

Black-Scholes goes hypergeometric

Claudio Albanese, Giuseppe Campolieti, Peter Carr and Alexander Lipton introduce a general pricing formula that extends Black-Scholes and contains as particular cases most analytically solvable models in the literature, including the quadratic and the constant elasticity of variance models for European-style and barrier options. In addition, large families of new solutions are found, containing as many as seven free parameters

It has been known since the 1970s that Black-Scholes (1973) pricing formulas are a special case of more general families of pricing formulas with more than just the volatility as an adjustable parameter. The list of the classical extensions includes affine, quadratic and constant elasticity of variance (CEV) models. These models admit up to three adjustable parameters and have been used to solve pricing problems for equity, foreign exchange, interest rate and credit derivatives. In a series of working papers, the authors have recently developed new mathematical techniques that allow much further extensions. Several new families of pricing formulas, with up to seven adjustable parameters in the stationary driftless case, are obtained while attaining additional flexibility in the general time dependent case. The formulas extend to barrier options and have a similar structure to the Black-Scholes formulas, the most notable difference being that error functions (or cumulative normal distributions) are replaced by (confluent) hypergeometric functions, ie, the special transcendental functions of applied mathematics and mathematical physics.

Let F denote a generic financial observable that we know is driftless. Examples are the forward price of a stock or foreign currency under the forward measure, a Libor forward rate or a swap rate with appropriate choice of numeraire asset. Black or Black-Scholes formulas are obtained by postulating that the time evolution of F obeys a stochastic differential equation of the form:

$$dF_t = \sigma(F_t) dW_t \quad (1)$$

where $\sigma(F) = \sigma_1 F$ is linear and W_t is a standard Brownian or Wiener process. In this case, pricing formulas of calls and puts for both plain vanilla and barrier options can be written in exact analytical form in terms of the error function. Interestingly, quadratic volatility models with:

$$\sigma(F) = \sigma_0 + \sigma_1 F + \sigma_2 F^2 \quad (2)$$

also allow for pricing formulas that reduce to the evaluation of an error function. This is because quadratic volatility models can be reduced to a Wiener process by means of a simple measure change and variable transformation of the form:

$$F_t = F(x_t) \quad (3)$$

where the underlying x_t follows:

$$dx_t = dW_t \quad (4)$$

The transformation (3) applies under a pricing measure where assets are valued in terms of a suitably defined numeraire $g_t = g(x, t)$. For a review of change in numeraire methods in pricing theory, we refer to Geman, El Karoui & Rochet (1995), Schroder (1999) and Borodin & Salminen (1996). Both functions $F(x)$ and $g(x, t)$ can be derived explicitly for any choice of parameters $\sigma_0, \sigma_1, \sigma_2$.

Quadratic volatility models are the only stationary, driftless models for which the combination of a non-linear transformation of the form (3) and a change of numeraire reduces the problem to a Wiener process x_t as the one in (4). Carr, Lipton & Madan (2000) consider whether it is possible to relax the condition of stationarity and find more general processes with drift and volatility both dependent on calendar time and F that still reduce to the Wiener process. They find that the general solution admits as many as 11 time-dependent functions.

A related line of reasoning leading to extensions of the Black-Scholes formula starts from the observation that the CEV models with state-dependent volatility specified as follows:

$$\sigma(F) = \sigma_0 (F - \bar{F})^{1+\theta} \quad (5)$$

with constants θ and \bar{F} , reduces to the Bessel process:

$$dx_t = \lambda_0 dt + v_0 \sqrt{x_t} dW_t \quad (6)$$

by means of a non-linear transformation combined with a measure change. Lipton (2000) derives general reducibility conditions to the more general processes:

$$dx_t = (\lambda_0 + \lambda_1 x_t) dt + v_0 x_t^\beta dW_t \quad (7)$$

which are solvable for $\beta = 0, 1, 1/2$. The case $\beta = 1$ is the lognormal (or affine) model leading to the Black-Scholes formula. The two cases $\beta = 0, 1/2$ correspond to well-known solvable short-rate models, namely the Vasicek and the Cox-Ingersoll-Ross (CIR, 1985) models. Albanese & Campolieti (2001a, 2001b) find a general solution to the reducibility conditions of Carr, Lipton & Madan (2000) for stationary, driftless processes. In this article, we summarise our findings by presenting the general solution formula, and illustrate its use in a few particular cases. Since the original derivation is somewhat lengthy, we refer to our other papers for a constructive derivation.

General pricing formula

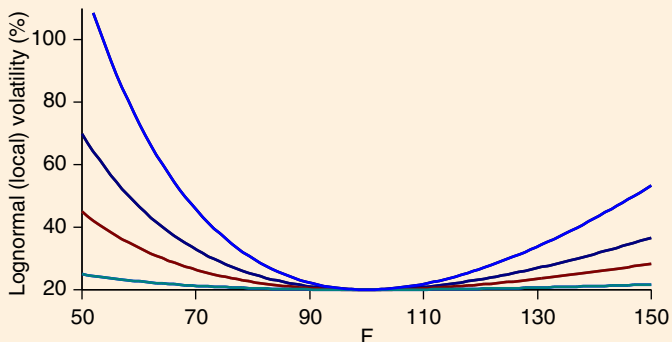
We present a general pricing formula for the models, which are solvable by the reduction method. Assume that the state variable x has a drift $\lambda(x)$ for which one can find the pricing kernel for the process:

$$dx_t = \lambda(x_t) dt + v(x_t) dW_t \quad (8)$$

The pricing kernel is the function $u(x, t; x_0, t_0)$, which solves the forward Kolmogorov equation in the first pair of arguments and the backward Kolmogorov (ie, Black-Scholes) equation in the second pair. The latter equation can be written as follows:

$$u_{t_0}(x, t; x_0, t_0) + \frac{v(x_0)^2}{2} u_{x_0 x_0}(x, t; x_0, t_0) + \lambda(x_0) u_{x_0}(x, t; x_0, t_0) = 0 \quad (9)$$

1. Examples of local volatility functions $\sigma(F)/F$ for the quadratic model



for $t \geq t_0$, with final time condition at terminal time $t_0 = t$ given by $u(x, t; x_0, t) = \delta(x - x_0)$ (a Dirac delta function). The processes in (7) are examples of analytically solvable models for which one can calculate the pricing kernel.

Solvable pricing models can be constructed starting from a solution $v(x, t)$ of the Black-Scholes equation in (9) with an arbitrary final time condition at $t = 0$. The Laplace transform of such a function:

$$\hat{v}(x, \rho) = \int_0^\infty e^{\rho(t_0-t)} v(x, t_0 - t) dt \quad (10)$$

is usually referred to as the 'time-independent Green's function' and satisfies a second-order ordinary differential equation with Dirac delta function source term $\delta(x - x_0)$. Let us consider the homogeneous part of this equation as given by:

$$-\rho \hat{v}(x, \rho) + \frac{v(x)^2}{2} \hat{v}_{xx}(x, \rho) + \lambda(x) \hat{v}_x(x, \rho) = 0 \quad (11)$$

We find that functions $\hat{v}(x, \rho)$ solving this equation can be taken as the elementary building blocks to construct solvable pricing models for the F space processes. We therefore call $\hat{v}(x, \rho)$ the 'generating function'.

Armed with a solution $\hat{v}(x, \rho)$, we define a volatility function $\sigma(F)$ and an invertible monotonic transformation $F = F(x)$ and its inverse $x = X(F)$ such that:

$$\sigma(F) = \frac{\sigma_0 v(X(F)) \exp\left(-2 \int^{X(F)} \frac{\lambda(s) ds}{v(s)^2}\right)}{\hat{v}(X(F), \rho)^2} \quad (12)$$

with arbitrary constant σ_0 and where:

$$\frac{dx}{v(x)} = \pm \frac{dF}{\sigma(F)} \quad (13)$$

The two signs correspond to either monotonic increasing or monotonic decreasing transformations. The freedom in choosing the sign gives rise to two families of solutions that are different in the general case. As is verified in the appendix, the process:

$$g_t = \frac{e^{\rho t}}{\hat{v}(x_t, \rho)} \quad (14)$$

can be regarded as a forward price process and, under the measure with g as a numeraire, the state variable x_t drifts at rate $\lambda(x)$. Hence, the pricing kernel $U(F, t; F_0, 0)$ for the underlying forward price F at time t can be evaluated in closed form as the expected reward from a limit butterfly spread contract with delta function payout:

$$U(F, t; F_0, 0) = E_0 \left[\frac{g_0}{g_t} \delta(F(x_t) - F) \right] \quad (15)$$

conditional on the price having value F_0 at initial time $t = 0$. Here, the expectation is calculated assuming that g_t is the numeraire and that the state variable x drifts at rate $\lambda(x)$. The final formula for the pricing kernel in F space is related to the kernel in the underlying x space as follows:

$$U(F, t; F_0, 0) = \frac{v(X(F)) \hat{v}(X(F), \rho)}{\sigma(F) \hat{v}(X(F_0), \rho)} e^{-\rho t} u(X(F), t; X(F_0), 0) \quad (16)$$

Ignoring discounting, a European-style call option written on the forward price F_0 at current time $t = 0$, struck at K and maturing at time $t = T$ can be priced in this model by calculating the following integral:

$$C(K, T; F_0) = e^{-\rho T} \int_{X(K)}^\infty dx \frac{\hat{v}(x, \rho)}{\hat{v}(X(F_0), \rho)} (F(x) - K) u(x, T; X(F_0), 0) \quad (17)$$

Barrier and lookback options can be handled by modifying the underlying kernel in x -space to account for the appropriate boundary conditions. This is accomplished by means of either integral representations or eigenfunction expansion methods, ie, Green's function methods that are standard in the theory of Sturm-Liouville equations. See Davydov & Linetsky (1999) for a discussion in an option pricing context.

Four families of solvable models

The case $\beta = 1$ is the usual lognormal (or affine) model. Of interest here are the other families with $\beta = 0, 1/2$ in equation (7). This provides four examples of our methodology to generate exactly solvable models.

If $\beta = 0$ and $\lambda_1 = 0$, we recover the Wiener process with constant drift, which is readily transformed into a driftless Wiener process and thus supports only quadratic volatility functions in F space, including the lognormal Black-Scholes model as a special subcase. If $\beta = 0$ and $\lambda_1 \neq 0$, then the kernel for $x \in [-\lambda_0/\lambda_1, \infty)$ can be written in terms of hyperbolic trigonometric functions and the generating function solves Hermite's equation.

If $\beta = 1/2$ and $\lambda_1 = 0$, then the pricing kernel for the state variable is expressed in terms of modified Bessel functions as follows:

$$u(x, t; x_0, 0) = \left(\frac{x}{x_0}\right)^{\frac{1}{2}\left(\frac{2\lambda_0}{v_0^2}-1\right)} \frac{e^{-2(x+x_0)/v_0^2 t}}{v_0^2 t / 2} I_{\frac{2\lambda_0}{v_0^2}-1} \left(\frac{4\sqrt{x x_0}}{v_0^2 t}\right) \quad (18)$$

The generating function is:

$$\hat{v}(x, \rho) = x^{\frac{1}{2}\left(1-\frac{2\lambda_0}{v_0^2}\right)} \left[q_1 I_{\frac{2\lambda_0}{v_0^2}-1} \left(\sqrt{\frac{8\rho x}{v_0^2}}\right) + q_2 K_{\frac{2\lambda_0}{v_0^2}-1} \left(\sqrt{\frac{8\rho x}{v_0^2}}\right) \right] \quad (19)$$

with arbitrary constants q_1, q_2 . Here $I_\nu(z)$ is the modified Bessel function of order ν and $K_\nu(z)$ is the associated McDonalds function. In this case, we obtain two families (one for each choice of sign in (13)) of exact solutions with six adjustable parameters.

The case $\beta = 1/2$ and $\lambda_1 < 0$ gives the pricing kernel for the state variable x corresponding to that of the short rate CIR model, and can still be expressed in terms of modified Bessel functions as follows:

$$u(x, t; x_0, 0) = c_t \left(\frac{x e^{-\lambda_1 t}}{x_0}\right)^{\frac{1}{2}\left(\frac{2\lambda_0}{v_0^2}-1\right)} \exp\left[-c_t (x_0 e^{\lambda_1 t} + x)\right] I_{\frac{2\lambda_0}{v_0^2}-1} \left(2c_t \sqrt{x x_0} e^{\lambda_1 t}\right) \quad (20)$$

where $c_t \equiv 2\lambda_1/(v_0^2(e^{\lambda_1 t} - 1))$. For a derivation, see Giorno *et al* (1986) and Kent (1978). The general solution of equation (11) reduces to Whittaker's equation and generating functions have the general form:

$$\hat{v}(x, \rho) = x^{-\lambda_0/v_0^2} e^{-\lambda_1 x/v_0^2} \left[q_1 W_{k,m} \left(-\frac{2\lambda_1}{v_0^2} x\right) + q_2 M_{k,m} \left(-\frac{2\lambda_1}{v_0^2} x\right) \right] \quad (21)$$

for arbitrary constants q_1, q_2 . Here $W_{k,m}(\cdot)$ and $M_{k,m}(\cdot)$ are Whittaker functions that can also be expressed in terms of confluent hypergeometric functions or in terms of Kummer functions (Abramowitz & Stegun, 1972). This construction gives rise to a dual family with seven free parameters (ie, ρ ,

$\lambda_0, \lambda_1, v_0, q_1, q_2$ and an additional constant of integration for the mapping from x space to forward price space), where:

$$k = \frac{\lambda_0}{v_0^2} + \frac{\rho}{\lambda_1}, \quad m = \frac{\lambda_0}{v_0^2} - \frac{1}{2} \quad (22)$$

The seven-parameter family that reduces to the CIR model has a local volatility function defined on either a finite interval or on a half line, and behaves asymptotically as the CEV volatility on one side and as a quadratic model on the other. This hybrid shape allows for a great deal of flexibility in reproducing observed volatility skews. There is a way of gaining a visual understanding of the geometric meaning of the seven parameters that perhaps oversimplifies the picture, but is intriguing. The allowed shapes, when confined to a finite interval, can be regarded as hybrids between the quadratic and the CEV model. The support of the volatility function can be either a finite or an infinite interval. On one side of the interval, the volatility behaves asymptotically as that of a CEV model. On the other side of the interval, the volatility behaves as in a quadratic model. Hunchback shapes with a local minimum and a local maximum are possible. The seven parameters single out the interval endpoints, the blow up or decay rate at one end and the location of the local minimum and the local maximum. This representation is an oversimplification, as the minimum and maximum disappear in certain parameter ranges while only inflection points persist. The inflection points also disappear in other parameter ranges. Hence, our seven-parameter model supports a varied zoology of skews, smiles, frowns and smirks. It also supports both cases with, and without, absorption.

Additional extensions are possible. For instance, one can apply a deterministic time change and still retain solvability. We refer to forthcoming articles for a discussion of this and other related topics.

Rediscovering exact solutions in the literature

We show that the known exact solutions in the literature, namely quadratic and CEV models, can all be rediscovered as particular cases of our general formula for the Bessel family where we make use of the above solutions to the underlying x space process with $\beta = 1/2, \lambda_1 = 0$ and $\lambda \equiv \lambda_0$. Without loss of generality, we can fix $v_0 = 2$. We specialise further to the case where:

$$F(x) = \bar{F} - a \frac{K_{\frac{\lambda}{2}-1}(\sqrt{2\rho x})}{I_{\frac{\lambda}{2}-1}(\sqrt{2\rho x})} \quad (23)$$

which leads to a process for the forward price F with volatility:

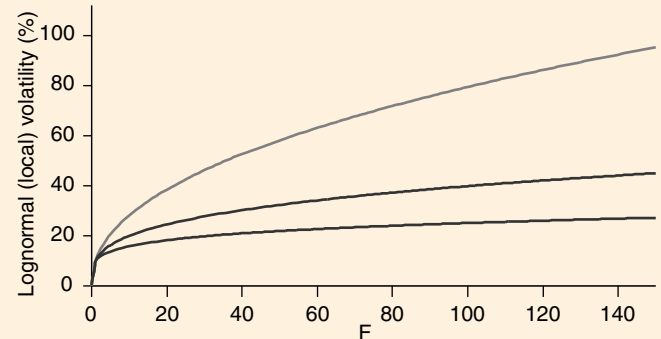
$$\sigma(F) = \frac{a}{\sqrt{x(F)} \left[I_{\frac{\lambda}{2}-1}(\sqrt{2\rho x(F)}) \right]^2} \quad (24)$$

where $x = X(F)$ is the inverse of the function in equation (23). In this family, a and ρ are positive, \bar{F} is arbitrary and $\lambda > 2$. The function $F(x)$ maps the half line $x \in [0, \infty)$ into $F \in (-\infty, \bar{F}]$, where $F(x)$ is a strictly monotonically increasing function with $dF(x)/dx = \sigma(F(x))/v(x)$. This solution region can be inverted so that $F \in [\bar{F}, \infty)$. This is accomplished by either replacing a by $-a$ in equation (23) or by applying a linear change of variables that maps F into $2\bar{F} - F$. In this special case, we make use of the generating function in equation (19), with the choice $q_2 = 0$, and formula (16) reduces to:

$$U(F, t; F_0, 0) = \frac{e^{-\rho t - (X(F) + X(F_0))/2t}}{at} \frac{X(F) \left[I_{\frac{\lambda}{2}-1}(\sqrt{2\rho X(F)}) \right]^3}{I_{\frac{\lambda}{2}-1}(\sqrt{2\rho X(F_0)})} I_{\frac{\lambda}{2}-1} \left(\frac{\sqrt{X(F)X(F_0)}}{t} \right) \quad (25)$$

We note that this density integrates exactly to unity in F space (ie, no absorption).

2. Examples of local volatility functions $\sigma(F)/F$ for the CEV model ($\theta = 3$)



□ **Quadratic volatility models.** Pricing kernels for quadratic volatility models are readily obtained as a subset of the above general family with the special choice of parameter $\lambda = 3$. After making the substitution $F \rightarrow 2\bar{F} - F$ and setting $a = (\bar{F} - \bar{\bar{F}})/\pi$ the transformation function $F(x)$ becomes:

$$F(x) = \bar{F} + \frac{(\bar{F} - \bar{\bar{F}}) K_{\frac{1}{2}}(\sigma_0 \sqrt{x}/2)}{\pi I_{\frac{1}{2}}(\sigma_0 \sqrt{x}/2)} = \bar{F} + \frac{(\bar{F} - \bar{\bar{F}})}{\exp(\sigma_0 \sqrt{x}) - 1} \quad (26)$$

where $\sigma_0 > 0$. Here, we assume that $\bar{F} > \bar{\bar{F}}$. The inverse transformation $X(F)$ is given by:

$$X(F) = \left(\frac{1}{\sigma_0^2} \right) \log^2 \left[1 + \frac{(\bar{F} - \bar{\bar{F}})}{(F - \bar{\bar{F}})} \right] \quad (27)$$

and the volatility function $\sigma(F)$ is obtained by insertion into equation (24) while using the Bessel function of order 1/2:

$$\sigma(F) = \frac{\sigma_0}{(\bar{F} - \bar{\bar{F}})} (F - \bar{\bar{F}}) \quad (28)$$

Inserting the expression (27) into (25), one obtains the pricing kernel:

$$U(F, t; F_0, 0) = \frac{2e^{-\sigma_0^2 t/8}}{\sigma(F)\sqrt{2\pi t}} \sqrt{\frac{(F_0 - \bar{\bar{F}})(F_0 - \bar{F})}{(F - \bar{\bar{F}})(F - \bar{F})}} e^{-(\phi(F)^2 + \phi(F_0)^2)/2\sigma_0^2 t} \sinh\left(\frac{\phi(F_0)\phi(F)}{\sigma_0^2 t}\right) \quad (29)$$

where $\phi(F) \equiv \log((F - \bar{\bar{F}})/(F - \bar{F}))$. In the special case of a volatility function with a double root, ie:

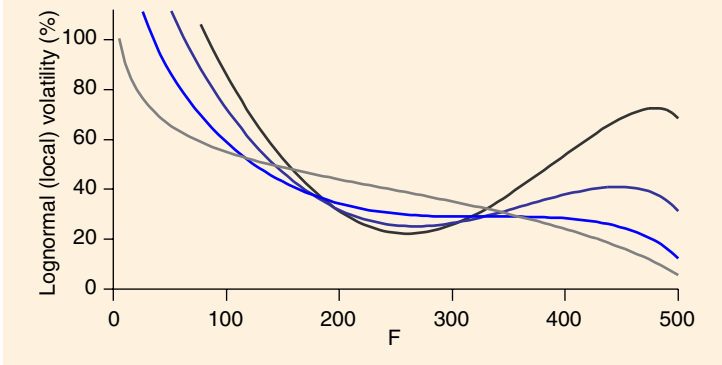
$$\sigma(F) = \sigma_0 (F - \bar{F})^2 \quad (30)$$

the pricing kernel is calculated by taking the limit as $\bar{\bar{F}} \rightarrow \bar{F}$, and one finds:

$$U(F, t; F_0, 0) = \frac{1}{\sigma_0 \sqrt{2\pi t}} \frac{(F_0 - \bar{F})}{(F - \bar{F})^3} \left[e^{-((F-\bar{F})^{-1} - (F_0-\bar{F})^{-1})^2 / 2\sigma_0^2 t} - e^{-((F-\bar{F})^{-1} + (F_0-\bar{F})^{-1})^2 / 2\sigma_0^2 t} \right] \quad (31)$$

□ **Lognormal models.** The pricing kernel for the lognormal Black-Scholes model with $\sigma(F) = \sigma_0 F$ is a particular case of the above formula for the quadratic model. The derivative with respect to F of the quadratic volatility function in (28), evaluated at $F = \bar{F}$, is σ_0 . Taking the limit $\bar{\bar{F}} \rightarrow -\infty$ (or $\bar{\bar{F}} \ll \bar{F}$), while holding the other parameters fixed,

3. Examples of local volatility functions $\sigma(F)/F$ for the CIR family of solvable models



one obtains $\sigma(F) = \sigma_0(F - \bar{F})$. The pricing kernel in (29) gives the kernel for the lognormal model in the limit $\bar{F} \rightarrow -\infty$, ie:

$$U(F, t; F_0, 0) = \frac{1}{(F - \bar{F})\sigma_0\sqrt{2\pi t}} \exp\left[-\left(\log\left(\frac{F_0 - \bar{F}}{F - \bar{F}}\right) - \frac{\sigma_0^2 t}{2}\right)^2 / 2\sigma_0^2 t\right] \quad (32)$$

□ **CEV model.** The CEV model is recovered in the limiting case as $\rho \rightarrow 0$. Assume $\lambda > 2$ and let $\theta > 0$ be defined so that $\lambda = \theta^{-1} + 2$. The transformation $F = F(x)$:

$$F(x) = \bar{F} + (\sigma_0^2 x)^{-(2\theta)} \quad (33)$$

has inverse $x = X(F)$ given by:

$$X(F) = \sigma_0^{-2} (\bar{F} - F)^{2\theta} \quad (34)$$

for any constant \bar{F} . The volatility function for this model is:

$$\sigma(F) = \frac{\sigma_0}{|\theta|} (\bar{F} - F)^{1+\theta} \quad (35)$$

In the limit $\rho \rightarrow 0$, the Laplace transform $\hat{v}(X(F), 0) = 1$, which implies that the numeraire change is trivial in this case. The pricing kernel can be evaluated by substitution into the general formula (16) and, after collecting terms, it turns out to be:

$$U(F, t; F_0, 0) = \frac{|\theta|}{\sigma_0^2 t} \frac{(F_0 - \bar{F})^{\frac{1}{2}}}{(F - \bar{F})^{\frac{3}{2} + 2\theta}} e^{-\left(\frac{(F - \bar{F})^{-2\theta} + (F_0 - \bar{F})^{-2\theta}}{2\sigma_0^2 t}\right)} I_{\frac{1}{2\theta}}\left(\frac{((F - \bar{F})(F_0 - \bar{F}))^{-\theta}}{\sigma_0^2 t}\right) \quad (36)$$

This formula was derived in the case $\theta > 0$, for which the limiting value $F = \bar{F}$ is not attained and the density is easily shown to integrate to unity (ie, no absorption occurs and the density also vanishes at the endpoint $F = \bar{F}$). We note that the same formula solves the forward pricing equation for $\theta < 0$, leading to the same Bessel equation of order $\pm(2\theta)^{-1}$. In the range $\theta < 0$, however, the properties of the above pricing kernel are generally more subtle. In particular, one can show that the density integrates to unity for all values $\theta < -1/2$, hence no absorption occurs for $\theta \in (-\infty, -1/2)$. The boundary conditions for the density can be shown to be vanishing at $F = \bar{F}$ (ie, paths do not attain the lower endpoint) for all $\theta < -1$. In contrast, for $\theta \in (-1, -1/2)$ the density becomes singular at the lower endpoint $F = \bar{F}$ (hence this corresponds to the case that the density has an integrable singularity for

which paths can also attain the lower endpoint, but are not absorbed). For the special case of $\theta = -1/2$, the formula gives rise to absorption. (Note that for the range $\theta \in (-1/2, 0)$ the above pricing kernel is not useful since it gives rise to a density that has a non-integrable singularity at $F = \bar{F}$. In this case, however, another solution that is integrable is obtained by only replacing the order $(2\theta)^{-1}$ by $-(2\theta)^{-1}$ in the Bessel function. The latter solution for the density does not integrate to unity and hence gives rise to absorption, which can be useful to price options in a credit setting.) The special case of $\theta = -1$ gives a non-zero constant value at the lower endpoint, and recovers the Wiener process with reflection and no absorption on the interval $[\bar{F}, \infty)$ with:

$$U(F, t; F_0, 0) = \frac{1}{\sigma_0\sqrt{2\pi t}} \left(e^{-(F-F_0)^2/2\sigma_0^2 t} + e^{-(F+F_0-2\bar{F})^2/2\sigma_0^2 t} \right) \quad (37)$$

Barrier options

The original motivation of two of us, Claudio Albanese and Giuseppe Campolieti, as we engaged in this project, was to streamline the derivation of pricing formulas for barrier options for our class of financial engineering master students. The general expression for the pricing kernel gives in fact a simple derivation of pricing formulas for barrier options, by allowing a reduction to standard Brownian motion in x space.

Consider, as an example, a down-and-out option with barrier at $F = H$ within the Black-Scholes model with $\sigma(F) = \sigma_0 F$. This reduces to the driftless Wiener process with volatility $v(x) = \sqrt{2}$, by means of the transformation where:

$$x = X(F) = (\sqrt{2}/\sigma_0)\log F \quad (38)$$

with inverse $F = F(x) = e^{\sigma_0 x/\sqrt{2}}$. Specialising equation (16) gives:

$$U(F, t; F_0, 0) = \frac{\sqrt{2}}{\sigma_0 F} \exp\left[\frac{1}{2}\log(F_0/F) - \frac{\sigma_0^2 t}{8}\right] u(X(F), t; X(F_0), 0) \quad (39)$$

The region $x \in (-\infty, \infty)$ maps into $F \in (0, \infty)$. A barrier located at $F = H$ corresponds to $H = F(x_H) = e^{\sigma_0 x_H/\sqrt{2}}$, so $x_H = X(H) = (\sqrt{2}/\sigma_0)\log H$. The upper region $F \in [H, \infty)$ maps into $x \in [x_H, \infty)$. The x -space kernel with absorbing boundary condition at $x = x_H$ is obtained by the method of images, as:

$$u(x, t; x_0, 0) = \frac{1}{\sqrt{4\pi t}} \left(e^{-(x-x_0)^2/4t} - e^{-(x+x_0-2x_H)^2/4t} \right) \quad (40)$$

Inserting this kernel into the general pricing formula in (39) immediately gives the pricing kernel in F space:

$$U^H(F, t; F_0, 0) = U(F, t; F_0, 0) \left[1 - \exp\left[-\frac{\log(F/H)\log(F_0/H)}{\sigma_0^2 t/2}\right] \right] \quad (41)$$

where $U(F, t; F_0, 0)$ is the barrier-free pricing kernel:

$$U(F, t; F_0, 0) = \frac{1}{\sigma_0 F\sqrt{2\pi t}} \exp\left[-\left(\log(F_0/F) - \frac{\sigma_0^2 t}{2}\right)^2 / 2\sigma_0^2 t\right] \quad (42)$$

Ignoring discounting, a down-and-out call maturing at time T and struck at $K > H$ has the price at time $t = 0$ given by the integral:

$$C^{DO}(F_0, K, T) = \int_H^\infty dF U^H(F, T; F_0, 0) (F - K)^+ \quad (43)$$

where F_0 is the current forward price of maturity T . This integral can be evaluated in terms of cumulative normal distribution functions as follows:

$$C^{DO}(F_0, K, T) = F_0 N(d_1(F_0/K)) - KN(d_2(F_0/K)) - HN(d_1(H^2/F_0K)) + (KF_0/H)N(d_2(H^2/F_0K)) \quad (44)$$

where:

$$d_1(x) = \frac{\log x + \frac{1}{2}\sigma_0^2 T}{\sigma_0\sqrt{T}} \quad (45)$$

and $d_2(x) = d_1(x) - \sigma_0\sqrt{T}$. Note that, since the risk-neutral drift is absent,

pricing formulas are more compact when written on forward prices instead of stock prices.

In the more general case of the other solvable models, one can also obtain analytic closed-form solutions for various exotic payouts, including barrier options. Based on our general results, the derivation of pricing formulas is straightforward and will be presented elsewhere. ■

Claudio Albanese and Giuseppe Campolieti are at Math Point and the

University of Toronto. Peter Carr is a senior consultant and visiting professor at the Courant Institute of New York University. Alexander Lipton is in the forex product development group at Deutsche Bank in New York. Claudio Albanese was supported in part by the National Science and Engineering Council of Canada. We thank Dilip Madan, Stephan Lawi, Vadim Linetsky and Andrei Zavidonov for discussions. Any remaining errors are our own

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Appendix

Here we verify the main formula, equation (16). Consider a generic pricing measure where the process for x_t obeys the equation:

$$dx_t = \mu(x_t)dt + v(x_t)dW_t \quad (46)$$

for some drift $\mu(x)$. Then, by Itô's lemma, the process g_t defined in (14) satisfies the equation:

$$dg = \left(\rho - \mu \frac{\hat{v}_x}{\hat{v}} + v^2 \left[\left(\frac{\hat{v}_x}{\hat{v}} \right)^2 - \frac{1}{2} \frac{\hat{v}_{xx}}{\hat{v}} \right] \right) g dt + \sigma^g g dW \quad (47)$$

where:

$$\sigma^g = -\frac{\hat{v}_x v}{\hat{v}} \quad (48)$$

is defined as the lognormal volatility of g . Substituting equation (11):

$$\frac{v^2}{2} \hat{v}_{xx} = \rho \hat{v} - \lambda \hat{v}_x \quad (49)$$

into the above stochastic differential equation, we find:

$$\frac{dg}{g} = \left(\frac{\mu - \lambda}{v} \sigma^g + (\sigma^g)^2 \right) dt + \sigma^g dW_t \quad (50)$$

To demonstrate that g defines a forward price process, consider this equation in the original forward measure where the forward price F follows a martingale process. Then, using Itô's lemma on the mapping $x_t = X(F_t)$ and equation (1) we arrive at equation (46) with drift:

$$\mu(x) = \frac{\sigma(F)^2}{2} \frac{d}{dF} \frac{dX(F)}{dF} = \frac{\sigma(F)^2}{2} \frac{d}{dF} \frac{v(x)}{\sigma(F)} \quad (51)$$

Expressing all functions in terms of x , we then have:

$$\mu(x) = \frac{\sigma v}{2} \frac{d}{dx} \left(\frac{v}{\sigma} \right) = \frac{v}{2} \left[v_x - \frac{v}{\sigma} \sigma_x \right] \quad (52)$$

where $\sigma \equiv \sigma(F(x))$ is the volatility function for the forward price F . Hence, by substitution the drift of g in the forward measure is:

$$\frac{\mu - \lambda}{v} \sigma^g + (\sigma^g)^2 = \left[\lambda + \frac{v^2}{2} \frac{\sigma_x}{\sigma} - \frac{1}{2} v v_x \right] \frac{\hat{v}_x}{\hat{v}} + v^2 \left(\frac{\hat{v}_x}{\hat{v}} \right)^2 \quad (53)$$

Equation (12) gives:

$$\frac{\sigma_x}{\sigma} = \frac{v_x}{v} - \frac{2\lambda}{v^2} - \frac{2\hat{v}_x}{\hat{v}} \quad (54)$$

Substituting into (53), we find that the drift of g under the forward measure vanishes. Next, consider the measure having g as numeraire. Under this pricing measure the price of risk $q^g = \sigma^g$. Indeed, by Itô's lemma, it is known that if one changes from a measure in which any asset A_t has a drift r_t (ie, the risk-free rate in the risk-neutral measure or zero in the forward measure) into a new measure with g_t as numeraire, then the drift of A_t in this new measure is $\mu^A = r + q^g \sigma^A$, where $q^g = \sigma^g$ is the price of risk and σ^A the lognormal volatility of A . Hence, in changing from the forward measure into the measure having g_t as numeraire $\mu^A = \sigma^g \sigma^A$. The choice $A_t = g_t$ gives $\mu^g = (\sigma^g)^2$ and:

$$dg = \mu^g g dt + \sigma^g g dW_t = (\sigma^g)^2 g dt + \sigma^g g dW_t \quad (55)$$

Comparison with equation (50) shows that the drift μ of the process x_t is λ , as stated. This implies that the representation (16) for the pricing kernel is correct.

We refer the reader interested in gaining further insight into our main formula to our article (Albanese & Campolieti, 2001b). There we provide a direct partial differential equation proof, which is more elaborate and fully constructive, and is not based on the above stochastic analysis argument. ■

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