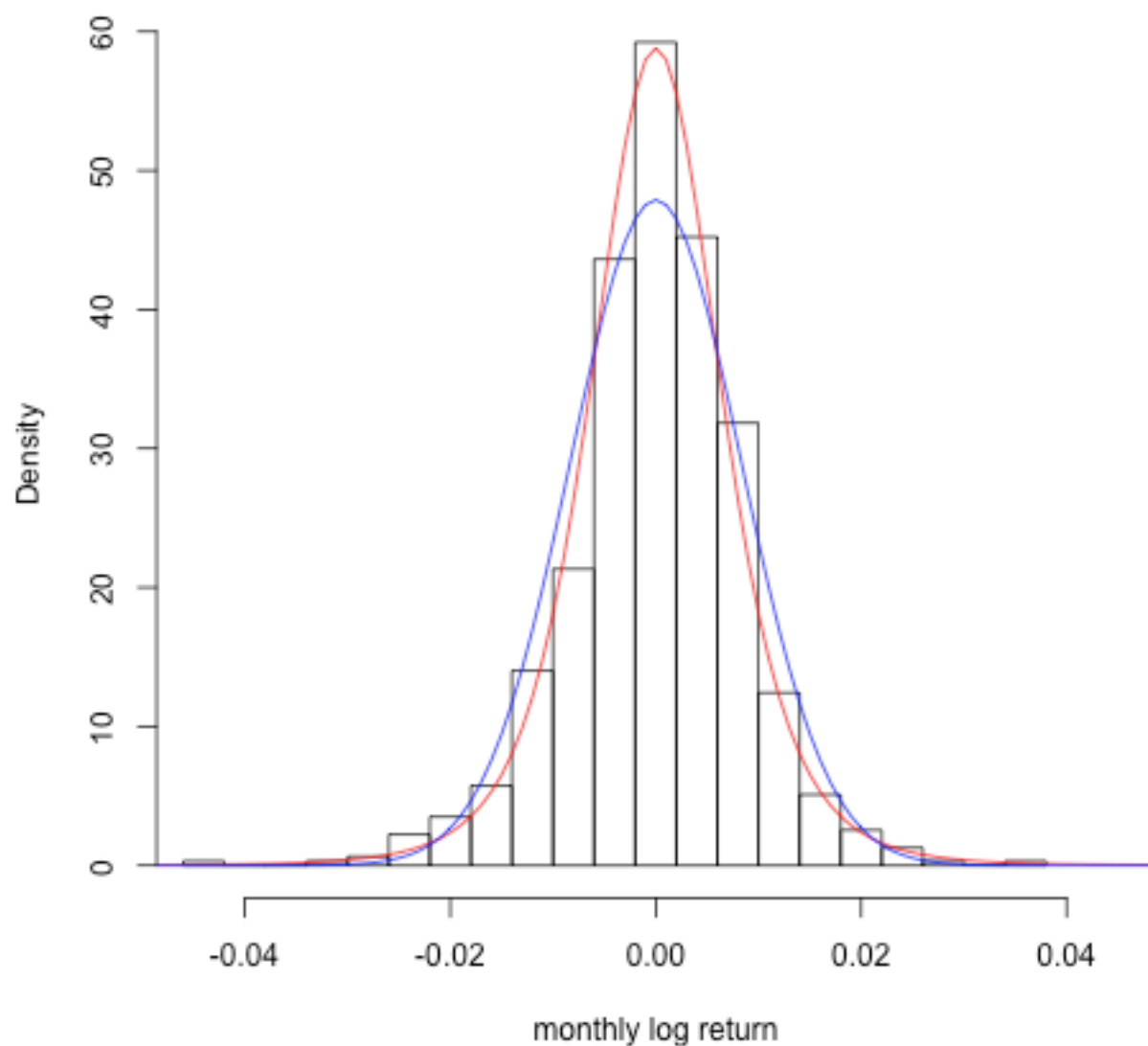


Figure 6: Student-t with  $\nu = 5$  in red. Normal fit is in blue. Aggregational Gaussianity again?

## SPX log-returns: Volatility clustering

```
In [15]:
```

```
%%R
```

```
plot(ret.spx,main=NA)
```

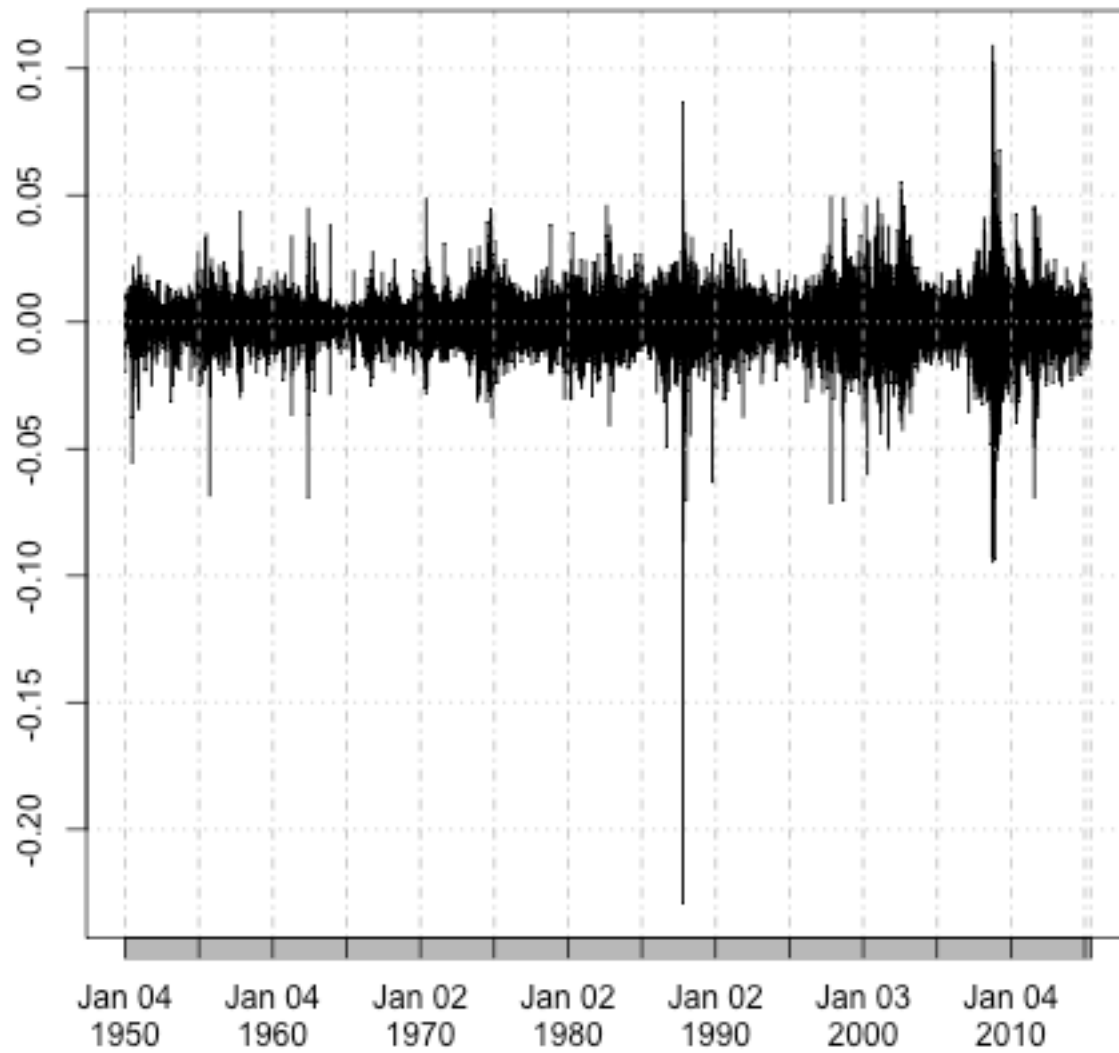


Figure 7: Daily log-returns of SPX

## Autocorrelation of SPX returns

```
In [16]:
```

```
%%R
```

```
acf.r <- acf(ret.spx,main=NA)
```

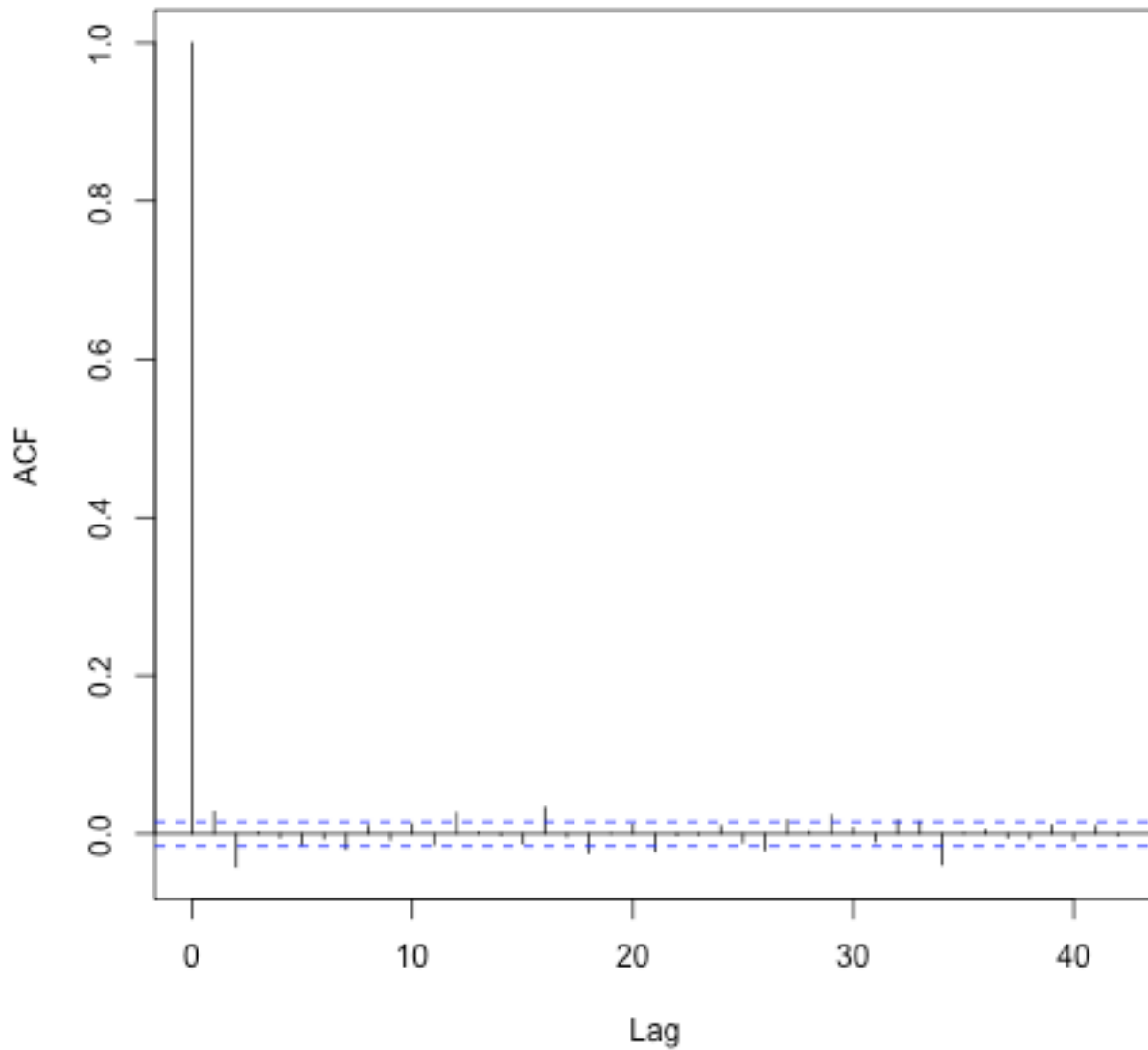


Figure 8: No significant autocorrelation in returns.

## Autocorrelation of SPX absolute returns

It is a stylized fact that the autocorrelation function (ACF) of absolute log-returns decays as a power-law. However, we will see below that this widespread belief is probably not justified.

In [17]:

```
##R
# Autocorrelation of absolute returns
acf.rabs <- acf(abs(ret.spx),main=NA,plot=F)
logacf.rabs <- log(acf.rabs$acf)[-1]
loglag.rabs <- log(acf.rabs$lag)[-1]
plot(loglag.rabs,logacf.rabs,xlab="log(lag)",ylab="log(acf)",pch=20,col="blue")
print(acfrabs.lm <- lm(logacf.rabs[-(1:15)]~loglag.rabs[-(1:15)]) )
abline(acfrabs.lm,col="red",lwd=2)
```

Call:

```
lm(formula = logacf.rabs[-(1:15)] ~ loglag.rabs[-(1:15)])
```

Coefficients:

(Intercept)	loglag.rabs[-(1:15)]
-0.7497	-0.2912

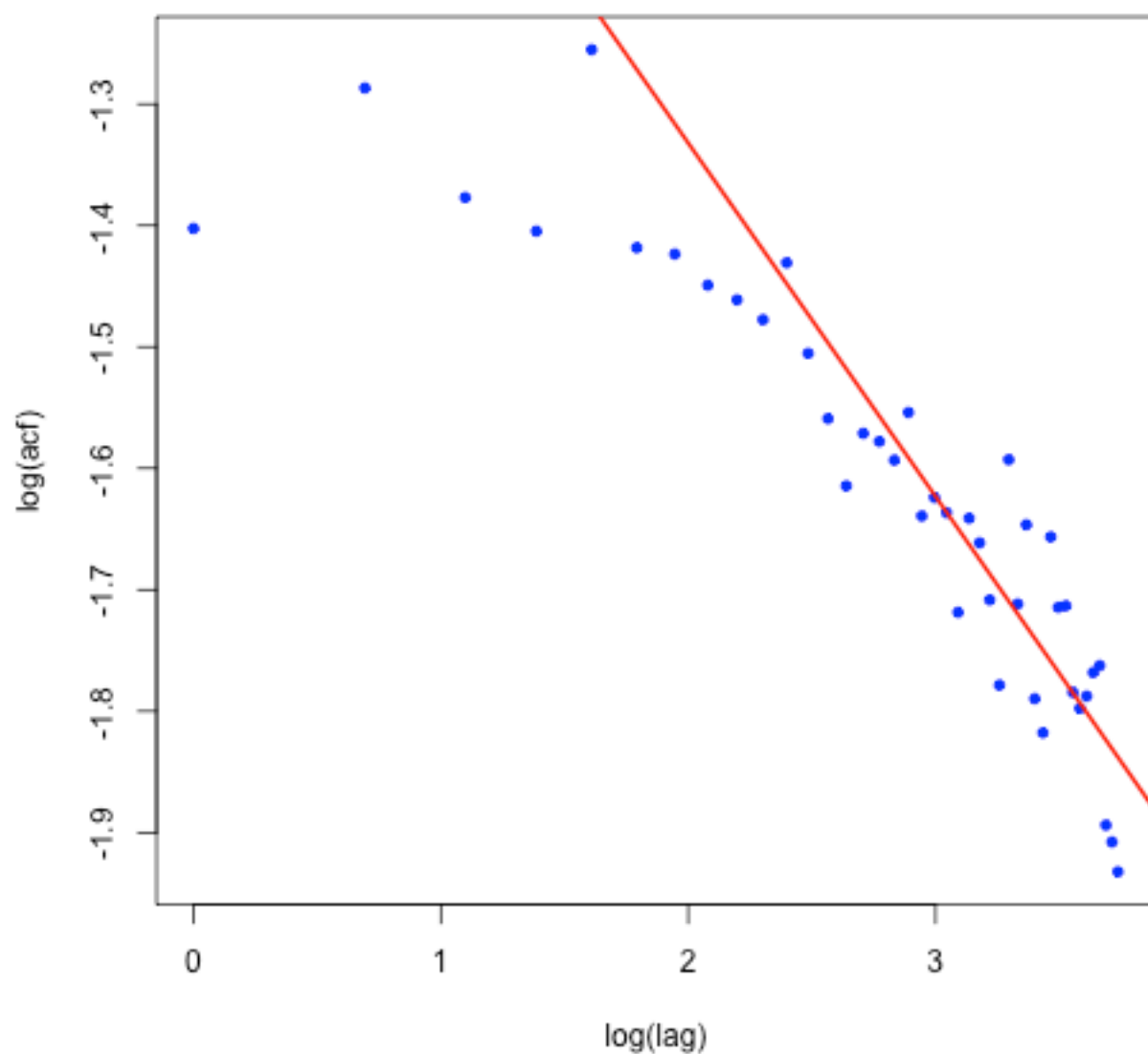


Figure 9: Slope of fit is around  $-0.29$ .

- It is often concluded from fits like this that the autocorrelation function of volatility decays roughly as  $t^{-\alpha}$  with  $\alpha \approx 0.3$ .

## Absolute and squared log-returns vs realized variance

Absolute daily log-returns are (very) noisy proxies for daily volatilities  $\sigma_t$  and squared daily log-returns are (very) noisy proxies for daily variances  $\sigma_t^2$ . Gatheral and Oomen [6] (for example) show how tick data can be used to obtain more accurate estimates of realized variance  $\sigma_t^2$ .

The Oxford-Man Institute of Quantitative Finance makes historical realized variance estimates for 21 different stock indices freely available at <http://realized.oxford-man.ox.ac.uk> (<http://realized.oxford-man.ox.ac.uk>). These estimates are updated daily. We may then investigate the time series properties of  $\sigma_t^2$  empirically.

## Decay of the volatility ACF

According to our recent work using such realized variance time series, which I will present in Lecture 3, the empirical ACF of volatility does not decay as a power-law.

In fact, SPX realized variance has the following amazingly simple scaling property:

$$m(q, \Delta) := \langle |\log \sigma_{t+\Delta} - \log \sigma_t|^q \rangle = A \Delta^{qH}$$

where  $\langle \cdot \rangle$  denotes a sample average.

This simple scaling property holds for all 21 indices in the Oxford-Man dataset. We have also checked that it holds for crude oil, gold and Bund futures. For SPX over 14 years,  $H \approx 0.14$  and  $A \approx 0.38$ .

As a consequence it may be shown that the autocorrelation function should take the form:

(1)

$$\rho(\Delta) \sim e^{-\frac{1}{2} \nu^2 \Delta^{2H}}.$$

## Predicted vs empirical autocorrelation function

In [18]:

```
##R

h.spx <- 0.14

y <- logacf.rabs
x <- acf.rabs$lag[-1]^(2*h.spx)

fit.lm <- lm(y[-1]~x[-1])
a <- fit.lm$coef[1]
b <- fit.lm$coef[2]

plot(loglag.rabs,logacf.rabs,xlab="log(lag)",ylab="log(acf)",pch=20,col="blue",x
lim=c(0,4.5))
abline(acfrabs.lm,col="red",lwd=2)
curve(a+b*exp(x*2*h.spx),from=0,to=4.5,add=T,col="green4",lwd=2)
```

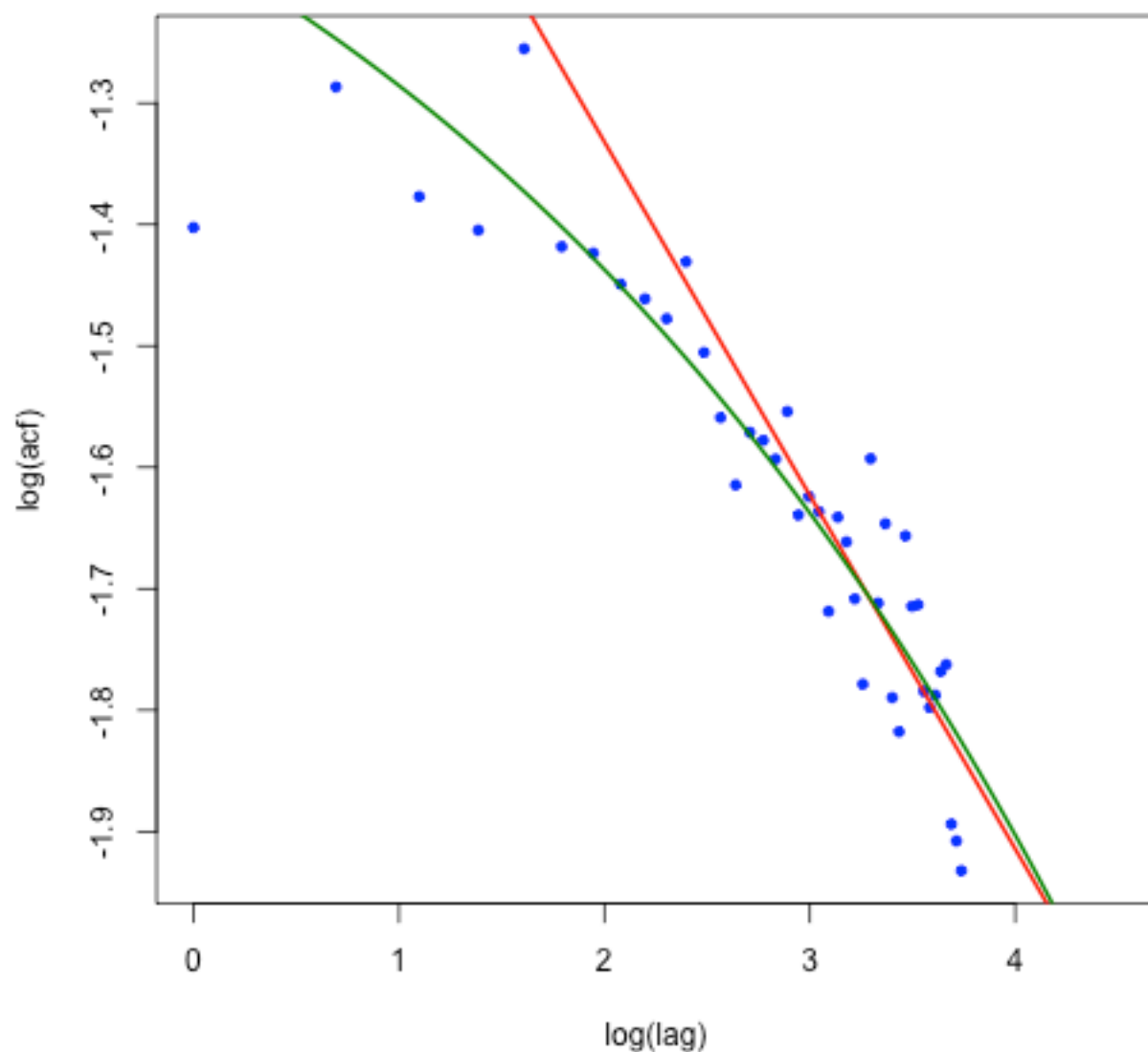


Figure 10: The red line is the conventional linear fit; the green line is the rough volatility prediction (1).

## Log returns of VIX

VIX can be thought of as a measure of volatility smoothed over one month. Let's look at the distribution of VIX log-returns.

In [19]:

```
##R

vix <- spxVixData[, "Cl.VIX"]
spx <- spxVixData[, "Cl.GSPC"]

retVIX <- as.numeric(diff(log(vix))[-1])
retSPX <- as.numeric(diff(log(spx))[-1])

sdVIX <- as.numeric(sd(retVIX))

hist(retVIX, breaks = 100, freq=F)
scale <- function(nu){sqrt(nu/(nu-2))}
curve(dt(x*scale(4)/sdVIX, df=4)*scale(4)/sdVIX, from=-.3, to=.3, col="blue", lwd=2,
add=T)
curve(dnorm(x, mean=0, sd=sdVIX), from=-.3, to=.3, col="red", lwd=2, add=T)
```

Histogram of retVIX

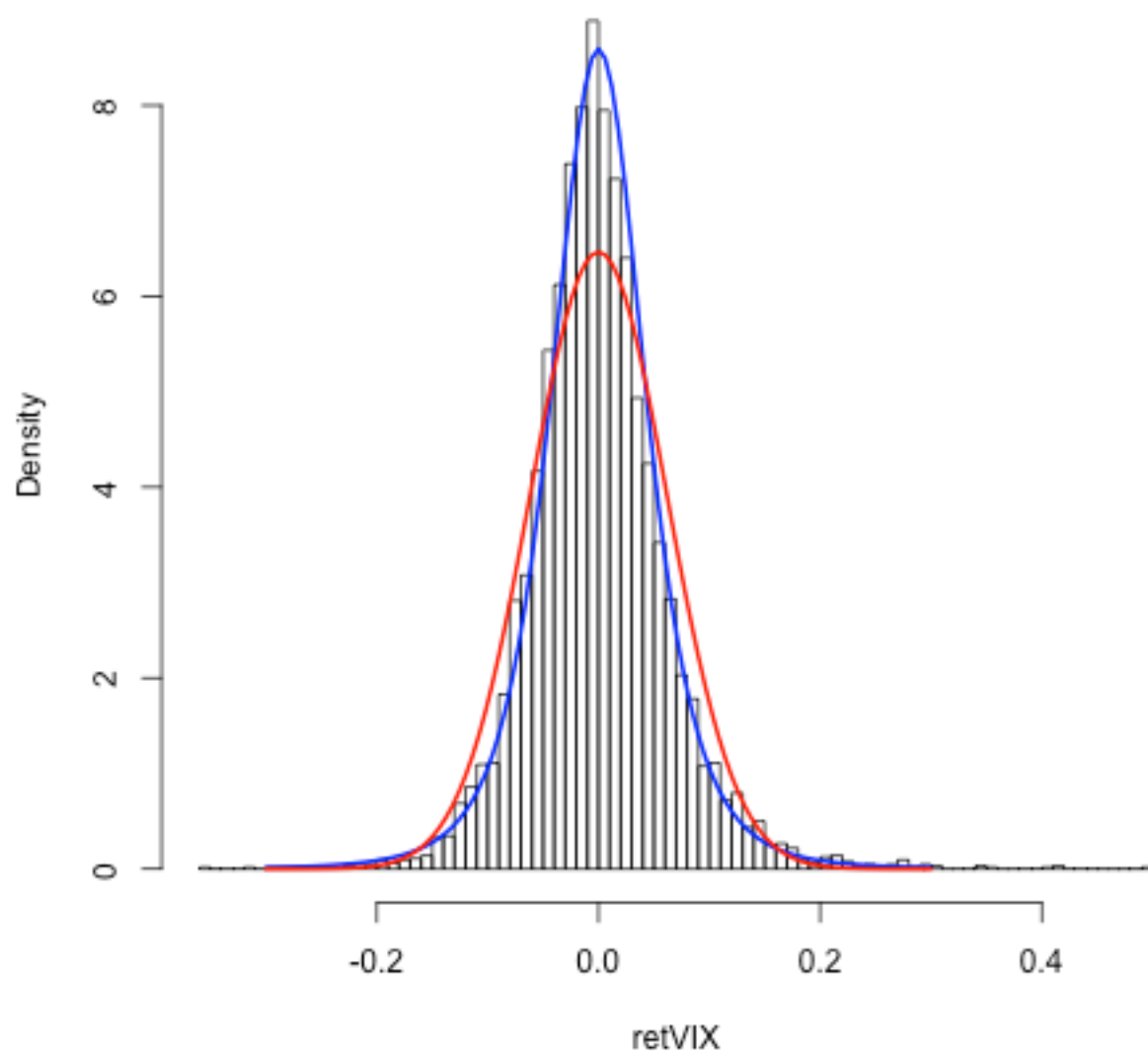


Figure 11: Log-returns of VIX are somewhat less fat-tailed than log-returns of SPX.

## VIX vs SPX

We now regress log-returns of VIX against log-returns of SPX.

In [20]:

```
##R

fit.spxvix <- lm(retVIX~retSPX)
fit.spxvix2 <- lm(retSPX~retVIX)

# Scatter plot + fit
plot(retSPX,retVIX,xlab="SPX log returns",ylab="VIX log returns");
abline(fit.spxvix,col="red",lwd=2);

print(cor(retVIX,retSPX)) # Gets correlation

[1] -0.7105517
```

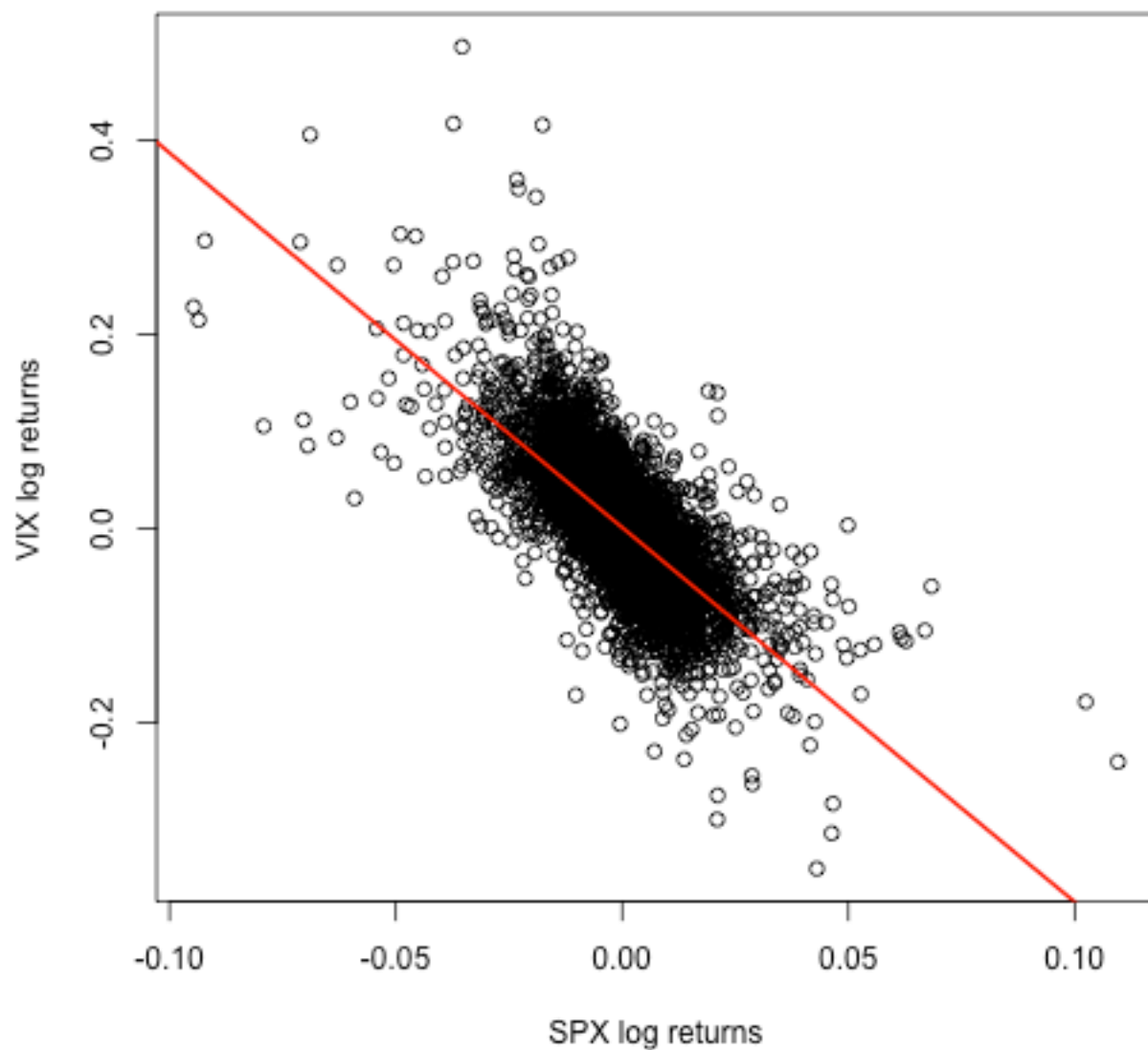




Figure 12: Regression of VIX log-returns vs SPX log-returns.

- Note the negative correlation  $\approx -0.7$  - similar to the correlation implied from fits of stochastic volatility models to option prices.

## Stochastic volatility

- In the stochastic volatility paradigm, the above stylized facts may be qualitatively reproduced by modeling volatility as a continuous stochastic process.
- Formally,

$$\frac{dS_t}{S_t} = \sigma_t dZ_t$$

with stochastic  $\sigma_t$ .

- Variance moves should be negatively autocorrelated and (anti-)correlated with index moves.

## Stochastic volatility examples

### The Heston model

$$\begin{aligned}\frac{dS_t}{S_t} &= \sqrt{v_t} dZ_t \\ dv_t &= -\kappa(v_t - \bar{v}) dt + \eta \sqrt{v_t} dW_t\end{aligned}$$

with  $\langle dZ_t dW_t \rangle = \rho dt$ .

### The SABR model (with $\beta = 1$ )

$$\begin{aligned}dS_t &= \sigma_t S_t^\beta dZ_t \\ d\sigma_t &= \alpha \sigma_t dW_t\end{aligned}$$

with  $\langle dZ_t dW_t \rangle = \rho dt$ .

- The Heston model is popular because there is a (complicated) closed-form expression for the characteristic function. It is therefore easy and fast to price European options. It follows that calibration to the market is fast.
- The SABR model is popular because there is a closed-form accurate asymptotic approximation to implied volatility. Again, calibration is easy.

## Variance curve models

Let  $x_t = \log S_t/F$  denote log-moneyness. If  $v_t = \sigma_t^2$  is instantaneous variance, the forward variance curve is given by

$$\xi_t(u) = \mathbb{E} [v_u | \mathcal{F}_t].$$

- Forward variances are conditional expectations and thus martingales in  $t$ .
- Forward variance swaps are effectively tradable in practice as variance swap spreads.
- Forward variances are thus natural fundamental objects to consider for volatility modeling.

Bergomi and Guyon consider models written in the following variance curve form:

$$\begin{aligned} dx_t &= -\frac{1}{2} \xi_t(t) dt + \sqrt{\xi_t(t)} dZ_t^1 \\ d\xi_t(u) &= \lambda(t, u, \xi_t) \cdot dZ_t, \quad \xi_0(u) = \xi(u). \end{aligned}$$

Here, the instantaneous forward variance curve at time  $t$  is represented by

$$\xi_t := \{ \xi_t(u) : u \geq t \}$$

and  $Z = \{Z_1, \dots, Z_d\}$  is a  $d$  – dimensional Brownian motion.

Obviously,  $\xi_t(t) = v_t$ .

## The Bergomi and Guyon expansion

Using a technique from quantum mechanics, [Bergomi and Guyon]<sup>[3]</sup> compute an expansion of the volatility smile up to second order in volatility of volatility for stochastic volatility models written in variance curve form.

The Bergomi-Guyon expansion of implied volatility takes the form

(2)

$$\sigma_{\text{BS}}(k, t) = \hat{\sigma}_T + S_T k + C_T k^2 + O(\epsilon^3)$$

Here

$$\begin{aligned}\hat{\sigma}_T &= \sqrt{\frac{w}{T}} \left\{ 1 + \frac{1}{4w} C^{x\xi} \right. \\ &\quad \left. + \frac{1}{32w^3} (12(C^{x\xi})^2 + w(w+4)C^{\xi\xi} + 4w(w-4)C^\mu) \right\} \\ S_T &= \sqrt{\frac{w}{T}} \left\{ \frac{1}{2w^2} C^{x\xi} + \frac{1}{8w^3} (4wC^\mu - 3(C^{x\xi})^2) \right\} \\ C_T &= \sqrt{\frac{w}{T}} \frac{1}{8w^4} (4wC^\mu + wC^{\xi\xi} - 6(C^{x\xi})^2)\end{aligned}$$

where  $w = \int_0^T \xi_0(s) ds$  is total variance to expiration  $T$ .

## Bergomi and Guyon correlation functionals

The various correlation functionals appearing in the BG expansion are:

$$\begin{aligned}C^{x\xi} &= \int_0^T dt \int_t^T du \frac{\mathbb{E}[dx_t d\xi_t(u)]}{dt} \\ C^{\xi\xi} &= \int_0^T dt \int_t^T ds \int_t^T du \frac{\mathbb{E}[d\xi_t(s) d\xi_t(u)]}{ds} \\ C^\mu &= \int_0^T dt \int_t^T du \frac{\mathbb{E}[dx_t dC_t^{x\xi}]}{dt}\end{aligned}$$

- $C^{x\xi}$  is an integral of the term structure of covariances between returns and forward variances.
- $C^{\xi\xi}$  is an integral of the variance curve autocovariance function.
- $C^\mu$  is somewhat more complicated. Something like the covariance between the skew and underlying returns.

In principle, we could compute these terms from the time series of implied volatility surfaces. The BG expansion gives us a direct correspondence between the implied volatility surface and the joint dynamics of the underlying and the implied volatilities.

## Example: The Heston model

It is straightforward to compute that

$$\xi_t(u) = (v_t - \bar{v}) e^{-\kappa(u-t)} + \bar{v}.$$

It follows that

$$d\xi_t(u) = e^{-\kappa(u-t)} \eta \sqrt{v_t} dW_t.$$

Then

$$\mathbb{E} [dx_t d\xi_t(u)] = \rho \eta v_t e^{-\kappa(u-t)} dt.$$

With  $v_0 = \bar{v}$  to simplify computations, we obtain

$$\begin{aligned} C^{x\xi} &= \rho \eta \bar{v} \int_0^T dt \int_t^T e^{-\kappa(u-t)} du \\ &= \rho \eta \bar{v} \int_0^T dt \int_t^T e^{-\kappa(u-t)} du \end{aligned}$$

## Term structure of ATM skew in the Heston model

Define the at-the-money (ATM) volatility skew

$$\psi(T) = \partial_k \sigma_{BS}(k, T)|_{k=0}$$

It follows from (2) (again with  $v_0 = \bar{v}$ ) that to first order in  $\eta$ ,

$$\begin{aligned} \psi(T) = \mathcal{S}_T &= \sqrt{\frac{w}{T}} \frac{1}{2w^2} C^{x\xi} \\ &= \frac{\rho \eta}{2\sqrt{\bar{v}}} \frac{1}{\kappa T} \left\{ 1 - \frac{1 - e^{-\kappa T}}{\kappa T} \right\}. \end{aligned}$$

- In the Heston model,
  - $\psi(T) \rightarrow$  a constant as  $T \rightarrow 0$ ,
  - $\psi(T) \sim \frac{1}{T}$  as  $T \rightarrow \infty$ .

## Example: The SABR model

$\mathbb{E} [\sigma_u | \mathcal{F}_t] = \sigma_t$  so to first order in the volatility of volatility  $\alpha$ ,

$$\xi_t(u) = v_t.$$

It follows that, to first order again,

$$d\xi_t(u) = 2 \sigma_t d\sigma_t = 2 \alpha \sigma_t^2 dW_t.$$

Then

$$\mathbb{E} [dx_t d\xi_t(u)] = 2 \rho \alpha \sigma_t^{3/2} dt.$$

Again to first order in  $\alpha$ , we obtain

$$C^{x\xi} = 2 \rho \alpha \sigma_0^{3/2} \int_0^T dt \int_t^T du = \rho \alpha \sigma_0^{3/2} T^2$$

and so

$$\psi(T) = S_T = \frac{\rho \alpha}{2}.$$

- $\psi(T)$  is independent of  $T$  in the SABR model!
  - There is no mean reversion and so no term structure of ATM skew.

## The Bergomi model

The  $n$ -factor Bergomi variance curve model<sup>[1]</sup> reads:

(3)

$$\xi_t(u) = \xi_0(u) \exp \left\{ \sum_{i=1}^n \eta_i \int_0^t e^{-\kappa_i (t-s)} dW_s^{(i)} + \text{drift} \right\}.$$

The Bergomi model generates a term structure of volatility skew  $\psi(T)$  that looks something like

$$\psi(\tau) \sim \sum_i \frac{1}{\kappa_i T} \left\{ 1 - \frac{1 - e^{-\kappa_i T}}{\kappa_i T} \right\}.$$

- This functional form is related to the term structure of the functional  $C^{x\xi}$ .
  - Which is in turn driven by the exponential kernel in the exponent in (3).

## Hedging European options

To hedge options using the Black-Scholes formula (say), market makers need to hedge two effects:

- The explicit spot effect

$$\frac{\partial C}{\partial \sigma} \delta S$$

and

- The change in implied volatility conditional on a change in the spot

$$\frac{\partial C}{\partial \sigma} \mathbb{E} [\delta \sigma | \delta S].$$

ATM implied volatilities  $\sigma(T)$  and stock prices are both observable so market makers can estimate the second component using a simple regression:

$$\delta\sigma(T) = \alpha + \beta(T) \frac{\delta S}{S} + \text{noise}.$$

Then  $\beta(T) = \mathbb{E} [\delta\sigma(T) | \delta S/S]$ .

## The skew-stickiness ratio

[Bergomi]<sup>[2]</sup> calls

$$\mathcal{R}(T) = \frac{\beta(T)}{\psi(T)}$$

the *skew-stickiness ratio* or *SSR*.

In the old days, traders would typically make one of two assumptions:

- **Sticky strike**  $\mathcal{R}(T) = 0$  where the implied volatility for a fixed strike is independent of stock price

or

- **Sticky delta**  $\mathcal{R}(T) = 1$  where the ATM volatility is independent of stock price.

Listed options were thought of as sticky strike and OTC options as sticky delta.

- **Empirically**,  $\mathcal{R}(T) \approx 1.5$ , independent of  $T$ .

## Regress volatility changes vs spot returns

Let's check the skew-stickiness ratio over the period June 1, 2010 to June 1, 2011, reproducing a figure from an article in the Encyclopedia of Quantitative Finance<sup>[7]</sup>.

In [21]:

```
%%R

library(stinepack)
load("spxAtmVolSkew2010.rData")

# First we need the time series of SPX returns:
spx2010 <- spx["2010-06-01::2011-06-01"]
ret.spx2010 <- diff(log(as.numeric(spx2010)))

n <- length(volSkewList2010)

vol.res <- array(dim=c(n,8))

for (i in 1:n){
  dat <- volSkewList2010[[i]]
  vol.res[i,1:4] <- stinterp(x=dat$texp,y=dat$atmVol,xout=c(1,3,6,12)/12)$y
  vol.res[i,5:8] <- stinterp(x=dat$texp,y=dat$atmSkew,xout=c(1,3,6,12)/12)$y
}

vol.skew.atm <- as.data.frame(vol.res)
colnames(vol.skew.atm) <- c("vol.1m","vol.3m","vol.6m","vol.12m","skew.1m","skew
.3m","skew.6m","skew.12m")

print(head(vol.skew.atm))

# Finally, create matrix of volatility changes
del.vol <- apply(vol.skew.atm[,1:4],2,function(x){diff(x)})
```

	vol.1m	vol.3m	vol.6m	vol.12m	skew.1m	skew.3m	skew.6m
1	0.3116781	0.2892093	0.2840357	0.2848998	-0.9775653	-0.6994318	-0.5033279
2	0.2585816	0.2569657	0.2637964	0.2674395	-1.0391103	-0.6308754	-0.5047893
3	0.2593993	0.2532371	0.2590840	0.2633074	-1.0543336	-0.6167919	-0.4882985
4	0.3217950	0.2916457	0.2932080	0.2841629	-0.9936603	-0.6787185	-0.5012572
5	0.3240469	0.2982968	0.2859462	0.2874102	-1.0006733	-0.6137535	-0.4395970
6	0.2963918	0.2821198	0.2833307	0.2822569	-1.1808506	-0.6769495	-0.5147833
	skew.12m						
1	-0.3469127						
2	-0.3334118						
3	-0.3472150						
4	-0.3454572						
5	-0.3411661						
6	-0.3498540						

**1-month SSR**

In [22]:

```
##R

y <- del.vol[,1]
x <- ret.spx2010*vol.skew.atm[-n,5]
fit.lm1 <- lm(y~x)
print(summary(fit.lm1))

plot(x,y,xlab=expression(psi(tau)*delta*x),ylab=expression(Delta*sigma ),main="1
m SSR",pch=20,col="blue")
abline(fit.lm1,col="red",lwd=2)
text(x=0.025,y=-0.0,"Slope is 1.47")
abline(coef=c(0,1),lty=2,lwd=2,col="green4")
```

Call:

```
lm(formula = y ~ x)
```

Residuals:

Min	1Q	Median	3Q	Max
-0.039160	-0.004263	0.000181	0.004642	0.024866

Coefficients:

	Estimate	Std. Error	t value	Pr(> t )
(Intercept)	0.000265	0.000466	0.569	0.57
x	1.474092	0.054832	26.884	<2e-16 ***

---

Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.007394 on 251 degrees of freedom

Multiple R-squared: 0.7422, Adjusted R-squared: 0.7412

F-statistic: 722.7 on 1 and 251 DF, p-value: < 2.2e-16



### 1m SSR

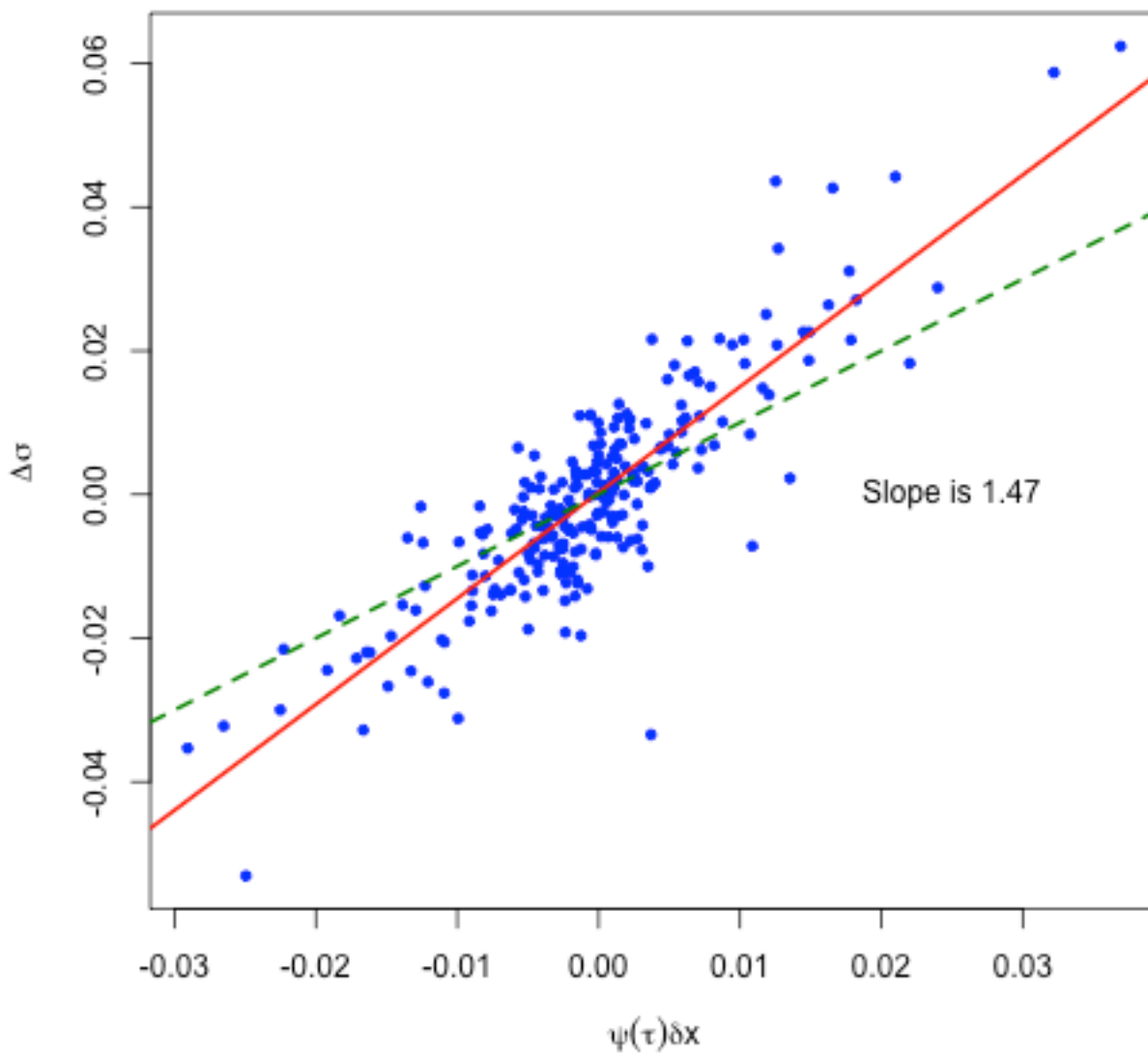


Figure 13: The 1-month skew-stickness ratio (SSR). The "sticky strike" green line with slope 1 clearly doesn't fit.

### 3-month SSR

In [23]:

```
##R

y <- del.vol[,2]
x <- ret.spx2010*vol.skew.atm[-n,6]
fit.lm3 <- lm(y~x)
print(summary(fit.lm3))

plot(x,y,xlab=expression(psi(tau)*delta*x),ylab=expression(Delta*sigma ),main="3
m SSR",pch=20,col="blue")
abline(fit.lm3,col="red",lwd=2)
text(x=0.015,y=-0.0,"Slope is 1.45")
abline(coef=c(0,1),lty=2,lwd=2,col="green4")
```

Call:  
lm(formula = y ~ x)

Residuals:  
Min 1Q Median 3Q Max  
-0.0180521 -0.0023352 -0.0000552 0.0024244 0.0106338

Coefficients:  
Estimate Std. Error t value Pr(>|t|)  
(Intercept) 0.0001939 0.0002445 0.793 0.428  
x 1.4509581 0.0432965 33.512 <2e-16 \*\*\*

---  
Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.003876 on 251 degrees of freedom  
Multiple R-squared: 0.8173, Adjusted R-squared: 0.8166  
F-statistic: 1123 on 1 and 251 DF, p-value: < 2.2e-16

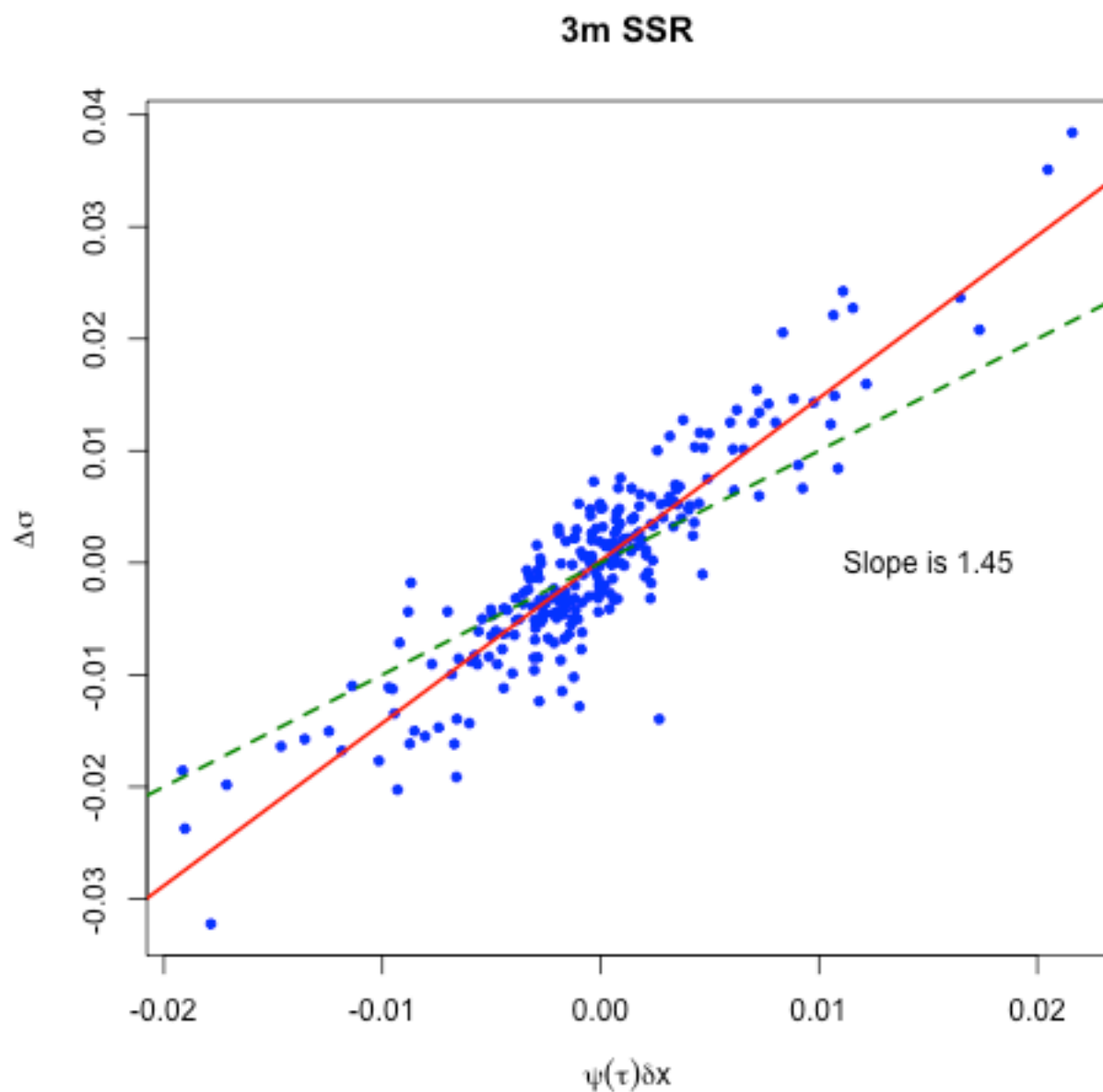


Figure 14: The 3-month skew-stickiness ratio (SSR). The "sticky strike" green line with slope 1 clearly doesn't fit.

## 6-month SSR

In [24]:

```
%%R

y <- del.vol[,3]
x <- ret.spx2010*vol.skew.atm[-n,7]
fit.lm6 <- lm(y~x)
print(summary(fit.lm6))

plot(x,y,xlab=expression(psi(tau)*delta*x),ylab=expression(Delta*sigma ),main="6
m SSR",pch=20,col="blue")
abline(fit.lm6,col="red",lwd=2)
text(x=0.01,y=-0.0,"Slope is 1.512")
abline(coef=c(0,1),lty=2,lwd=2,col="green4")
```

Call:

```
lm(formula = y ~ x)
```

Residuals:

	Min	1Q	Median	3Q	Max
	-0.0176677	-0.0017325	-0.0000075	0.0016376	0.0089884

Coefficients:

	Estimate	Std. Error	t value	Pr(> t )
(Intercept)	8.003e-05	1.894e-04	0.423	0.673
x	1.512e+00	4.522e-02	33.436	<2e-16 ***

---

Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.003004 on 251 degrees of freedom

Multiple R-squared: 0.8166, Adjusted R-squared: 0.8159

F-statistic: 1118 on 1 and 251 DF, p-value: < 2.2e-16

## 6m SSR

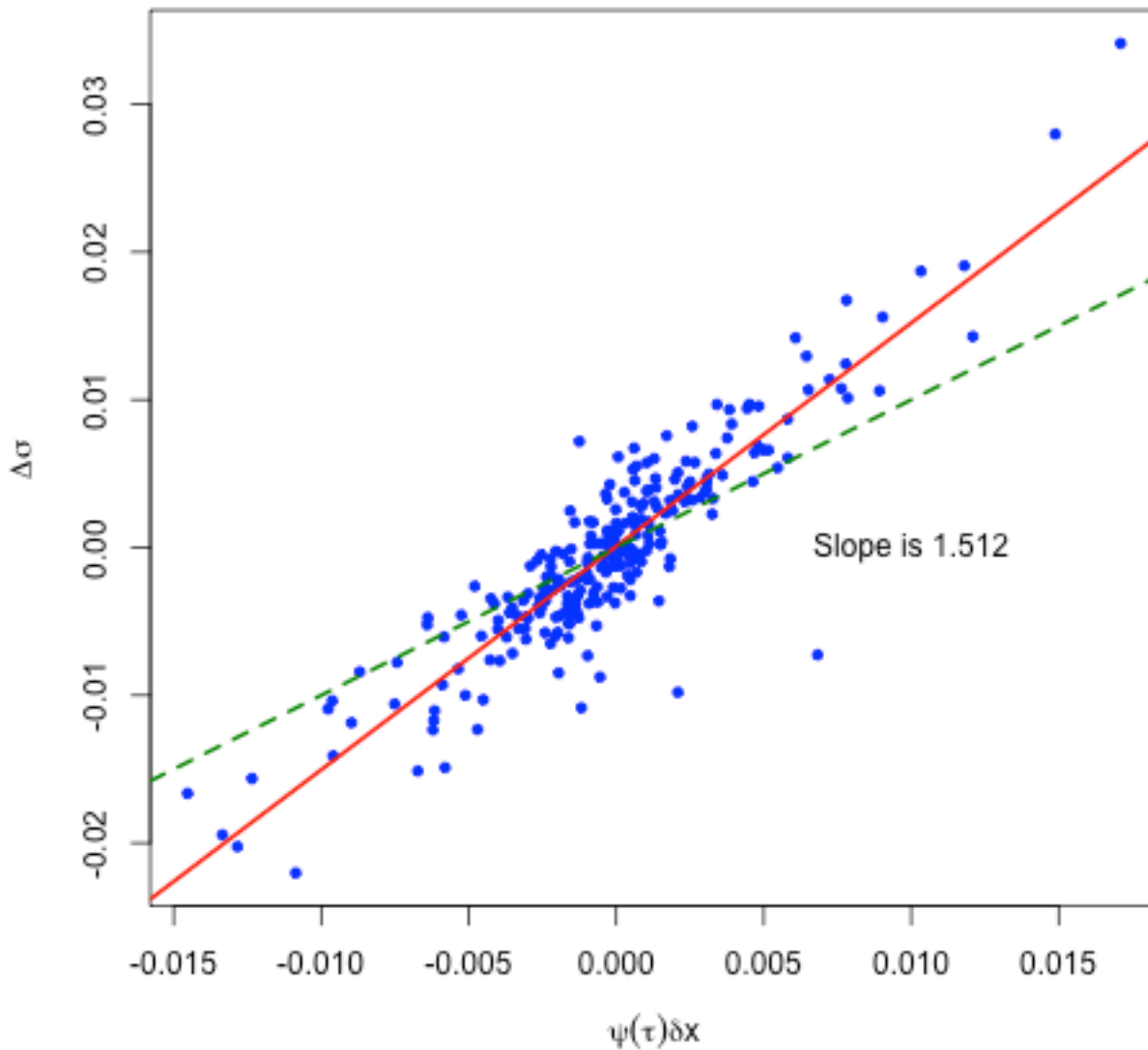


Figure 15: The 6-month skew-stickiness ratio (SSR). The "sticky strike" green line with slope 1 clearly doesn't fit.

## 12-month SSR

In [25]:

```
%%R

y <- del.vol[,4]
x <- ret.spx2010*vol.skew.atm[-n,8]
fit.lm12 <- lm(y~x)
print(summary(fit.lm12))

plot(x,y,xlab=expression(psi(tau)*delta*x),ylab=expression(Delta*sigma ),main="1
2m SSR",pch=20,col="blue")
abline(fit.lm12,col="red",lwd=2)
text(x=0.007,y=0.0,"Slope is 1.601")
abline(coef=c(0,1),lty=2,lwd=2,col="green4")
```

Call:  
lm(formula = y ~ x)

Residuals:  
Min 1Q Median 3Q Max  
-0.007310 -0.001219 0.000014 0.001259 0.008139

Coefficients:  
Estimate Std. Error t value Pr(>|t|)  
(Intercept) 6.517e-05 1.401e-04 0.465 0.642  
x 1.601e+00 4.407e-02 36.343 <2e-16 \*\*\*

---  
Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.002221 on 251 degrees of freedom  
Multiple R-squared: 0.8403, Adjusted R-squared: 0.8397  
F-statistic: 1321 on 1 and 251 DF, p-value: < 2.2e-16

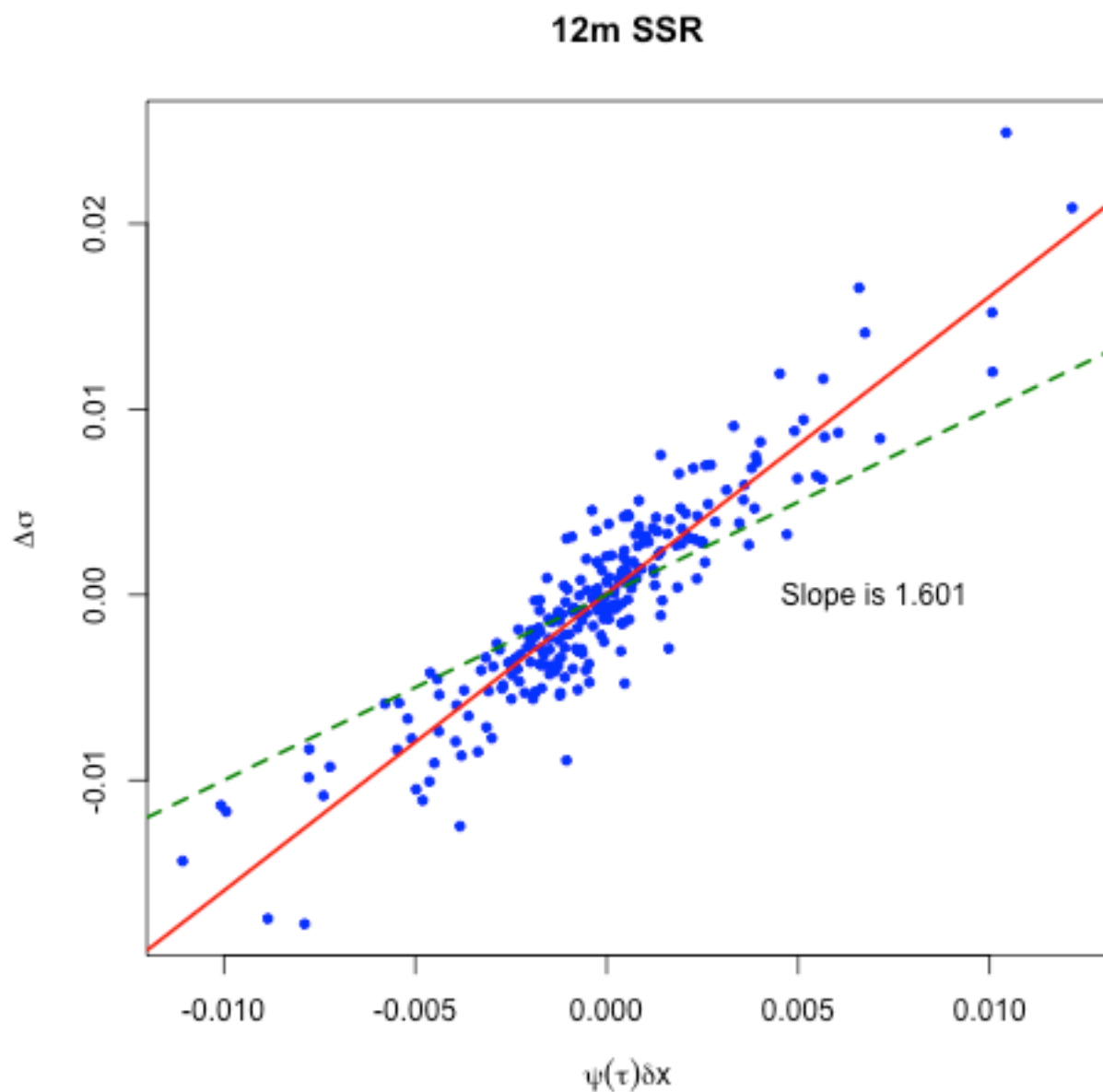


Figure 16: The 12-month skew-stickiness ratio (SSR). The "sticky strike" green line with slope 1 clearly doesn't fit.

## Computation of the regression coefficient $\beta(T) = \mathbb{E} [\delta\sigma(T)|\delta S/S]$

We approximate ATM volatility using the variance swap.

- By definition this is

$$\mathcal{V}_t(T) = \int_t^T \xi_t(u) du.$$

Then

$$2 \sigma(T) \delta\sigma(T) T \approx \int_t^T \delta\xi_t(u) du.$$

Thus

$$\beta(T) = \mathbb{E} [\delta\sigma(T)|\delta S/S] \approx \frac{1}{2 \sigma(T) T} \int_t^T \mathbb{E} \left[ \delta\xi_t(u) \left| \frac{\delta S}{S} \right. \right] du.$$

Also,

$$\mathbb{E} \left[ \delta\xi_t(u) \left| \frac{\delta S}{S} \right. \right] \approx \frac{\mathbb{E} [\delta\xi_t(u) \delta x_t]}{\mathbb{E} [\delta x_t^2]} = \frac{\rho(t, u)}{v_t}$$

with  $x_t = \log S_t$  and

$$\rho(t, u) = \frac{\mathbb{E} [d\xi_t(u) dx_t]}{dt}.$$

## $\beta(T)$ and $C^{x\xi}$

Then, restoring explicit dependence on  $t$  and  $T$ ,

$$C_t^{x\xi}(T) = \int_t^T ds \int_s^T \rho(s, u) du$$

and

$$\begin{aligned} \beta(T) &\approx \frac{1}{2 \sigma(T) T} \frac{1}{v_t} \int_t^T \rho(t, u) du \\ &\approx - \frac{1}{2 \sigma(T) T} \frac{1}{v_t} \frac{\partial C_t^{x\xi}(T)}{\partial t}. \end{aligned}$$

Assuming time-homogeneity,  $C^{x\xi}$  would be a function of  $\tau = T - t$  only and  $v_t \approx \sigma^2$  and so we further approximate

$$\beta(\tau) \approx \frac{1}{2\sigma^3\tau} \frac{\partial C^{x\xi}(\tau)}{\partial \tau}.$$

Also, from the Bergomi-Guyon expansion, the ATM skew is given by

$$\psi(\tau) \approx \sqrt{\frac{w}{T}} \frac{1}{2w^2} C_t^{x\xi}(T) \approx \frac{1}{2\sigma^3} \frac{1}{\tau^2} C^{x\xi}(\tau).$$

Thus

$$\mathcal{R}(\tau) = \frac{\beta(\tau)}{\psi(\tau)} \approx \tau \frac{d}{d\tau} \log C^{x\xi}(\tau).$$

For emphasis, to first order in the volatility of volatility, the SSR  $\mathcal{R}$  is given by the time derivative of the spot-volatility correlation functional  $C^{x\xi}$ .

## SSR examples

### The SABR model

In the SABR model,  $C^{x\xi}(\tau) \propto \tau^2$  so  $\mathcal{R}(\tau) \approx 2$ .

### The Heston model

$$C^{x\xi}(\tau) = \rho \eta \bar{v} \int_0^\tau dt \int_t^\tau e^{-\kappa(u-t)} du$$

- For  $\tau \ll 1/\kappa$ ,  $C^{x\xi}(\tau) \sim \tau^2$  and  $\mathcal{R}(\tau) \approx 2$ .
- For  $\tau \gg 1/\kappa$ ,  $C^{x\xi}(\tau) \sim \tau$  and  $\mathcal{R}(\tau) \approx 1$ .

### The $n$ -factor Bergomi model

Let  $\kappa_1$  be the shortest timescale (largest) mean reversion coefficient and  $\kappa_n$  be the longest timescale (smallest) mean reversion coefficient.

- For  $\tau \ll 1/\kappa_1$ ,  $C^{x\xi}(\tau) \sim \tau^2$  and  $\mathcal{R}(\tau) \approx 2$ .
- For  $\tau \gg 1/\kappa_n$ ,  $C^{x\xi}(\tau) \sim \tau$  and  $\mathcal{R}(\tau) \approx 1$ .

For stochastic volatility models in general,  $\mathcal{R}(\tau) \approx 2$  for  $\tau$  small and  $\mathcal{R}(\tau) \approx 1$  for  $\tau$  large.

## Implication for the "true" model

Empirically, we see  $\mathcal{R}(\tau) \sim \frac{3}{2}$  for all  $\tau$ . That is

$$\mathcal{R}(\tau) \approx \tau \frac{d}{d\tau} \log C^{x\xi}(\tau) \approx 2 - \gamma.$$

with  $\gamma \approx 1/2$ . Thus  $C^{x\xi}(\tau) \sim \tau^{2-\gamma}$ , and so  $\rho(t, u) \sim (u - t)^{-\gamma}$ .

An obvious model that would generate  $\rho(t, u) \sim (u - t)^{-\gamma}$  is

$$\frac{d\xi_t(u)}{\xi_t(u)} \propto \frac{dW_t}{(u - t)^\gamma}.$$

- In effect, replace all the exponential kernels in the Bergomi model (3) with a power-law kernel.
- Such a model would be non-Markovian. The price of an option would depend on the entire history  $\left\{W_s, s\right\}$
- To be consistent with  $\mathcal{R}(\tau) \sim \frac{3}{2}$ , we would need  $\gamma \approx \frac{1}{2}$ .

In Session 6, we will see that scaling properties of the time series of realized variance also suggest such a model.

## Observed term structure of ATM volatility skew

We study a period of history over which the ATM skew was relatively stable.



In [26]:

```
##R

vsl <- volSkewList2010
n <- length(names(vsl))

mycol <- rainbow(n)

plot(vsl[[1]]$texp,abs(vsl[[1]]$atmSkew),col=mycol[1],pch=20,cex=0.1,xlim=c(0,2.6),
      xlab=expression(paste("Expiration ",tau)),ylab=expression(psi(tau)))
for (i in 2:n){
  points(vsl[[i]]$texp,abs(vsl[[i]]$atmSkew),col=mycol[i],pch=20,cex=0.1)
}
```

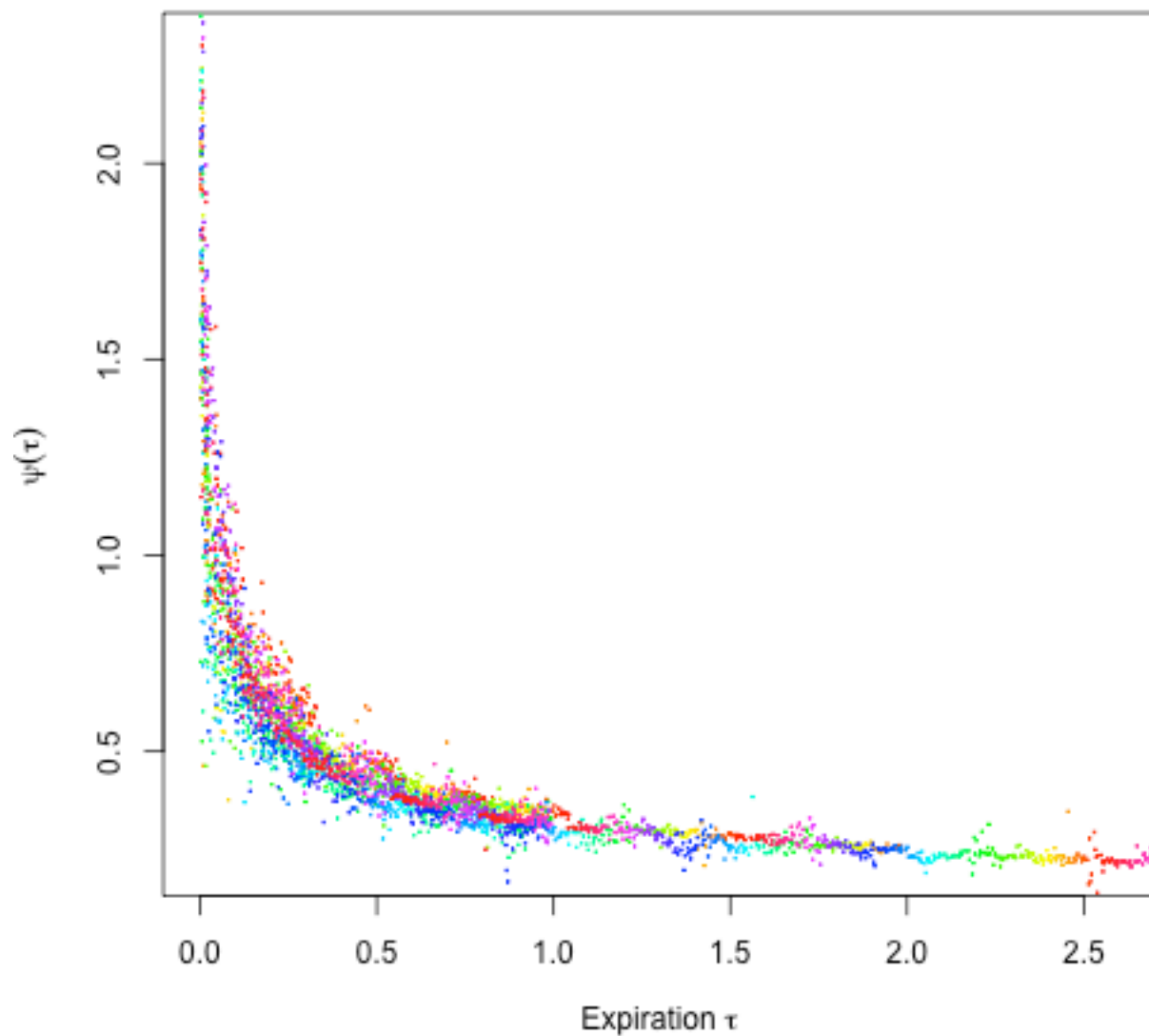


Figure 17: Decay of ATM skew (red dots) with respect to time to expiration. Data is SPX from 01-Jun-2010 to 01-Jun-2011.

**A shorter even more stable period**

In [27]:

```
%%R
vsl <- volSkewList2010[201:250]
n <- length(names(vsl))

plot(vsl[[1]]$texp,abs(vsl[[1]]$atmSkew),col="red",pch=20,cex=0.1,xlim=c(0,2.6),
      xlab=expression(paste("Expiration ",tau)),ylab=expression(psi(tau)))
for (i in 2:n){
  points(vsl[[i]]$texp,abs(vsl[[i]]$atmSkew),col="red",pch=20,cex=0.1)
}
```

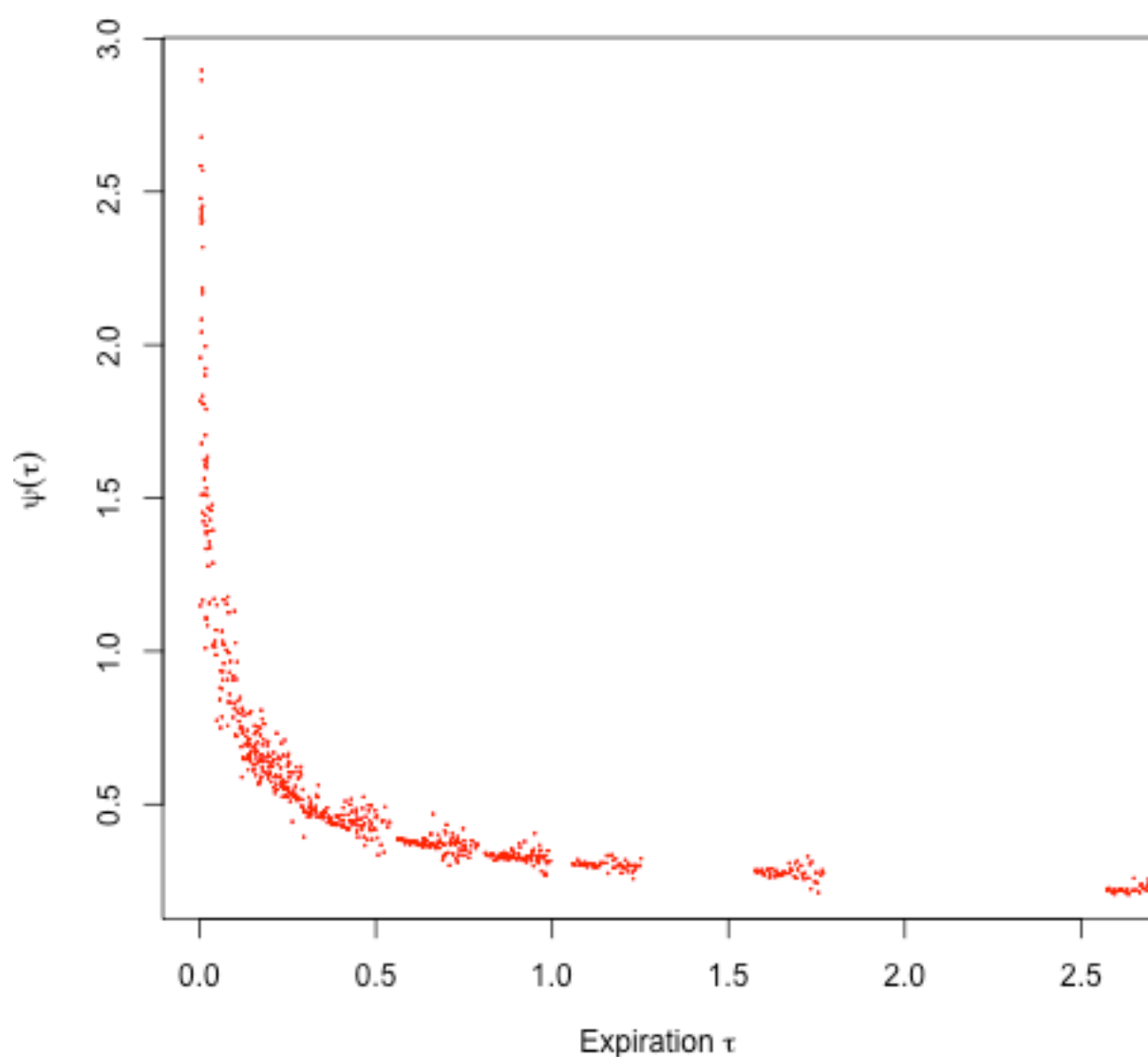


Figure 18: Decay of ATM skew (red dots) with respect to time to expiration. Data is SPX from 16-Mar-2011 to 25-May-2011.

## Log-log plot of empirical ATM skew

In [28]:

```
##R

lvsl.texp <- log(vsl[[1]]$texp)
lvsl.atmSkew <- log(abs(vsl[[1]]$atmSkew))

for (i in 2:n){
  lvsl.texp <- c(lvsl.texp,log(vsl[[i]]$texp))
  lvsl.atmSkew <- c(lvsl.atmSkew,log(abs(vsl[[i]]$atmSkew)))
}

plot(lvsl.texp,lvsl.atmSkew,col="red",pch=20,cex=0.5,
      xlab=expression(paste("log ",tau)),ylab=expression(paste("log ", psi(tau)))
)

pick <- (lvsl.texp > -3)

print(fit.lm <- lm(lvsl.atmSkew[pick] ~ lvsl.texp[pick]))
abline(fit.lm,col="blue",lwd=2)
```

Call:

```
lm(formula = lvsl.atmSkew[pick] ~ lvsl.texp[pick])
```

Coefficients:

(Intercept)	lvsl.texp[pick]
-1.1332	-0.3983

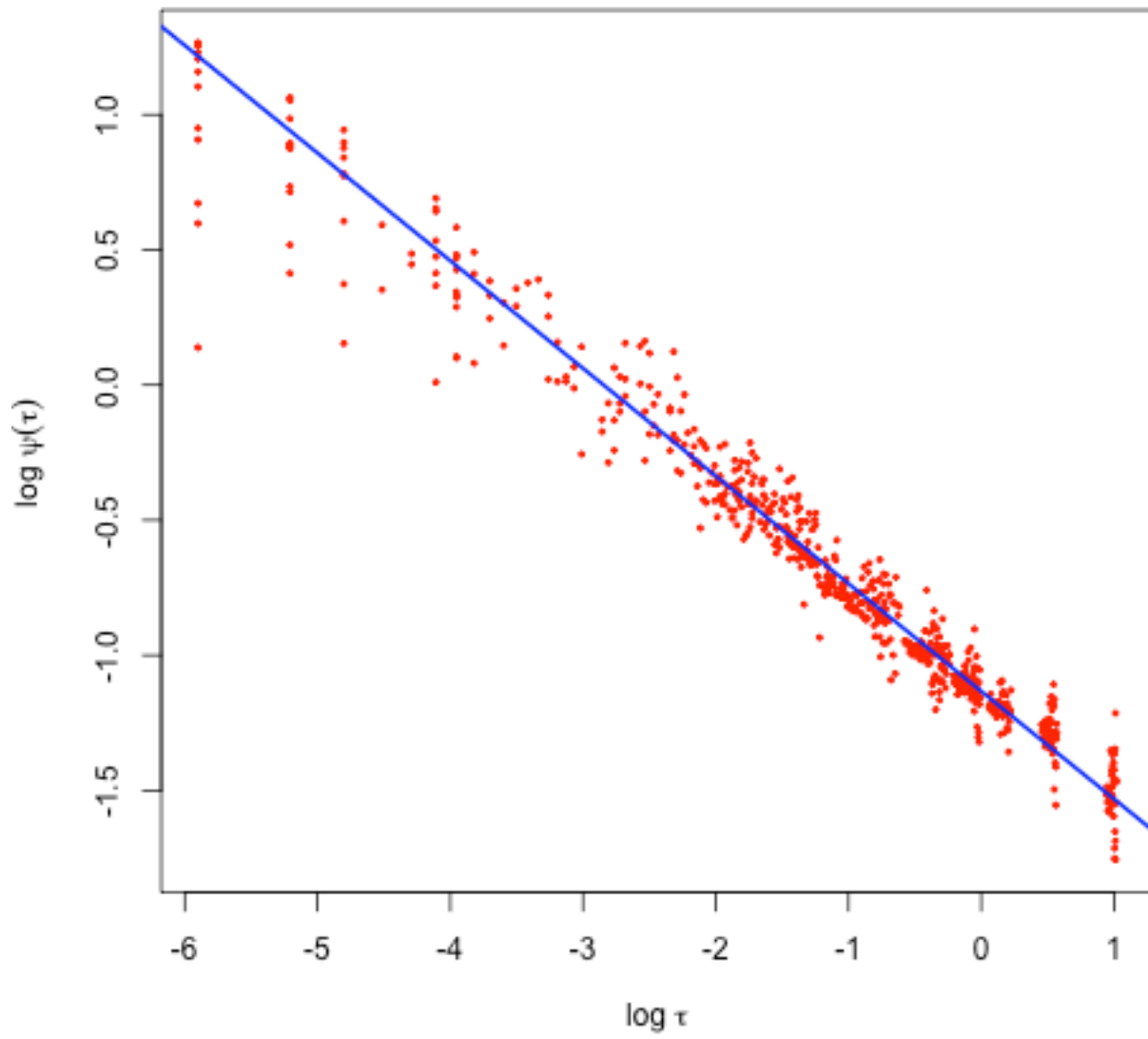


Figure 19: Log-log plot of decay of ATM skew (red dots) with respect to time to expiration.

### Plot of ATM skew with power-law fit

In [29]:

```
##R
plot(vsl[[1]]$texp,abs(vsl[[1]]$atmSkew),col="red",pch=20,cex=0.1,xlim=c(0,2.6),
      xlab=expression(paste("Expiration ",tau)),ylab=expression(psi(tau)))
for (i in 2:n){
  points(vsl[[i]]$texp,abs(vsl[[i]]$atmSkew),col="red",pch=20,cex=0.1)
}

a <- fit.lm$coef[1]; b <- fit.lm$coef[2]
curve(exp(a+b*log(x)),from=0,to=3,col="blue",add=T,n=1000,lwd=2)
```

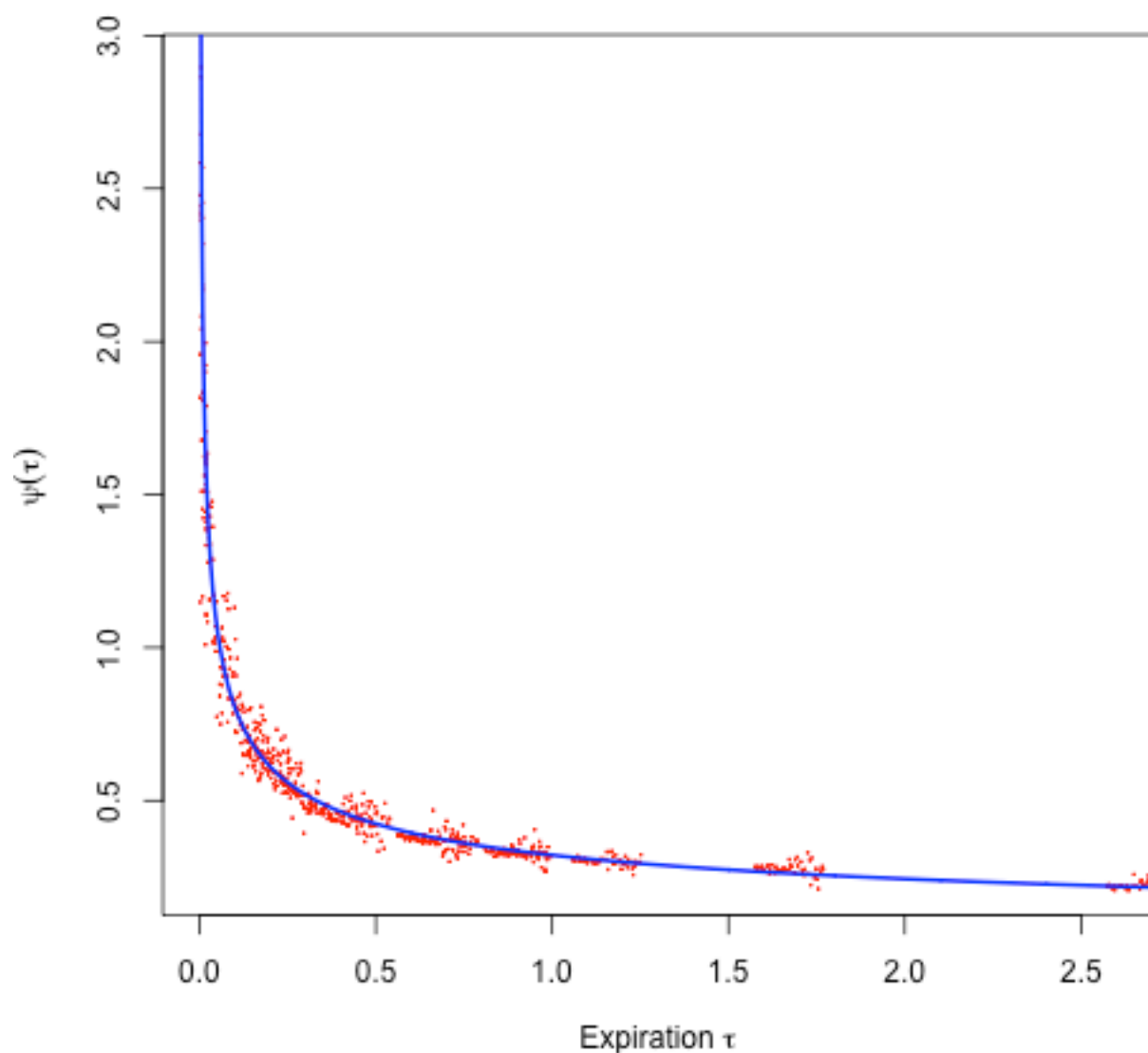


Figure 20: Log-log plot of decay of ATM skew (red dots) with respect to time to expiration. Power-law fit  $\psi(\tau) \sim \tau^{-.3983}$  in blue. Data is SPX from 16-Mar-2011 to 25-May-2011.

## The variance swap curve

Assuming the underlying diffuses, the fair value  $\mathcal{V}_t(T)$  of a variance swap with maturity  $T$  may be computed as follows:

$$\mathcal{V}_t(T) = \int_t^T \xi_t(u) du = \int_0^\infty \min[P(K, T), C(K, T)] \frac{dK}{K^2}$$

where  $C(K, T)$  and  $P(K, T)$  denote the prices of calls and puts with strike  $K$  and expiration  $T$  respectively. For details, see for example [The Volatility Surface]<sup>[5]</sup>. We will visit this again in Session 5.

Thus, with this continuity assumption, we may compute the fair value of a variance swap from an infinite strip of call and put option prices (the so-called log-strip).

- For a given maturity, the variance swap level depends on exactly how we interpolate and extrapolate option prices.
- We use the *arbitrage-free SVI parameterization* that I will explain in Session 3.
- In particular, we will analyze variance swap estimates from June 01, 2010 to June 10, 2011, the same period as before.

In [30]:

```
##R

load("spxVarSwapList2010.rData")

n <- length(names(varSwapList2010))

# Convert list to matrix
tmp <- array(dim=c(n,40))

for (i in 1:n){
  tmp[i,] <- varSwapList2010[[i]]$varSwap
}

varswap.mean <- apply(tmp,2,mean)
varswap.sd <- apply(tmp,2,sd)
tmat <- (1:40)*.05
```

## The average shape of the variance swap curve

In [31]:

```
%%R
```

```
plot(tmat, sqrt(varswap.mean), type="b", col="red", xlab=expression(paste("Maturity", tau)), ylab="Variance swap quote")
```

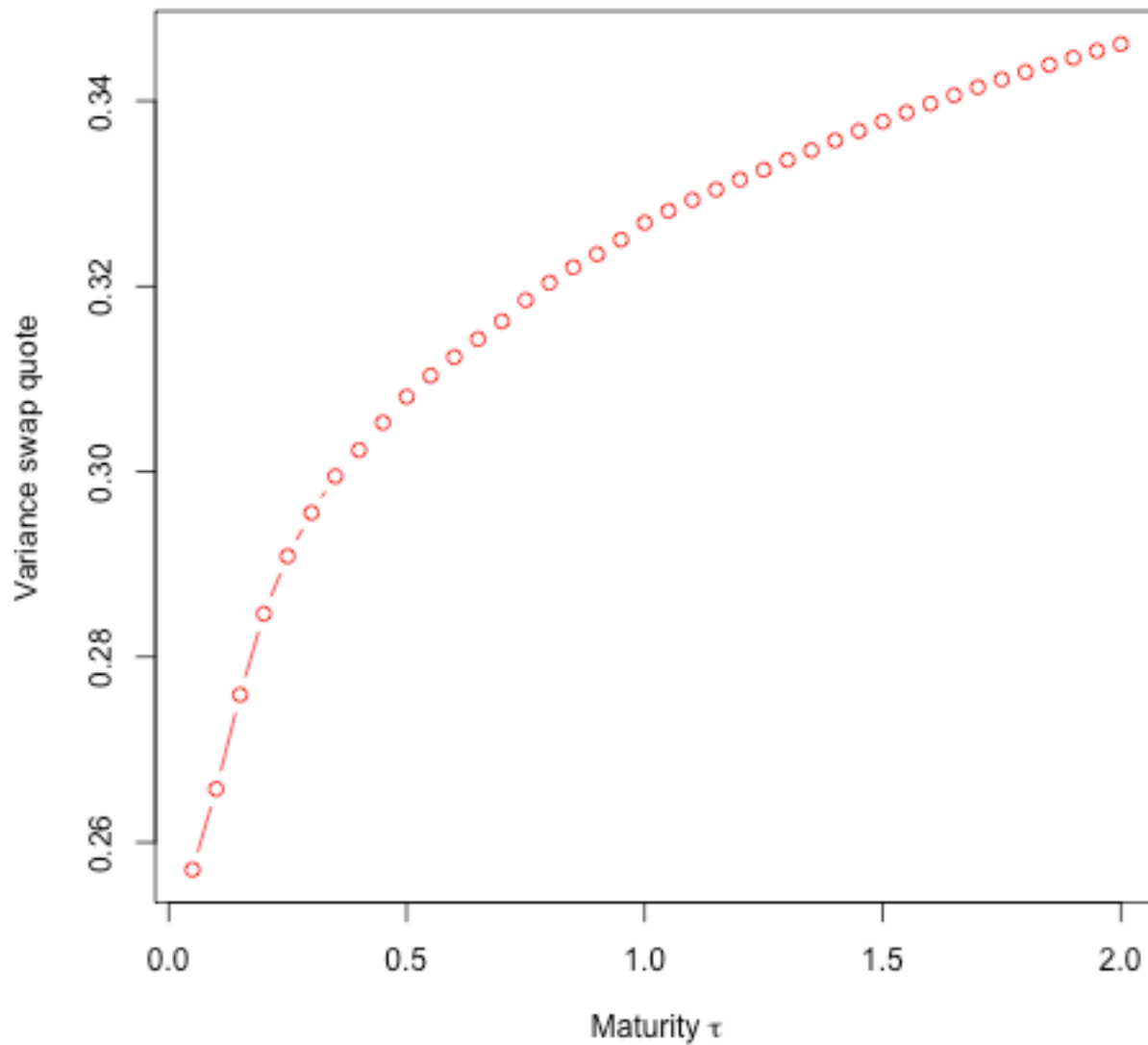


Figure 21: Average shape of the variance swap curve from 01-Jun-2010 to 01-Jun-2011.

## The volatility envelope

The *envelope* is the graph of volatilities of variance swap quotes vs time to maturity. First, we draw a log-log plot of standard deviation of log-differences of the curves.

In [32]:

```
##R

# Compute standard deviation of log-differences
sd.t <- function(x){sd(diff(log(x)))}

varswap.sd.t <- apply(tmp,2,sd.t)

# Log-log plot
x <- log(tmat)
y <- log(varswap.sd.t)
plot(x,y,col="red")
points(x[1:20],y[1:20],col="blue",pch=20)

fit.lm2 <- lm(y[1:20]~x[1:20])
print(summary(fit.lm2))
abline(fit.lm2,col="orange",lwd=2)
a2 <- fit.lm2$coef[1]; b2 <- fit.lm2$coef[2]
```

Call:

```
lm(formula = y[1:20] ~ x[1:20])
```

Residuals:

	Min	1Q	Median	3Q	Max
	-0.046099	-0.015713	0.004174	0.014588	0.048747

Coefficients:

	Estimate	Std. Error	t value	Pr(> t )	
(Intercept)	-2.885944	0.008890	-324.6	<2e-16	***
x[1:20]	-0.365137	0.007513	-48.6	<2e-16	***

---

Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.02662 on 18 degrees of freedom

Multiple R-squared: 0.9924, Adjusted R-squared: 0.992

F-statistic: 2362 on 1 and 18 DF, p-value: < 2.2e-16



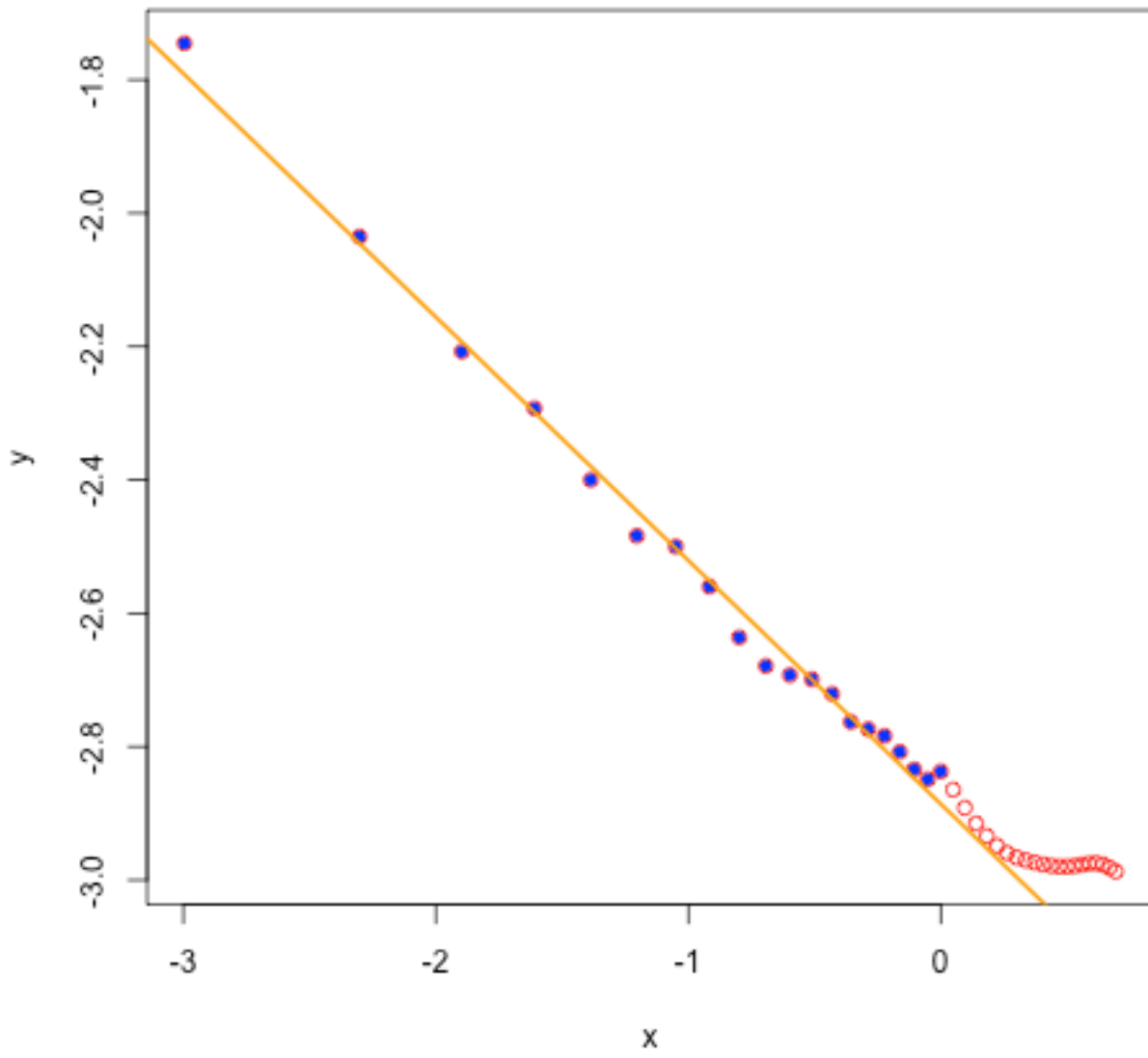


Figure 22: Log-log plot of volatility envelope with linear fit.

## Variance swap envelope with power-law fit

In [33]:

```
%%R
```

```
plot(tmat, varswap.sd.t, col="red", pch=20, xlab=expression(paste("Maturity ", tau)),  
ylab="sd(Variance swap quote)", ylim=c(0.04, .2))  
curve(exp(a2+b2*log(x)), from=0, to=3, col="blue", add=T, n=1000, lwd=2)
```

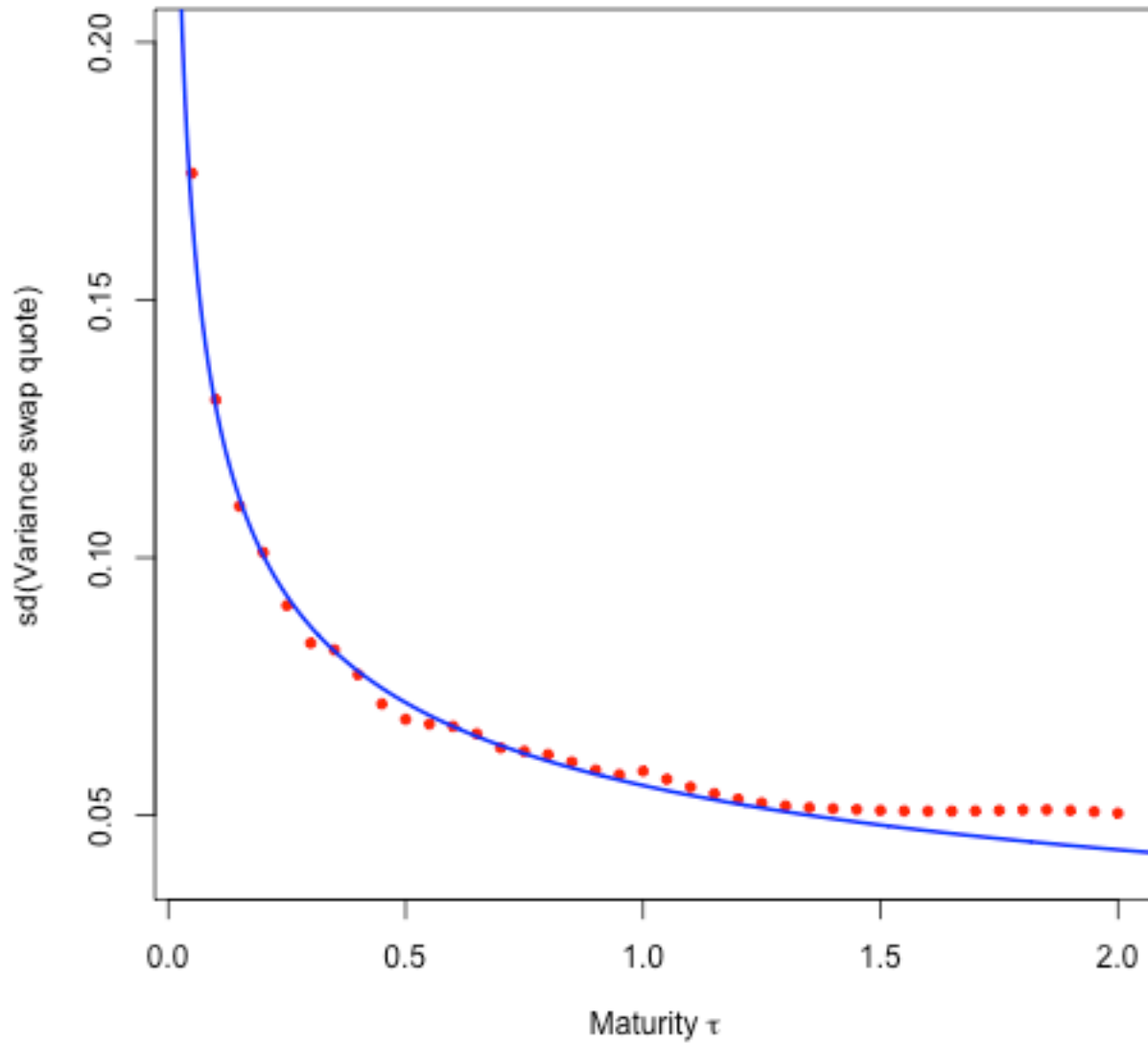


Figure 23: Variance swap envelope from 01-Jun-2010 to 01-Jun-2011. The blue line corresponds to the fit  $\tau^{-0.365}$ .

## Hand-waving computation

Consider once again a model of the form

$$\frac{d\xi_t(u)}{\xi_t(u)} \propto \frac{dW_t}{(u-t)^\gamma}.$$

Then

$$d\mathcal{V}_t(T) = \int_t^T d\xi_t(u) du \sim \int_t^T \xi_t(u) \frac{dW_t}{(u-t)^\gamma} du.$$

Then

$$\mathbb{E} [d\mathcal{V}_t(T)^2] \sim \left( \int_t^T \xi_t(u) \frac{1}{(u-t)^\gamma} du \right)^2 dt \sim \mathcal{V}_t(T)^2 (T-t)^{2-2\gamma}.$$

The standard deviation of log-differences computed above should then scale as

$$\left( \frac{1}{\tau^2} \tau^{2-2\gamma} \right)^{1/2} = \frac{1}{\tau^\gamma}.$$

- the same scaling as that of the ATM volatility skew
- a relationship also confirmed in the data.

## Moral of the story

Conventional stochastic volatility models are *normative*.

- We write down underlying dynamics as if to say “suppose the underlying stochastic drivers were to satisfy the following...”.
- Dynamics are invariably Markovian, in contrast to the real world.
- The state space is typically very small.

Conventional models of volatility are engineering models, not physics models.

- Conventional stochastic volatility models are Markovian approximations to a non-Markovian reality.

## Motivating observation

A Bergomi model with dynamics of the form

$$\frac{d\xi_t(u)}{\xi_t(u)} \propto \frac{dW_t}{(u-t)^\gamma}$$

might be consistent with our empirical observations.

# References

1. <sup>^</sup> Lorenzo Bergomi, Smile dynamics II, *Risk Magazine* 67-73 (October 2005).
2. <sup>^</sup> Lorenzo Bergomi, Smile dynamics IV, *Risk Magazine* 94-100 (December 2009).
3. <sup>^</sup> Lorenzo Bergomi and Julien Guyon, Stochastic volatility's orderly smiles. *Risk Magazine* 60-66, (May 2012).
4. <sup>^</sup> Rama Cont, Empirical properties of asset returns: stylized facts and statistical issues, *Quantitative Finance* **1** 223-236 (2001).
5. <sup>^</sup> Jim Gatheral, *The Volatility Surface: A Practitioner's Guide*, John Wiley and Sons, Hoboken, NJ (2006).
6. <sup>^</sup> Jim Gatheral and Roel Oomen, Zero-intelligence realized variance estimation, *Finance and Stochastics* **14**(2) 249-283 (2010).
7. <sup>^</sup> Michael Kamal and Jim Gatheral, Implied Volatility Surface, *Encyclopedia of Quantitative Finance*, Wiley (2010).