

Diamonds and forward variance models

Peter Friz, TU and WIAS Berlin.
friz@math.tu-berlin.de

Jim Gatheral, Baruch College, CUNY,
jim.gatheral@baruch.cuny.edu

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Abstract

In this non-technical introduction to diamond trees and forests, we focus on their application to computation in stochastic volatility models written in forward variance form, rough volatility models in particular.

1 Diamond trees and forests

From [AGR20, FGR22], we have the following definition of the diamond product.

Definition 1.1. *Given two continuous semimartingales A, B with integrable covariation process $\langle A, B \rangle$, the diamond product is defined by*

$$(A \diamond B)_t(T) := \mathbb{E} [\langle A, B \rangle_{t,T} | \mathcal{F}_t] = \mathbb{E} [\langle A, B \rangle_T | \mathcal{F}_t] - \langle A, B \rangle_t .$$

We shall assume that all martingales admit a continuous version, it is then clear that $A \diamond B$ is another continuous semimartingale (but in general, due to the covariation term $\langle A, B \rangle$, not a martingale). The diamond product has the following properties.

- It is commutative: $A \diamond B = B \diamond A$.
- It is non-associative: $(A \diamond B) \diamond C \neq A \diamond (B \diamond C)$.
- $A \diamond B$ depends only on the respective local martingale parts of A and B .

Diamond products of (sufficiently integrable, continuous) semimartingales naturally lead to binary “diamond” trees such as $(A \diamond B) \diamond C$. Diagrammatically, the diamond product of two trees \mathbb{T}_1 and \mathbb{T}_2 is represented by *root joining*,

$$\mathbb{T}_1 \diamond \mathbb{T}_2 = \bullet \blacklozenge \circ,$$

where the two binary trees \mathbb{T}_1 and \mathbb{T}_2 are represented as the single leaves \bullet and \circ . We regard linear combinations of diamond trees, as seen in (1.4) below, as *forests*. This tree formalism is extremely convenient when it comes to doing explicit computations as we will see later on.

The following theorem, adapted from [FGR22], expresses the cumulant generating function of any continuous semimartingale and its quadratic variation as a sum of forests of diamond trees.

Theorem 1.1. *Let Y_T be a real-valued, \mathcal{F}_T -measurable random variable with associated martingale $Y_t = \mathbb{E}_t[Y_T]$. Under natural integrability conditions, with a, b small enough, there is a.s. convergence of*

$$\log \mathbb{E} \left[e^{aY_T + b\langle Y \rangle_T} \mid \mathcal{F}_t \right] = aY_t + b\langle Y \rangle_t + \sum_{k \geq 2} \mathbb{G}_t^k(T), \quad (1.1)$$

where

$$\begin{aligned} \mathbb{G}^2 &= \left(\frac{1}{2}a^2 + b \right) (Y \diamond Y)_t(T), \\ \mathbb{G}^k &= \frac{1}{2} \sum_{j=2}^{k-2} \mathbb{G}^{k-j} \diamond \mathbb{G}^j + (aY \diamond \mathbb{G}^{k-1}) \text{ for } k > 2. \end{aligned} \quad (1.2)$$

For the multivariate case, $Y = (Y^1, \dots, Y^d)$ and $a \in \mathbb{R}^d, b \in \mathbb{R}^{d \times d}$, we replace aY and $b\langle Y \rangle$ by $\sum_i a_i Y^i$ and $\sum_{i,j} b_{ij} \langle Y^i, Y^j \rangle$, and further have $\mathbb{G}^2 = \sum_{i,j} (\frac{1}{2}a_i a_j + b_{ij}) (Y^i \diamond Y^j)_t(T)$.

Proof of Theorem 1.1. (Sketch) For a generic (continuous) semimartingale Z , sufficiently integrable, let

$$\Lambda_t^T = \log \mathbb{E}_t \left[e^{Z_{t,T}} \right].$$

Then, noting that $\Lambda_T^T = 0$,

$$\mathbb{E}_t \left[e^{Z_T} \right] = \mathbb{E}_t \left[e^{Z_T + \Lambda_T^T} \right] = e^{Z_t + \Lambda_t^T}.$$

The stochastic logarithm $\mathcal{L}(\mathbb{E}_\bullet(Z_T)) = Z + \Lambda^T + \frac{1}{2}\langle Z + \Lambda^T \rangle$ is a martingale. Thus,

$$\begin{aligned} \Lambda_t^T &= \mathbb{E}_t \left[Z_{t,T} + \frac{1}{2}\langle Z + \Lambda^T \rangle_{t,T} \right] \\ &= \mathbb{E}_t \left[Z_{t,T} \right] + \frac{1}{2}((Z + \Lambda^T) \diamond (Z + \Lambda^T))_t(T). \end{aligned} \quad (1.3)$$

Now with¹ $Z = \epsilon aY + \epsilon^2 b\langle Y \rangle$ we get

$$\Lambda_t^T(\epsilon) = \epsilon a \mathbb{E}_t \left[Y_{t,T} \right] + \epsilon^2 b (Y \diamond Y)_t(T) + \frac{1}{2} (\epsilon aY + \Lambda_t^T(\epsilon))_t^{\diamond 2}(T).$$

Put $\Lambda_t^T(\epsilon) = \epsilon^2 \mathbb{G}_t^2 + \epsilon^3 \mathbb{G}_t^3 + \dots$, and match coefficients of ϵ^n to obtain the result.

□

¹Recall that terms of bounded variation such as $\langle Y \rangle$ do not contribute to diamond products.

The process Λ^T is the correction required for $e^{Z+\Lambda^T}$ to be a martingale, with $\Lambda_T^T = 0$. In particular, if e^Z is already a martingale, then $\Lambda^T \equiv 0$ on $[0, T]$.

Remark 1.1. Note that the convergent \mathbb{G} -sum is exactly equal to Λ^T , which satisfies the “abstract Riccati” equation (1.3),

$$\Lambda_t^T = \mathbb{E}_t Z_{t,T} + \frac{1}{2} \left((Z + \Lambda^T) \diamond (Z + \Lambda^T) \right)_t (T).$$

To make the recursion (1.2) more concrete, consider the first few forests written diagrammatically with \bullet as a short-hand for Y , interpreted as single leaf:

$$\begin{aligned} \mathbb{G}^2 &= \left(\frac{1}{2}a^2 + b\right) \bullet \swarrow \searrow \\ \mathbb{G}^3 &= a \left(\frac{1}{2}a^2 + b\right) \bullet \swarrow \bullet \searrow \\ \mathbb{G}^4 &= \frac{1}{2} \left(\frac{1}{2}a^2 + b\right)^2 \bullet \swarrow \bullet \searrow + a^2 \left(\frac{1}{2}a^2 + b\right) \bullet \swarrow \bullet \bullet \searrow \\ \mathbb{G}^5 &= a \left(\frac{1}{2}a^2 + b\right)^2 \bullet \swarrow \bullet \bullet \searrow + \frac{1}{2} a \left(\frac{1}{2}a^2 + b\right)^2 \bullet \swarrow \bullet \bullet \bullet \searrow + a^3 \left(\frac{1}{2}a^2 + b\right) \bullet \swarrow \bullet \bullet \bullet \bullet \searrow \end{aligned} \quad (1.4)$$

We note that when $b = -\frac{1}{2}a^2$, all of the corrector terms \mathbb{G}^k vanish, as they must, because in this case $e^Z = \exp \left\{ aY - \frac{1}{2}a^2 \langle Y \rangle \right\}$ is the exponential martingale and $\Lambda = 0$. With $a = -b/2$, the \mathbb{G} -recursion (1.2) becomes precisely the \mathbb{F} -recursion, Equation (3.1) of [AGR20] derived in the context of stochastic volatility modeling, whereas the case $b = 0$ yields the \mathbb{K} (cumulant) recursion, Equations (3.4), (3.9) of [LRV22], derived in the context of renormalisation of the sine-Gordon model in quantum physics.

In [FGR22], a number of applications of the \mathbb{K} expansion are given, including a neat derivation of the moment generating function of the Lévy area. Other applications include the computation of likelihood functions in statistics, the computation of correlation functions in statistical physics, and the computation of amplitudes in quantum field theory.

However, in the context of (rough) stochastic volatility modeling that originally gave rise to diamond expansions, the \mathbb{G} -expansion and its special case, the \mathbb{F} -expansion (with $b = 0$), are the relevant ones.

2 Model-free results under stochastic volatility

Consider a stochastic volatility model written in forward variance form. Specifically, let S be a strictly positive continuous martingale (the stock price). Then $X := \log S/S_0$ is a continuous semimartingale with quadratic variation process $\langle X \rangle$, which is assumed continuously differentiable such as to have a well-defined instantaneous variance, defined via

$$V_t dt := d\langle X \rangle_t,$$

and then forward variance as the conditional expectation of future instantaneous variance

$$\xi_t(T) = \mathbb{E} [V_T | \mathcal{F}_t].$$

Forward variances are tradable assets (unlike spot variance), constituting a family of martingales indexed by their individual time horizons T .

The fair strike of a (total) variance swap maturing at time T is given by

$$M_t(T) = \int_t^T \xi_t(u) du$$

and the fair strike of a forward-starting variance swap, starting at time T and maturing at time $T + \Delta$ by

$$\zeta_T(T) = \int_T^{T+\Delta} \xi_T(u) du = \mathbb{E}_T \int_T^{T+\Delta} V_u du = \mathbb{E}_T \langle X \rangle_{T, T+\Delta}.$$

Its price process $\zeta_t(T) = \mathbb{E}_t \langle X \rangle_{T, T+\Delta}$ for $t \in [0, T]$ defines a martingale $\zeta = \zeta(T)$.

We now show how a straightforward application of Theorem 1.1 gives a *model-free* expression for the moment generating function (mgf) of $\log S$, the variance swap, and the forward-starting variance swap. As for the practical interest, if $S = S_0 e^X$ represents the SPX index, and Δ is 30 days, we get the joint mgf of SPX, the variance swap and VIX^2 .

Corollary 2.1. *Under natural integrability conditions, for $a, b, c \in \mathbb{R}$ sufficiently small,*

$$\mathbb{E} \left[e^{aX_T + b \langle X \rangle_T + c \zeta_T(T)} \middle| \mathcal{F}_t \right] = \exp \left\{ aX_t + b \langle X \rangle_t + c \zeta_t(T) + \sum_{k=2}^{\infty} \mathbb{G}_t^k(T; a, b, c) \right\}, \quad (2.1)$$

with $0 \leq t \leq T$, where the \mathbb{G}^k 's are given recursively by (1.2), starting with

$$\mathbb{G}^2 = \left(\frac{1}{2}a(a-1) + b \right) X \diamond X + ac X \diamond \zeta + \frac{1}{2}c^2 \zeta \diamond \zeta. \quad (2.2)$$

Proof. Re-express the exponent at terminal time T in terms of the martingale $Y = X + \frac{1}{2} \langle X \rangle$,

$$aX_T + b \langle X \rangle_T + c \zeta_T(T) = aY_T + c \zeta_T(T) + \left(b - \frac{1}{2}a \right) \langle Y \rangle_T.$$

This is the inner product of $(a, c) \in \mathbb{R}^2$ with the 2-dimensional martingale $(Y, \zeta) = (Y, \zeta(T))$, plus $\text{diag}(b - a/2, 0)$ contracted against its (2×2) -covariation matrix, evaluated at time T . The bivariate formulation of Theorem 1.1 now gives the right-hand side exponent

$$aY_t + c \zeta_t(T) + \left(b - \frac{1}{2}a \right) \langle Y \rangle_t = aX_t + b \langle X \rangle_t + c \zeta_t(T)$$

plus the \mathbb{G} -series, started with

$$\mathbb{G}^2 = \frac{1}{2}a^2 Y \diamond Y + ac Y \diamond \zeta + \frac{1}{2}c^2 \zeta \diamond \zeta + (b - a/2) Y \diamond Y.$$

It remains to collect terms, noting that Y can be safely replaced by X , since they only differ by a bounded variation term invisible to the diamond product. \square

Remark 2.1. Martingality of $S = S_0 e^X$ is seen in (2.1) upon setting $b = c = 0$ and $a = 1$. This is perfectly reflected in our \mathbb{G} expansion, which in this case, and also when $a = 0$, vanishes altogether, as seen directly from (2.2). In this sense, martingality and total probability constraints are preserved at arbitrary truncation of the \mathbb{G} expansion of [FGR22] and its \mathbb{F} -predecessor in [AGR20]. We note that this is not achieved by the (cumulant) recursion of [LRV22]: applied in its bivariate form, with $X = (1, -1/2) \cdot (Y, \langle Y \rangle)$, the martingality constraint is only seen upon summing a non-trivial infinite sum, whose value turns out to be zero. (In [FGR22] this is explained by forest reordering.)

Let $X \equiv \circ$ and $\zeta \equiv \bullet$ so that we can write the first term of the \mathbb{G} -series in Corollary 2.1 as

$$\mathbb{G}^2 = \left(\frac{1}{2}a(a-1) + b\right) \circ \vee \circ + ac \circ \vee \bullet + \frac{1}{2}c^2 \bullet \vee \bullet.$$

We could define $\circ \vee \circ =: \circ$, meaning $(X \diamond X) = M = \circ$, resulting in trees with leaves of three different colors, in which case X_t would represent the log-stock price and $M_t(T)$, the variance swap. Then

$$\mathbb{G}^2 = \left(\frac{1}{2}a(a-1) + b\right) \circ + ac \circ \vee \bullet + \frac{1}{2}c^2 \bullet \vee \bullet.$$

In general, we can always identify subtrees in this way and assign them a new variable name (and leaf color).

Upon setting $b = c = 0$ in Corollary 2.1, we get the \mathbb{F} -expansion of [FGR22]. Upon Wick rotation, replacing a by ia , and shifting the integer index, we recover the $\tilde{\mathbb{F}}$ -expansion of [AGR20].

Corollary 2.2. The conditional cumulant generating function (CGF) is given by

$$\psi_t(T; a) = \log \mathbb{E}_t \left[e^{ia X_{t,T}} \right] = -\frac{1}{2}a(a+i) M_t(T) + \sum_{k=1}^{\infty} \tilde{\mathbb{F}}_k(a). \quad (2.3)$$

where the $\tilde{\mathbb{F}}_k$ satisfy the recursion $\tilde{\mathbb{F}}_0 = -\frac{1}{2}a(a+i) M_t = -\frac{1}{2}a(a+i) \circ$ and for $k > 0$,

$$\tilde{\mathbb{F}}_k = \frac{1}{2} \sum_{j=0}^{k-2} (\tilde{\mathbb{F}}_{k-2-j} \diamond \tilde{\mathbb{F}}_j) + ia (X \diamond \tilde{\mathbb{F}}_{k-1}). \quad (2.4)$$

Having generated a model-free expression for the cumulant generating function for any stochastic volatility model written in forward variance form, we can compute model-free expressions for attainable claims, such as variance and gamma swaps. (See e.g. [Lee10].)

As is well-known, the fair value of the variance swap is given by the fair value of the log-strip:

$$\mathbb{E} \left[X_{t,T} \mid \mathcal{F}_t \right] = (-i) \psi'_t(T; 0) = -\frac{1}{2} M_t(T)$$

and the fair value of the gamma swap is given by the fair value of the entropy contract:

$$\mathbb{E} \left[X_{t,T} e^{X_{t,T}} \mid \mathcal{F}_t \right] = -i \psi'_t(T; -i).$$

From (2.3) and the recursion (2.4), it is easy to see that only trees containing a single \circ leaf will survive in the sum after differentiation when $a = -i$ so that

$$\begin{aligned} \sum_{k=1}^{\infty} \tilde{\mathbb{F}}'_k(-i) &= \frac{i}{2} \sum_{k=1}^{\infty} X^{\circ k} M \\ &= \frac{i}{2} \left\{ \begin{array}{c} \circ \\ \swarrow \searrow \end{array} + \begin{array}{c} \circ \\ \swarrow \circ \searrow \\ \circ \end{array} + \begin{array}{c} \circ \\ \swarrow \circ \searrow \\ \circ \circ \end{array} + \dots \right\}. \end{aligned}$$

Then the fair value of a gamma swap is given by

$$\mathcal{G}_t(T) = 2 \mathbb{E} [X_{t,T} e^{X_{t,T}} | \mathcal{F}_t] = \begin{array}{c} \circ \\ \swarrow \searrow \end{array} + \begin{array}{c} \circ \\ \swarrow \circ \searrow \\ \circ \end{array} + \begin{array}{c} \circ \\ \swarrow \circ \searrow \\ \circ \circ \end{array} + \dots \quad (2.5)$$

We deduce that the fair value of a leverage swap is given by

$$\begin{aligned} \mathcal{L}_t(T) &:= \mathcal{G}_t(T) - M_t(T) = \sum_{k=1}^{\infty} X^{\circ k} M \\ &= \begin{array}{c} \circ \\ \swarrow \searrow \end{array} + \begin{array}{c} \circ \\ \swarrow \circ \searrow \\ \circ \end{array} + \begin{array}{c} \circ \\ \swarrow \circ \searrow \\ \circ \circ \end{array} + \dots \end{aligned} \quad (2.6)$$

The completely model-free and explicit expression (2.6) for the leverage swap is in terms of diamond products of products of the spot and volatility processes. In particular, if spot and volatility processes are uncorrelated, $\begin{array}{c} \circ \\ \swarrow \searrow \end{array}$ and higher order terms vanish, and we see that the fair value of the leverage swap is zero.

In [Fuk14], Fukasawa shows that

$$\mathbb{E}_t \left[\int_t^T d\langle S, M(T) \rangle_r \right] = S_t \{ \mathcal{G}_t(T) - M_t(T) \}.$$

Thus, by definition of the diamond product and by the definition (2.6) of the leverage swap,

$$(S \diamond M)_t(T) = S_t \mathcal{L}_t(T). \quad (2.7)$$

In other words, the fair value of a leverage swap is given by the quadratic covariation of the underlying and the variance swap, for any admissible stochastic volatility model.

Comparing equation (2.6) with equation (2.7), we are led to the following identity, for which we offer a direct diamond proof.

Lemma 2.1.

$$(S \diamond M)_t(T) = (e^X \diamond M)_t(T) = S_t \sum_{k=1}^{\infty} (X^{\circ k} M)_t(T).$$

Proof. Note that

$$(S \diamond M)_t(T) = \mathbb{E} \left[\int_t^T d\langle S, M \rangle_u \middle| \mathcal{F}_t \right] = \mathbb{E} \left[\int_t^T S_u d\langle X, M \rangle_u \middle| \mathcal{F}_t \right].$$

Now define $U_t := \mathbb{E} \left[\int_t^T d\langle X, M \rangle_u \middle| \mathcal{F}_t \right] = (X \diamond M)_t(T)$. Applying Itô's Formula to the process SU we get

$$\begin{aligned} & \mathbb{E} \left[\int_t^T S_u d\langle X, M \rangle_u \middle| \mathcal{F}_t \right] \\ &= S_t \mathbb{E} \left[\int_t^T d\langle X, M \rangle_u \middle| \mathcal{F}_t \right] + \mathbb{E} \left[\int_t^T S_u d\langle X, U \rangle_u \middle| \mathcal{F}_t \right] \\ &= S_t (X \diamond M)_t(T) + \mathbb{E} \left[\int_t^T S_u d\langle X, (X \diamond M) \rangle_u \middle| \mathcal{F}_t \right]. \end{aligned}$$

Then a recursive argument gives us that

$$\mathbb{E} \left[\int_t^T d\langle S, M \rangle_u \middle| \mathcal{F}_t \right] = S_t \sum_{k=1}^{\infty} (X^{k \diamond} M)_t(T).$$

□

What is this good for? Since the leverage swap is an attainable claim, it can be expressed as a portfolio of European options whose value is given in principle by the market – only in principle, because in practice interpolation and extrapolation are required, market prices being available only certain discrete strikes and expirations. Since (2.6) gives the model value of the leverage swap for any given maturity T , and we can estimate the value of the leverage swap for each T from the market, models can be efficiently calibrated.

However there is a catch. This model calibration procedure works only when we know how to compute or easily approximate diamond trees.

2.1 The Bergomi-Guyon smile expansion

Following [BG12], consider a forward variance model of the form

$$\begin{aligned} \frac{dS_t}{S_t} &= \sqrt{V_t} dZ_t \\ d\xi_t(u) &= \lambda(t, u, \xi_t) dW_t. \end{aligned} \tag{2.8}$$

To expand such a model for small volatility of volatility, Bergomi and Guyon scale the volatility of volatility function $\lambda(\cdot)$ by a dimensionless parameter ε so that $\lambda \mapsto \varepsilon \lambda$. Setting $\varepsilon = 1$ at the end then gives the required expansion.

Let $k = \log K/S$ be the log-strike and T be the option expiration. For ease of notation and without loss of generality, set $t = 0$ and drop references to the initial time. The Bergomi-Guyon smile expansion (Equation (14) of [BG12]) then reads

$$\sigma_{BS}(k, T) = \hat{\sigma}_T + \mathcal{S}_T k + \mathcal{C}_T k^2 + \mathcal{O}(\varepsilon^3) \tag{2.9}$$

where the coefficients $\hat{\sigma}_T$, \mathcal{S}_T and C_T are expressed in terms of what Bergomi and Guyon call autocorrelation functionals. The beauty of the Bergomi-Guyon smile expansion is that it relates observed properties of the implied volatility surface such as at-the-money volatility $\hat{\sigma}_T$ and the at-the-money volatility skew \mathcal{S}_T directly to quantities that are computable from the formulation of the model, for any model written in forward variance form.

On the other hand the $\tilde{\mathbb{F}}$ -expansion (2.3) for the cgf can be expressed as a formal power series in (our own parameter) ϵ whose power counts the forest index ℓ . That is, from (2.3),

$$\psi(T; a) = \log \mathbb{E} \left[e^{i a X_T} \right] = -\frac{1}{2} a (a + i) M(T) + \sum_{\ell=1}^{\infty} \epsilon^\ell \tilde{\mathbb{F}}_\ell(a). \quad (2.10)$$

It is well-known [CM99, Lew00] that the price of a European option may be computed by inverting the characteristic function of a given stochastic process. Equivalently, equ. (5.7) of [Gat06] expresses the the implied total variance smile $\Sigma(k, T) := \sigma_{\text{BS}}^2(k, T) T$ in terms of the characteristic function:

$$\int_0^\infty \frac{du}{u^2 + \frac{1}{4}} \operatorname{Re} \left[e^{-iuk} \left(e^{\psi(T; u-i/2)} - e^{-\frac{1}{2}(u^2 + \frac{1}{4})\Sigma(k, T)} \right) \right] = 0. \quad (2.11)$$

Dropping explicit references to T , let

$$\Sigma(k) = \sum_{\ell=0}^{\infty} \epsilon^\ell a_\ell(k).$$

Then, substituting from (2.10), (2.11) becomes

$$\int_0^\infty \frac{du}{u^2 + \frac{1}{4}} \operatorname{Re} \left[e^{-iuk} \left\{ e^{-\frac{1}{2}(u^2 + \frac{1}{4})M(T) + \sum_{\ell=1}^{\infty} \epsilon^\ell \tilde{\mathbb{F}}_\ell(u-i/2)} - e^{-\frac{1}{2}(u^2 + \frac{1}{4})\sum_{\ell=0}^{\infty} \epsilon^\ell a_\ell(k)} \right\} \right] = 0.$$

Setting $\epsilon = 0$ gives $a_0(k) = M(T)$, as expected. Then matching powers of ϵ to first order gives

$$a_1(k) = \left(\frac{k}{M} + \frac{1}{2} \right) (X \diamond M). \quad (2.12)$$

where we have put $M = M(T)$ for short. In Appendix A of [AGR20], an explicit algorithm using Hermite and Bell polynomials that matches coefficients of ϵ to any desired order is given. This algorithm gives the second order coefficient as

$$\begin{aligned} a_2(k) = & \frac{1}{4}(X \diamond M)^2 \left\{ -\frac{5k^2}{M^3} - \frac{2k}{M^2} + \frac{3}{M^2} + \frac{1}{4M} \right\} \\ & + \frac{1}{4}(M \diamond M) \left\{ \frac{k^2}{M^2} - \frac{1}{M} - \frac{1}{4} \right\} + (X \diamond (X \diamond M)) \left\{ \frac{k^2}{M^2} + \frac{k}{M} - \frac{1}{M} + \frac{1}{4} \right\}. \end{aligned}$$

It is straightforward to verify that the resulting expansion $\Sigma(k) = M + \epsilon a_1(k) + \epsilon^2 a_2(k) + \mathcal{O}(\epsilon^3)$ agrees with the Bergomi Guyon smile expansion to second order in ϵ , up to the identification $\epsilon \rightsquigarrow \varepsilon$. In this sense, the $\tilde{\mathbb{F}}$ -expansion allows us to extend the Bergomi-Guyon expansion to all orders.

3 Affine forward variance models

Diamond trees turn out to be particularly easy to compute in affine forward variance (AFV) models. As shown in Chapter ??, affine forward variance (AFV) models must take the form

$$\begin{aligned} \frac{dS_t}{S_t} &= \sqrt{V_t} dZ_t \\ d\xi_t(u) &= \kappa(u-t) \sqrt{V_t} dW_t, \quad t \leq u, \end{aligned} \quad (3.1)$$

with $d\langle W, Z \rangle_t = \rho dt$. AFV models are therefore essentially all Heston models with different choices of the kernel function κ . For example, the classical Heston model corresponds to the choice

$$\kappa(\tau) = \nu e^{-\lambda\tau},$$

and the rough Heston model of [EER19] to the choice

$$\kappa(\tau) = \nu \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda \tau^\alpha),$$

with $\nu, \lambda > 0$ and $\alpha = H + \frac{1}{2} \in (\frac{1}{2}, 1]$. Integrating the forward variance equation gives

$$d\xi_t(T) = \left(\int_T^{T+\Delta} \kappa(u-t) du \right) \sqrt{V_t} dW_t =: \bar{\kappa}(T-t) \sqrt{V_t} dW_t \quad (3.2)$$

which has the same form as (3.1).

Lemma 3.1. *In the affine forward variance model (3.1) all diamond trees (with leaves of two types $X = \circ$ and $\zeta = \bullet$, respectively), and hence all forests terms \mathbb{G}_t^k in (2.1) are of the form*

$$\int_t^T \xi_t(u) h(T-u) du \quad (3.3)$$

for some integrable function h .

Proof. As we computed earlier, $\circ = \circ \circ = \int_t^T \xi_t(s) ds$, and from (3.2),

$$\circ \bullet = \rho \int_t^T \xi_t(u) \bar{\kappa}(T-u) du, \quad \bullet \bullet = \int_t^T \xi_t(u) \bar{\kappa}(T-u)^2 du. \quad (3.4)$$

We thus see that the claim holds for all diamond trees with two leaves and proceed by induction. Consider two trees

$$\mathbb{T}_t^i = \int_t^T \xi_t(u) h^i(T-u) du, \quad i = 1, 2$$

of the supposed form. Then

$$\begin{aligned}
(\mathbb{T}^1 \diamond \mathbb{T}^2)_t(T) &= \mathbb{E} \left[\int_t^T d\langle \mathbb{T}^1, \mathbb{T}^2 \rangle_u \middle| \mathcal{F}_t \right] \\
&= \mathbb{E} \left[\int_t^T \int_u^T \int_u^T h^1(T-s) h^2(T-r) ds dr d\langle \xi(s), \xi(r) \rangle_u \middle| \mathcal{F}_t \right] \\
&= \mathbb{E} \left[\int_t^T V_u \kappa(s-u) \kappa(r-u) du \int_u^T h^1(T-s) ds \int_u^T h^2(T-r) dr \middle| \mathcal{F}_t \right] \\
&= \int_t^T \xi_t(u) h^{12}(T-u) du,
\end{aligned}$$

and the induction step is completed upon setting

$$h^{12}(T-u) = \int_u^T h^1(T-s) \kappa(T-s) ds \int_u^T h^2(T-r) \kappa(T-r) dr.$$

□

At this stage it is tempting to combine Lemma 3.1 with Theorem 2.1 to compute the triple-joint mgf of X_T , $\langle X \rangle_{t,T}$, and $\zeta_T(T)$ by summing the full \mathbb{G} -expansion for an affine forward variance model. Since each tree in the \mathbb{G} -expansion has the form $\int_t^T \xi_t(u) h(T-u) du$, it follows that the mgf is necessarily of the convolutional form

$$\log \mathbb{E} \left[e^{aX_T + b\langle X \rangle_{t,T} + c\zeta_T(T)} \middle| \mathcal{F}_t \right] = aX_t + b\langle X \rangle_t + c\zeta_t(T) + \int_t^T \xi_t(u) g(T-u; a, b, c, \Delta) du,$$

which amounts to an infinite-dimensional version of the classical affine ansatz. Inserting $\Lambda_t(T) = \int_t^T \xi_t(u) g(T-u; a, b, c, \Delta) du$ directly into the “abstract Riccati” equation (1.3), we readily obtain that the triple-joint MGF satisfies a convolution Riccati equation of the type considered in [AJLP19, GKR19]. We summarise this in the following theorem.

Theorem 3.1. *Let*

$$\begin{aligned}
dX_t &= -\frac{1}{2} V_t dt + \sqrt{V_t} dZ_t \\
d\xi_t(T) &= \kappa(T-t) \sqrt{V_t} dW_t,
\end{aligned}$$

with $d\langle W, Z \rangle_t = \rho dt$ and let $\langle X \rangle_{t,T} = \langle X \rangle_T - \langle X \rangle_t$. Further let $\tau = T-t$, $\bar{\kappa}(\tau) = \int_\tau^{\tau+\Delta} \kappa(u) du$, and define the convolution integral

$$(\kappa \star g)(\tau) = \int_0^\tau \kappa(\tau-s) g(s) ds.$$

Then

$$\mathbb{E} \left[e^{aX_T + b\langle X \rangle_{t,T} + c\zeta_T(T)} \middle| \mathcal{F}_t \right] = \exp \{ aX_t + c\zeta_t(T) + (\xi \star g)(T-t; a, b, c, \Delta) \}$$

where $g(\tau; a, b, c, \Delta)$ satisfies the convolution Riccati integral equation

$$g(\tau; a, b, c, \Delta) = b - \frac{1}{2}a + \frac{1}{2}(1-\rho^2)a^2 + \frac{1}{2}[\rho a + c\bar{\kappa}(\tau) + (\kappa \star g)(\tau; a, b, c, \Delta)]^2, \quad (3.5)$$

with the boundary condition $g(0; a, b, c, \Delta) = b + \frac{1}{2}a(a-1) + \rho ac\bar{\kappa}(0) + \frac{1}{2}c^2\bar{\kappa}(0)^2$.

Proof. From (3.2), $d\zeta_t(T) = \sqrt{V_t} dW_t \bar{\kappa}(T-t)$. As before, let $\Lambda_t = \int_t^T \xi_t(u)g(T-u; a, b, c, \Delta)du$. Then dropping the arguments a, b, c, Δ for ease of notation,

$$\begin{aligned} d\Lambda_t &= -\xi_t(t)g(T-t)dt + \int_t^T d\xi_t(s)g(T-s)ds \\ &= -V_t g(T-t)dt + \sqrt{V_t} dW_t \int_t^T \kappa(s-t)g(T-s)ds \\ &= -V_t g(T-t)dt + \sqrt{V_t} dW_t (\kappa \star g)(T-t). \end{aligned}$$

We compute

$$\begin{aligned} d\langle X \rangle_t &= V_t dt \\ d\langle X, \zeta \rangle_t &= \rho V_t \bar{\kappa}(T-t) dt \\ d\langle \zeta \rangle_t &= V_t \bar{\kappa}(T-t)^2 dt \\ d\langle X, \Lambda \rangle_t &= \rho V_t (\kappa \star g)(T-t) dt \\ d\langle \Lambda \rangle_t &= V_t [(\kappa \star g)(T-t)]^2 dt \\ d\langle \zeta, \Lambda \rangle_t &= V_t \bar{\kappa}(T-t) (\kappa \star g)(T-t) dt. \end{aligned}$$

Integrating these terms from t to T , then taking a time- t conditional expectation allows us to compute all diamond products in the “abstract Riccati” equation (1.3)

$$\Lambda = \left(\frac{1}{2}a(a-1) + b\right) \circlearrowleft + ac \circlearrowright + a \circ \diamond \Lambda + \frac{1}{2} [c \bullet + \Lambda] \diamond^2$$

to yield

$$g(\tau) = \left(\frac{1}{2}a(a-1) + b\right) + \rho ac \bar{\kappa}(\tau) + a\rho (\kappa \star g)(\tau) + \frac{1}{2} [c \bar{\kappa}(\tau) + (\kappa \star g)(\tau)]^2,$$

which upon rearrangement gives (3.5). Finally, $(\kappa \star g)(0) = 0$ gives the boundary condition

$$g(0) = b + \frac{1}{2} a(a-1) + \rho ac \bar{\kappa}(0) + \frac{1}{2} c^2 \bar{\kappa}(0)^2.$$

□

4 Explicit computations under rough Heston

Explicit computations are easiest in the rough Heston model with $\lambda = 0$. In this case, with $\alpha = H + 1/2 \in (1/2, 1]$,

$$\begin{aligned} dX_t &= -\frac{1}{2}V_t dt + \sqrt{V_t} dZ_t \\ d\xi_t(u) &= \frac{\nu}{\Gamma(\alpha)} (u-t)^{\alpha-1} \sqrt{V_t} dW_t, \end{aligned}$$

with $d\langle W, Z \rangle = \rho dt$. Starting with the simplest tree, the fair strike of the (total) variance swap is given by

$$M_t(T) = \bullet = \circlearrowleft = (X \diamond X)_t(T) = \int_t^T \xi_t(u) du,$$

as we noted earlier. Thus, abbreviating bounded variation terms as ‘BV’, we have

$$\begin{aligned} dX_t &= \sqrt{V_t} dZ_t + \text{BV} \\ dM_t &= \int_t^T d\xi_t(u) du + \text{BV} \\ &= \frac{\nu}{\Gamma(\alpha)} \sqrt{V_t} \left(\int_t^T \frac{du}{(u-t)^\gamma} \right) dW_t + \text{BV} \\ &= \frac{\nu(T-t)^\alpha}{\Gamma(1+\alpha)} \sqrt{V_t} dW_t + \text{BV}. \end{aligned}$$

We now proceed to compute the first few $\tilde{\mathbb{F}}$ forests explicitly. There is only one tree in the forest $\tilde{\mathbb{F}}_1$.

$$\begin{aligned} \circlearrowleft = (X \diamond M)_t(T) &= \mathbb{E} \left[\int_t^T d\langle X, M \rangle_s \middle| \mathcal{F}_t \right] \\ &= \frac{\rho \nu}{\Gamma(1+\alpha)} \mathbb{E} \left[\int_t^T \nu_s (T-s)^\alpha ds \middle| \mathcal{F}_t \right] \\ &= \frac{\rho \nu}{\Gamma(1+\alpha)} \int_t^T \xi_t(s) (T-s)^\alpha ds. \end{aligned}$$

In order to continue to higher orders, it makes sense to define for $j \geq 0$

$$I_t^{(j)}(T) := \int_t^T ds \xi_t(s) (T-s)^{j\alpha}.$$

Then

$$\begin{aligned} dI_s^{(j)}(T) &= \int_s^T du d\xi_s(u) (T-u)^{j\alpha} + \text{BV} \\ &= \frac{\nu \sqrt{v_s}}{\Gamma(\alpha)} dW_s \int_s^T \frac{(T-u)^{j\alpha}}{(u-s)^\gamma} du + \text{BV} \\ &= \frac{\Gamma(1+j\alpha)}{\Gamma(1+(j+1)\alpha)} \nu \sqrt{v_s} (T-s)^{(j+1)\alpha} dW_s + \text{BV}. \end{aligned}$$

With this notation,

$$(X \diamond M)_t(T) = \frac{\rho \nu}{\Gamma(1+\alpha)} I_t^{(1)}(T).$$

There are two trees in $\tilde{\mathbb{F}}_2$:

$$\begin{aligned} \text{tree} &= (M \diamond M)_t(T) = \mathbb{E} \left[\int_t^T d\langle M, M \rangle_s \middle| \mathcal{F}_t \right] \\ &= \frac{\nu^2}{\Gamma(1 + \alpha)^2} \int_t^T \xi_t(s) (T - s)^{2\alpha} ds \\ &= \frac{\nu^2}{\Gamma(1 + \alpha)^2} I_t^{(2)}(T) \end{aligned}$$

and

$$\begin{aligned} \text{tree} &= (X \diamond (X \diamond M))_t(T) = \frac{\rho \nu}{\Gamma(1 + \alpha)} \mathbb{E} \left[\int_t^T d\langle X, I^{(1)} \rangle_s \middle| \mathcal{F}_t \right] \\ &= \frac{\rho^2 \nu^2}{\Gamma(1 + 2\alpha)} I_t^{(2)}(T). \end{aligned}$$

Note in passing that tree gives the variance of the variance swap. Continuing to the forest $\tilde{\mathbb{F}}_3$, we have the following.

$$\begin{aligned} \text{tree} &= (M \diamond (X \diamond M))_t(T) = \frac{\rho \nu^3}{\Gamma(1 + \alpha) \Gamma(1 + 2\alpha)} I_t^{(3)}(T) \\ \text{tree} &= (X \diamond (X \diamond (X \diamond M)))_t(T) = \frac{\rho^3 \nu^3}{\Gamma(1 + 3\alpha)} I_t^{(3)}(T) \\ \text{tree} &= (X \diamond (M \diamond M))_t(T) = \frac{\rho \nu^3 \Gamma(1 + 2\alpha)}{\Gamma(1 + \alpha)^2 \Gamma(1 + 3\alpha)} I_t^{(3)}(T). \end{aligned}$$

In particular, we readily identify the pattern

$$(X^{\circ k} M)_t(T) = \frac{(\rho \nu)^k}{\Gamma(1 + k\alpha)} I_t^{(k)}(T), \quad (4.1)$$

which gives us a simple closed-form expression for a tree with k \bullet leaves and only one \circ leaf.

4.1 Time scaling of the $\tilde{\mathbb{F}}$ -expansion

By inspection, fixing model parameters, we see that $\tilde{\mathbb{F}}_k$ scales as

$$I_t^{(k)}(T) = \int_t^T ds \xi_t(s) (T - s)^{k\alpha} \sim (T - t)^{k\alpha+1} \quad \text{as } T \downarrow t.$$

Thus, the $\tilde{\mathbb{F}}$ -expansion of Corollary 2.2 has the interpretation of a short-time expansion, the concrete powers of which depend on the roughness parameter $\alpha = H + 1/2 \in (1/2, 1)$. The resulting diamond expansions (which can be obtained by alternative methods in the rough Heston case) have been applied to construct efficient numerical schemes for the computation of the rough Heston characteristic function [CGP21, GR19].

4.2 The leverage swap

Recall that Equation (2.6) gives a model-free expression for the fair value of the leverage swap in terms of trees with only one \bullet leaf. Substituting from (4.1), we obtain

$$\begin{aligned} \mathcal{L}_t(T) &= \sum_{k=1}^{\infty} (X^{\diamond k} M)_t(T) \\ &= \sum_{k=1}^{\infty} \frac{(\rho \nu)^k}{\Gamma(1 + k \alpha)} \int_t^T du \xi_t(u) (T - u)^{k\alpha} \\ &= \int_t^T du \xi_t(u) \{E_{\alpha}(\rho \nu (T - u)^{\alpha}) - 1\} \end{aligned}$$

where $E_{\alpha}(\cdot)$ denotes the Mittag-Leffler function. We have thus obtained an explicit expression for the leverage swap under rough Heston with $\lambda = 0$. As discussed earlier in Section 2, since we can impute the leverage swap $\mathcal{L}_t(t)$ from the smile for each expiration T , fast calibration of the rough Heston model is then possible.

4.3 The rough Heston implied volatility skew

As in Section 2.1, fix $t = 0$ and define $\Sigma(k, T) = \sigma_{\text{BS}}^2(k, T) T$. Then, from (2.12), in a general stochastic volatility model, the short dated ($T \downarrow 0$) total variance skew is given by

$$\partial_k \Sigma(k, T)|_{k=0} = \frac{X \diamond M}{M}.$$

Recall that under rough Heston, $M_t(T) = \bullet = \circ \vee \circ = (X \diamond X)_t(T) = \int_t^T \xi_t(u) du$ and,

$$\circ \vee \bullet = (X \diamond M)_t(T) = \frac{\rho \nu}{\Gamma(H + 3/2)} \int_t^T \xi_t(s) (T - s)^{H+1/2} ds.$$

Setting $t = 0$ in these expressions, with $\xi_0(s) \rightarrow \xi_0(0) = V_0$, time-0 spot variance as $s \downarrow 0$, an easy computation gives

$$\left. \frac{\partial}{\partial k} \right|_{k=0} \sigma_{\text{BS}}^2(k, T) \sim \frac{\rho \nu}{(H + 3/2) \Gamma(H + 3/2)} T^{H-1/2} = \frac{\rho \nu}{\Gamma(H + 5/2)} T^{H-1/2}.$$

As a sanity check, with $H = 1/2$, we recover the classical Heston model with no mean reversion, and noting that $\Gamma(3) = 2! = 2$, the well-known short-dated Heston implied variance skew formula, cf. [Gat06, p.35]. For $H < 1/2$ we see skew explosion of order $T^{H-1/2}$, as expected.

5 Explicit computations under rough Bergomi

The skeptical reader might wonder whether explicit diamond tree computations are only possible in AFV models. We now show that explicit tree computations are also possible in the rough Bergomi model.

From [BFG16], reprinted as Chapter ??, the rough Bergomi model may be written as

$$\begin{aligned}\frac{dS_t}{S_t} &= \sqrt{V_t} \left\{ \rho dW_t + \sqrt{1-\rho^2} dW_t^\perp \right\} \\ V_u &= \xi_t(u) \mathcal{E} \left(\eta \sqrt{2H} \int_t^u \frac{dW_s}{(u-s)^\gamma} \right)\end{aligned}$$

with $\gamma = \frac{1}{2} - H$. In forward variance form

$$\frac{d_t \xi_t(u)}{\xi_t(u)} = \tilde{\eta} \frac{dW_t}{(u-t)^\gamma}$$

with $\tilde{\eta} := \eta \sqrt{2H}$. Then, again abbreviating bounded variation terms as ‘BV’, we have

$$\begin{aligned}dX_t &= \sqrt{V_t} dZ_t + \text{BV} \\ dM_t &= \tilde{\eta} \left(\int_t^T \xi_t(u) \frac{du}{(u-t)^\gamma} \right) dW_t + \text{BV}.\end{aligned}$$

As in the rough Heston case, we now proceed to compute the first few $\tilde{\mathbb{F}}$ forests explicitly.

5.1 The first order forest $\tilde{\mathbb{F}}_1$

There is only one tree in the forest $\tilde{\mathbb{F}}_1$.

$$\begin{aligned}\diamond \circ &= (X \diamond M)_t(T) = \mathbb{E} \left[\int_t^T d\langle X, M \rangle_s \middle| \mathcal{F}_t \right] \\ &= \rho \tilde{\eta} \mathbb{E} \left[\int_t^T ds \sqrt{v_s} \int_s^T \xi_s(u) \frac{du}{(u-s)^\gamma} \middle| \mathcal{F}_t \right] \\ &= \rho \tilde{\eta} \int_t^T ds \int_s^T \mathbb{E} \left[\sqrt{\xi_s(s)} \xi_s(u) \middle| \mathcal{F}_t \right] \frac{du}{(u-s)^\gamma}.\end{aligned}$$

Applying Lemma A.2, we have

$$\begin{aligned}&\mathbb{E} \left[\sqrt{\xi_s(s)} \xi_s(u) \middle| \mathcal{F}_t \right] \\ &= \sqrt{\xi_t(s)} \xi_t(u) \exp \left\{ \frac{\eta^2}{2} (s-t)^{2H} G_\gamma \left(\frac{u-t}{s-t} \right) \right\} \exp \left\{ -\frac{\eta^2}{8} (s-t)^{2H} \right\}.\end{aligned}$$

with $G_\gamma(\cdot)$ as defined in Appendix A. Thus

$$\begin{aligned} \circlearrowleft^\circ &= (X \diamond M)_t(T) \\ &= \rho \tilde{\eta} \int_t^T ds \sqrt{\xi_t(s)} \int_s^T \frac{du}{(u-s)^\gamma} \xi_t(u) \exp \left\{ \frac{\eta^2}{2} (s-t)^{2H} \left[G_\gamma \left(\frac{u-t}{s-t} \right) - \frac{1}{4} \right] \right\}. \end{aligned} \quad (5.1)$$

5.2 The second order forest $\tilde{\mathbb{F}}_2$

There are two trees in $\tilde{\mathbb{F}}_2$, $\circlearrowleft^\circ = (M \diamond M)$ and $\circlearrowleft^\circ \circlearrowleft^\circ = (X \diamond (X \diamond M))$. First we compute

$$\begin{aligned} \circlearrowleft^\circ &= (M \diamond M)_t(T) = \mathbb{E} \left[\int_t^T d\langle M, M \rangle_s \middle| \mathcal{F}_t \right] \\ &= \tilde{\eta}^2 \mathbb{E} \left[\int_t^T ds \int_s^T \xi_s(r) \frac{dr}{(r-s)^\gamma} \int_s^T \xi_s(u) \frac{du}{(u-s)^\gamma} \middle| \mathcal{F}_t \right] \\ &= 2 \tilde{\eta}^2 \mathbb{E} \left[\int_t^T ds \int_s^T \xi_s(r) \frac{dr}{(r-s)^\gamma} \int_r^T \xi_s(u) \frac{du}{(u-s)^\gamma} \middle| \mathcal{F}_t \right]. \end{aligned}$$

Another application of Lemma A.2 gives, for $u \geq r$,

$$\begin{aligned} &\mathbb{E} [\xi_s(r) \xi_s(u) | \mathcal{F}_t] \\ &= \xi_t(r) \xi_t(u) \exp \left\{ \eta^2 (s-t)^{2H} G_\gamma \left(\frac{u-t}{s-t}, \frac{r-t}{s-t} \right) \right\}. \end{aligned}$$

Thus

$$\begin{aligned} \circlearrowleft^\circ &= (M \diamond M)_t(T) \\ &= 2 \tilde{\eta}^2 \int_t^T ds \int_s^T \xi_t(r) \frac{dr}{(r-s)^\gamma} \int_r^T \xi_t(u) \frac{du}{(u-s)^\gamma} \exp \left\{ \eta^2 (s-t)^{2H} G_\gamma \left(\frac{u-t}{s-t}, \frac{r-t}{s-t} \right) \right\}. \end{aligned} \quad (5.2)$$

Reflecting the fact that \circlearrowleft° is the variance of \circ , this integral may be simplified as

$$\begin{aligned} \circlearrowleft^\circ &= (M \diamond M)_t(T) \\ &= 2 \int_t^T \xi_t(u) du \int_t^u \xi_t(r) dr \left[\exp \left\{ \eta^2 (r-t)^{2H} G_\gamma \left(\frac{u-t}{r-t} \right) \right\} - 1 \right]. \end{aligned} \quad (5.3)$$

Next we compute

$$\circlearrowleft^\circ \circlearrowleft^\circ = (X \diamond (X \diamond M))_t(T) = \mathbb{E} \left[\int_t^T d\langle X, (X \diamond M) \rangle_s \middle| \mathcal{F}_t \right].$$

To compute this, we need that

$$d(X \diamond M)_s = \rho \tilde{\eta} \{dI_s + dJ_s\}$$

where

$$\begin{aligned} dI_s &= \frac{1}{2} \int_s^T dr \frac{d\xi_s(r)}{\sqrt{\xi_s(r)}} \int_r^T \frac{du}{(u-r)^\gamma} \xi_s(u) \exp \left\{ \frac{\eta^2}{2} (r-s)^{2H} \left[G_\gamma \left(\frac{u-s}{r-s} \right) - \frac{1}{4} \right] \right\} \\ &= dW_s \frac{\tilde{\eta}}{2} \int_s^T \frac{dr}{(r-s)^\gamma} \sqrt{\xi_s(r)} \int_r^T \frac{du}{(u-r)^\gamma} \xi_s(u) \exp \left\{ \frac{\eta^2}{2} (r-s)^{2H} \left[G_\gamma \left(\frac{u-s}{r-s} \right) - \frac{1}{4} \right] \right\} \end{aligned}$$

and

$$\begin{aligned} dJ_s &= \int_s^T dr \sqrt{\xi_s(r)} \int_r^T \frac{du}{(u-r)^\gamma} d\xi_s(u) \exp \left\{ \frac{\eta^2}{2} (r-s)^{2H} \left[G_\gamma \left(\frac{u-s}{r-s} \right) - \frac{1}{4} \right] \right\} \\ &= dW_s \tilde{\eta} \int_s^T dr \sqrt{\xi_s(r)} \int_r^T \xi_s(u) \frac{du}{(u-s)^\gamma (u-r)^\gamma} \exp \left\{ \frac{\eta^2}{2} (r-s)^{2H} \left[G_\gamma \left(\frac{u-s}{r-s} \right) - \frac{1}{4} \right] \right\}. \end{aligned}$$

Also from Lemma A.2 again,

$$\begin{aligned} &\mathbb{E} \left[\sqrt{\xi_s(s)} \sqrt{\xi_s(r)} \xi_s(u) \middle| \mathcal{F}_t \right] \\ &= \sqrt{\xi_t(s)} \sqrt{\xi_t(r)} \xi_t(u) \\ &\quad \times \exp \left\{ \eta^2 (s-t)^{2H} \left[\frac{1}{4} G_\gamma \left(\frac{r-t}{s-t} \right) + \frac{1}{2} G_\gamma \left(\frac{u-t}{s-t} \right) + \frac{1}{2} G_\gamma \left(\frac{u-t}{s-t}, \frac{r-t}{s-t} \right) \right] \right\} \\ &\quad \times \exp \left\{ -\frac{1}{8} \eta^2 [(s-t)^{2H} + (r-t)^{2H} - (r-s)^{2H}] \right\}. \end{aligned}$$

Define

$$\begin{aligned} &\eta^2 F(s-t, r-t, u-t) \\ &= \eta^2 (s-t)^{2H} \left[\frac{1}{4} G_\gamma \left(\frac{r-t}{s-t} \right) + \frac{1}{2} G_\gamma \left(\frac{u-t}{s-t} \right) + \frac{1}{2} G_\gamma \left(\frac{u-t}{s-t}, \frac{r-t}{s-t} \right) \right] \\ &\quad + \frac{\eta^2}{2} (r-s)^{2H} G_\gamma \left(\frac{u-s}{r-s} \right) - \frac{1}{8} \eta^2 [(s-t)^{2H} + (r-t)^{2H}]. \end{aligned}$$

Then

$$(X \diamond (X \diamond M))_t(T) = (\rho \tilde{\eta})^2 \left\{ \frac{1}{2} I_t + J_t \right\}$$

with

$$I_t = \int_t^T ds \sqrt{\xi_t(s)} \int_s^T \frac{dr}{(r-s)^\gamma} \sqrt{\xi_t(r)} \int_r^T \frac{du}{(u-r)^\gamma} \xi_t(u) \exp \{ \eta^2 F(s-t, r-t, u-t) \} \quad (5.4)$$

and

$$J_t = \int_t^T ds \sqrt{\xi_t(s)} \int_s^T dr \sqrt{\xi_t(r)} \int_r^T \frac{du}{(u-s)^\gamma (u-r)^\gamma} \xi_t(u) \exp \{ \eta^2 F(s-t, r-t, u-t) \}. \quad (5.5)$$

One might ask whether it is possible to simplify these expressions using integration by parts as we did for $M \diamond M$. Unfortunately, it seems that we cannot.

In summary, though computations of diamond trees under rough Bergomi are possible in closed-form as shown above, computations get more and more complicated for higher order forests and numerical implementation is far from straightforward. To make further progress, smart approximations will doubtless be required.

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A Conditional expectations of products of forward variances under rough Bergomi

We will need the following Lemma which is obtained by explicit integration.

Lemma A.1. *Let $u_j > u_i \geq s > t$ and $\gamma = \frac{1}{2} - H$. Then*

$$\int_t^s \frac{dr}{(u_j - r)^\gamma (u_i - r)^\gamma} = \frac{1}{2H} (s - t)^{2H} G_\gamma \left(\frac{u_j - t}{s - t}, \frac{u_i - t}{s - t} \right)$$

where for $y \geq x \geq 1$,

$$\begin{aligned} G_\gamma(y, x) &= 2H \int_0^1 \frac{dr}{(y - r)^\gamma (x - r)^\gamma} \\ &= \frac{1 - 2\gamma}{(1 - \gamma)(y - x)} \left\{ x^{1-\gamma} y^{1-\gamma} {}_2F_1 \left(1, 2 - 2\gamma; 2 - \gamma; \frac{y}{y - x} \right) \right. \\ &\quad \left. - (x - 1)^{1-\gamma} (y - 1)^{1-\gamma} {}_2F_1 \left(1, 2 - 2\gamma; 2 - \gamma; \frac{y - 1}{y - x} \right) \right\}. \end{aligned} \tag{A.1}$$

Equation (A.1) has the special cases

$$G_\gamma(y) := G_\gamma(y, 1) = \frac{1 - 2\gamma}{(1 - \gamma)(y - 1)} y^{1-\gamma} {}_2F_1 \left(1, 2 - 2\gamma; 2 - \gamma; \frac{y}{y - 1} \right)$$

with $y > 1$. In particular $G_\gamma(1) = G_\gamma(1, 1) = 1$.

Lemma A.2. *Consider the Rough Bergomi model*

$$V_u = \xi_u(u) = \xi_t(u) \mathcal{E} \left(\tilde{\eta} \int_t^u \frac{dW_r}{(u - r)^\gamma} \right)$$

and let $u_n > \dots > u_i > u_{i-1} > \dots \geq s > t$, $i = 1, \dots, n$. Then,

$$\begin{aligned} & \mathbb{E} \left[\prod_{i=1}^n \xi_s(u_i)^{\alpha_i} \middle| \mathcal{F}_t \right] \\ &= \left(\prod_{i=1}^n \xi_t(u_i)^{\alpha_i} \right) \exp \left\{ \eta^2 (s-t)^{2H} \sum_{j>i} \alpha_i \alpha_j G_\gamma \left(\frac{u_j-t}{s-t}, \frac{u_i-t}{s-t} \right) \right\} \\ & \quad \times \exp \left\{ \frac{1}{2} \eta^2 \sum_{i=1}^n \alpha_i (\alpha_i - 1) [(u_i-t)^{2H} - (u_i-s)^{2H}] \right\}. \end{aligned}$$

Proof. Let X be a Gaussian random variable and $\alpha \in \mathbb{R}$. Then

$$\mathcal{E}(X)^\alpha = \mathcal{E}(\alpha X) \exp \left\{ \frac{\alpha(\alpha-1)}{2} \text{var}[X] \right\}.$$

Also, with $\gamma = \frac{1}{2} - H$, $\tilde{\eta} = \eta \sqrt{2H}$ and $u > s > t$,

$$\xi_s(u) = \xi_t(u) \mathcal{E} \left(\tilde{\eta} \int_t^s \frac{dW_r}{(u-r)^\gamma} \right)$$

so

$$\xi_s(u)^\alpha = \xi_t(u)^\alpha \mathcal{E} \left(\alpha \tilde{\eta} \int_t^s \frac{dW_r}{(u-r)^\gamma} \right) \exp \left\{ \frac{1}{2} \eta^2 \alpha (\alpha - 1) [(u-t)^{2H} - (u-s)^{2H}] \right\}.$$

Also, if $X_i, i = 1, \dots, n$ are zero mean Gaussian random variables, then

$$\prod_{i=1}^n \mathcal{E}(X_i) = \mathcal{E} \left(\sum_{i=1}^n X_i \right) \exp \left\{ \sum_{j>i} \text{cov}[X_i X_j] \right\}.$$

Finally, assuming wlog that $u_j > u_i$, we have from Lemma A.1 that

$$\begin{aligned} \mathbb{E} \left[\alpha_j \tilde{\eta} \int_t^s \frac{dW_r}{(u_j-r)^\gamma} \alpha_i \tilde{\eta} \int_t^s \frac{dW_{r'}}{(u_i-r')^\gamma} \middle| \mathcal{F}_t \right] &= \alpha_i \alpha_j \tilde{\eta}^2 \int_t^s \frac{dr}{(u_j-r)^\gamma (u_i-r)^\gamma} \\ &= \alpha_i \alpha_j \eta^2 (s-t)^{2H} G_\gamma \left(\frac{u_j-t}{s-t}, \frac{u_i-t}{s-t} \right). \end{aligned}$$

□