Singular Exotic Perturbation

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January 2021/This version December 2021

Abstract

We combine singular perturbation techniques with a price adjustment argument to analyze the impact of the smile dynamics i.e. the price difference between Local Stochastic Volatility (LSV) and Local Volatility (LV) on exotic products. We obtain an elegant formula that is exact on vanilla options and propose a set of well-chosen scenario, to compute the impact efficiently. We perform massive tests on Autocalls. [1]

1 Introduction

The main driver when selecting a model for pricing and risk management derivatives products is its capacity to explain the PnL evolution [18], [16] and [1]. Choosing the right model to successfully price and hedge financial instruments is based on a careful study of the financial structure to be considered and in which market it evolves. The quantitative finance literature has initially promoted the local volatility (LV) and then the pure stochastic volatility (SV) models as means of explaining the observed market smile. However, when we consider the dynamic hedging of exotic products such as autocalls, we rapidly conclude that matching the smile is not enough, one needs to also control the way the latter evolves when the spot moves. None of the (LV) or (SV) model can describe the smile and its evolution properly. However, a well fine-tuned mix between the two gives the flexibility to both fit the vanilla options and the way they evolves when the spot moves. This mix in known in the literature as Local Stochastic Volatility models (we say LSV). The literature on this topic is vast and covers a diversity of approaches in the definition of the models or the way to calibrate them. We can cite [3] and [4] for a universal diffusion model presentation with applications to the FX derivatives. In [5], Lipton et al make a survey of (LSV) models applied to a variety of first-generation exotics. Lots of papers also in the literature cover the calibration approach as it represents one of the most important building blocks of the computation. In particular, Monte Carlo based approaches for calibration Henry-Labordere [9], van der Stoep et al. [10], Guyon and Henry-Labordere [11], calibration based on McKean's particle method Guyon and Henry-Labordere [11, 12], hyperbolic-local model Jackel and Kahl [13].

[1]We would like to warmly thank anonymous referees for their insightful comments. We also would like to thank Pierre Henry-Labordere and Jean-Pierre Fouque for useful discussions.

Due to the numerical difficulties on this calibration some papers propose to rely on a parametric approach with a forward process for the construction of the leverage function Murex [14], adding local volatility component to an unspanned volatility term structure, model Halperin and Itkin [15]. In [6], Papanicolaou et al. perform a perturbative approach on SV models with one or two factors. However, the calibration of the vanilla is not considered. In [17], Reghai et al. introduce a mixing weight to control the correlation and the volatility of volatility of the process. The (LSV) impact is computed for exotics. However, this method does not span all possible stochastic volatility parameters and only works for mild parameters. In [8], Hagan et al. perform a singular perturbation analysis on term structure SABR model with fast varying parameters. However, this is done on a pure stochastic volatility model.

The objective of this paper is to apply singular perturbation approach in the case of the (LSV) model. For that purpose, we use a singular perturbation approach without focusing on the vanilla calibration as described in [6]. We then recover the vanilla fit using a price adjustment as described in the [7]. The obtained formulae are then computed effectively using well- chosen scenario as is done in the computation of the exotic theta in [1].

This is the main motivation of this work. We indeed propose an extremely fast algorithm that prices the (LSV) impact at a much lower computational cost than traditional (LSV) implementations. It is based on (LV) prices calculated on well- chosen volatility scenario. This is not only a game changer for real time risk management but also a powerful way to imply the stochastic volatility parameters in the presence of exotic prices. One last property of the proposed technique is that it reprices vanilla options perfectly removing all the known burden of (LSV) calibration. In the end, this formula offers a rapid, robust and easy implementation of an essential model in real time management. Also, we suggest that this technique is quite general and opens doors for other industrial applications which will make it possible to enhance all the existing perturbation formulas developed over the years and which did not find industrial applications due to lack of their equivalent in the presence of the smile.

The rest of the paper is organised as follows. In the next section, we clearly expose the obtained result and how a dimensional analysis could have predicted its form. In the section 2, we remind the Profit and Loss (PnL) explain equation and observe the fact the stochastic volatility parameters in equity are highly varying. This situation is convenient for applying singular perturbation approach. In section 3, we present the price adjustment technique as a general first order approach which improves the quality of prices based on a first order expansion in the direction of the calibration. In section 4, we perform the singular expansion combined with the adjustment. In section 5, we show how to design well-chosen scenario in order to compute the exotic greeks that appear in the perturbation expansion. In section 6, we perform massive tests on autocalls comparing the full Monte Carlo LSV impact calculation with the one obtained with the perturbation formula.

Problem formulation and main result

In order to ease the reading of the paper we propose to summarize the problem formulation in this section and give the main result of the paper.

Assume that the dynamic of the stock is given by the following LSV process:

$$\frac{dS_t}{S_t} = \frac{\sigma_D\,(t, S_t)}{f_\epsilon(t, S_t)} h(Y_t^\epsilon) dW_t \tag{1.1}$$

where, $h(x) = e^x$ and Y_t satisfies an Orstein-Uhlenbeck process:

$$dY_t^\epsilon = -\frac{\kappa}{\epsilon} Y_t^\epsilon dt + \frac{\nu}{\sqrt{\epsilon}} dB_t \tag{1.2}$$

with $< dB_t, dW_t >= \rho dt$

f is an adjustment of the Dupire local volatility $\sigma_D\,(t, S_t)$ in order to preserve the vanilla calibration. It satisfies the equation [9]:

$$f_\epsilon^2(t, S) = \mathbb{E}(h^2(Y_t^\epsilon)|S_t = S) \tag{1.3}$$

The particular choice of representation of κ the mean reversion and ν the volatility of volatility of the dynamic as $-\frac{\kappa}{\epsilon}$ and $\frac{\nu}{\sqrt{\epsilon}}$ expresses the fact that these parameters are large in practice, in order to fit the anticipated breakevens. At this stage we can note that κ^2 is homogeneous to ν. This choice is dictated by the fact that κ has a dimension inverse of time whereas ν is the inverse of square root of time.

The main contributions of the paper is first to provide a methodology that combines perturbation techniques with calibration and second to propose a computation strategy based on exotic greeks that is performing both theoretically and numerically.

The paper shows a detailed application to the stochastic volatility model as it derives a formula to compute efficiently the LSV impact $\pi_{LSV} - \pi_{LV}$. It is given by:

$$\pi_{LSV} \approx \pi_{LV} + \frac{1}{2}\sigma_y^2 \frac{\partial_E^2 \pi_{LV}^\beta}{\partial \beta^2}|_{\beta=0} + \frac{\rho\nu}{\kappa} \frac{\partial_E^2 \pi_{LV}}{\partial \ln S_0 \partial \sigma} \tag{1.4}$$

where:

- π_{LSV} is the price under the LSV :

$$\frac{dS_t}{S_t} = \sigma_D\,(t, S_t) \frac{e^{Y_t^\epsilon}}{\sqrt{\mathbb{E}(e^{2Y_t^\epsilon}|S_t)}} dB_t, dY_t^\epsilon = -\frac{\kappa}{\epsilon} Y_t^\epsilon dt + \frac{\nu}{\sqrt{\epsilon}} dB_t \tag{1.5}$$

- $\sigma_y^2 = \frac{\nu^2}{2\kappa}$ represents the variance of the invariant distribution of Y_t^ϵ when $\epsilon \longrightarrow 0$,

- Let π^{β}_{LV} := Local volatility pricing which depends on initial conditions β:

$$\frac{dS^{\beta}_t}{S^{\beta}_t} = \sigma_D\left(t, S^{\beta}_t\right)e^{\beta-\sigma^2_y}dW_t \tag{1.6}$$

- ∂_E means Exotic greek, We remind that any exotic greek ∂_{Exotic} of a payoff π is formed from the standard greek ∂ adjusted from the vanilla contribution. More precisely, if we define

$$q_{K,T} = \frac{\partial_{\sigma(K,T)}\pi}{\partial_{\sigma(K,T)}C_{K,T}}. \tag{1.7}$$

  $q_{K,T}$ represents the quantity of vanilla $C_{K,T}$ to be detained in order to hedge volatility surface movement.

$$\partial_{Exotic}(\pi) = \partial(\pi) - \int_0^T\int_0^{\infty} q_{K,T}\partial(C_{K,T})dTdK$$

We also show how to compute the exotic greek through well-chosen scenario at a very low computational cost. We end up by showing numerical results on the mostly traded instruments in equity derivatives i.e. Autocalls.

2 PnL explain

The most important feature of a model is its ability to explain the PnL evolution on a daily basis. For this exercise, it is based on **3 pillars**:

- **Option**, P, which gives the exposure,
- **Market**, $(\frac{dS}{S})^2, d\Sigma dS$, which gives the Break even of the stock volatility,
- and **Model**, $\sigma_D, \alpha, \kappa, \nu$... which gives the intrinsic property of the model.

A delta hedged position under Black & Scholes model gives the following PnL explanation formula:

$$\delta PL = \tfrac{1}{2}S^2\tfrac{d^2P}{dS^2}[(\tfrac{\delta S^2}{S^2})^{rlzd} - \sigma^2\delta t] \quad \textit{Gamma}$$

The fair price is obtained by putting the model parameter σ to its corresponding realized value or at least its anticipated level $(\frac{\delta S^2}{S^2})^{rlzd}$.

Likewise, if we use an advanced model such as (LV)or (LSV) model and perform uniquely a delta hedge strategy, we obtain the following PnL explanation formula is given by:

$$\begin{aligned}
\delta PL &= \tfrac{dP}{d\sigma_{KT}}[(\delta\sigma_{KT})^{rlzd} - (\delta\sigma_{KT})^{model}] && \text{Vega}\\
&+ \tfrac{1}{2}S^2\tfrac{d^2P}{dS^2}[(\tfrac{\delta S^2}{S^2})^{rlzd} - \sigma^2\delta t] && \text{Gamma}\\
&+ S\sigma_{KT}\tfrac{d^2P}{dSd\sigma_{KT}}[(\tfrac{\delta S\delta\sigma_{KT}}{S\sigma_{KT}})^{rlzd} - (\tfrac{\delta S\delta\sigma_{KT}}{S\sigma_{KT}})^{model}] && \text{Vanna}\\
&+ \tfrac{1}{2}\sigma^2_{KT}\tfrac{d^2P}{d\sigma^2_{KT}}[(\tfrac{\delta\sigma^2_{KT}}{\sigma^2_{KT}})^{rlzd} - (\tfrac{\delta\sigma^2_{KT}}{\sigma^2_{KT}})^{model}] && \text{Volga}
\end{aligned}$$

Before tackling the fair value pricing through matching the Break Even, one needs to cancel the Vega risk which is a first order magnitude and is in practice 20 times more important that second order terms. By doing so, the PnL explanation formula modifies with the first order term that disappears but also with exposures that are now adjusted by the vega hedging. The new exposures are called exotic exposures as they are nil for vanilla options or any product that is replicable with vanillas.

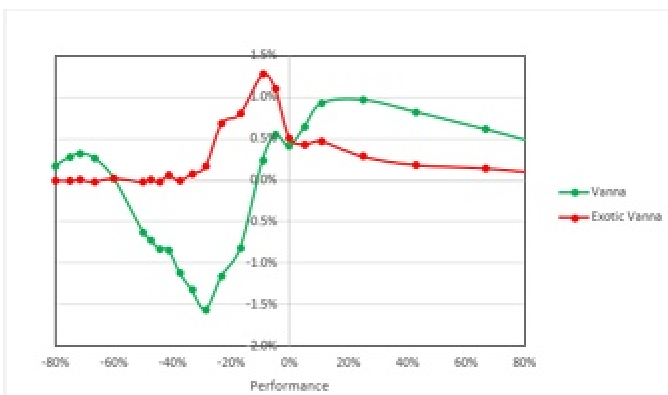
More precisely, to cancel **Vega Risk**, the trader needs to sell q_{KT} vanilla options C_{KT} (1.7).

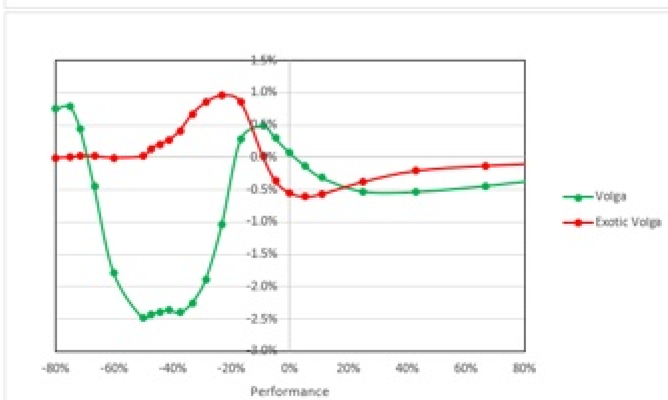
Her new PnL equation is given by:

- $P^H = P - q_{KT}C_{KT}$

$$\begin{array}{rcll} \delta PL^H & = & \frac{1}{2}S^2\frac{d^2P^H}{dS^2}[(\frac{\delta S^2}{S^2})^{rlzd} - \sigma^2\delta t] & \text{Exotic Gamma} \\ & + & S\sigma_{KT}\frac{d^2P^H}{dSd\sigma_{KT}}[(\frac{\delta S\delta\sigma_{KT}}{S\sigma_{KT}})^{rlzd} - (\frac{\delta S\delta\sigma_{KT}}{S\sigma_{KT}})^{model}] & \text{Exotic Vanna} \\ & + & \frac{1}{2}\sigma_{KT}^2\frac{d^2P^H}{d\sigma_{KT}^2}[(\frac{\delta\sigma_{KT}^2}{\sigma_{KT}^2})^{rlzd} - (\frac{\delta\sigma_{KT}^2}{\sigma_{KT}^2})^{model}] & \text{Exotic Volga} \end{array}$$

As an illustration, we can compute the exotic greeks of an Autocall product.





The exotic volga is obviously different from the classic one but presents compensations between the downside and the upside. The Exotic vanna for the autocall product (long position) remains positive regardless to the scenarii of the spot.

This has a strong consequence on the hedging with (LV) model for which the model Break Even is exactly -1 and therefore generates a negative carry for the seller of the Autocall.

At this stage, we can conclude that hedging with (LV) model an Autocall will generate systematic loss on the exotic vanna term. A way to compensate and have a fair pricing is to move to a Local Stochastic Volatility 1 Factor (LSV1F) model that fits the correlation break even. This is done through levels of stochastic volatility parameters that are extreme $\kappa, \nu >> 1$ and ρ close to -1.

3 Price Adjustment technique

We start with the following Newton modified lemma.

Lemma 1. *Modified Newton: Suppose you have a function f of a variable x. Suppose that $x^\star$ is set such that $g(x^\star) = 0$ i.e. g satisfies a given constraint. The value of f on $x^\star$ is given by the following formula:*

$$f(x^\star) = f(x) - \frac{\partial_x f(x)}{\partial_x g(x)} g(x) + O(x^\star - x)^2$$

Typically, $g(x) := \pi_{Model}(x) - \pi_{Market}$ is a way to encode a calibration.
In multi dimensions, the adjustment takes the following form

$$f(\pi, \overrightarrow{\beta^\star}) = f(\pi, \overrightarrow{\beta}) - \nabla_{\overrightarrow{\beta}} f_{nx1}.[\partial_i g_j]^{-1}_{nxn}.[g_i(\overrightarrow{\beta})]_{nx1} + O(||\overrightarrow{\beta^\star} - \overrightarrow{\beta}||^2) \tag{3.1}$$

The proof of this lemma goes like this. First we follow the lines of Newton's approach by searching for $x^\star$ as a perturbation of x, namely, $x^\star = x + \epsilon$. As $g(x^\star) = 0$ we can expand as follows:

$$g(x + \epsilon) = g(x) + \epsilon \partial_x g(x) = 0$$

Then, $\epsilon = -\frac{g(x)}{\partial_x g(x)}$ Now, We expand $f(x^\star)$ to obtain the final result. QED

This result is the basis of price adjustment in order to fit a given set of constraints. Fitting is another word for calibration. Indeed, the constraint $g(x) = 0$ is usually written in finance as follows:

$$g(x) := \pi_{Model}(x) - \pi_{Market}$$

where x plays the role of model parameters.

In more classical financial notations, we can adjust our (LV) model price in order to match exactly vanilla prices using the following vectorial formula:

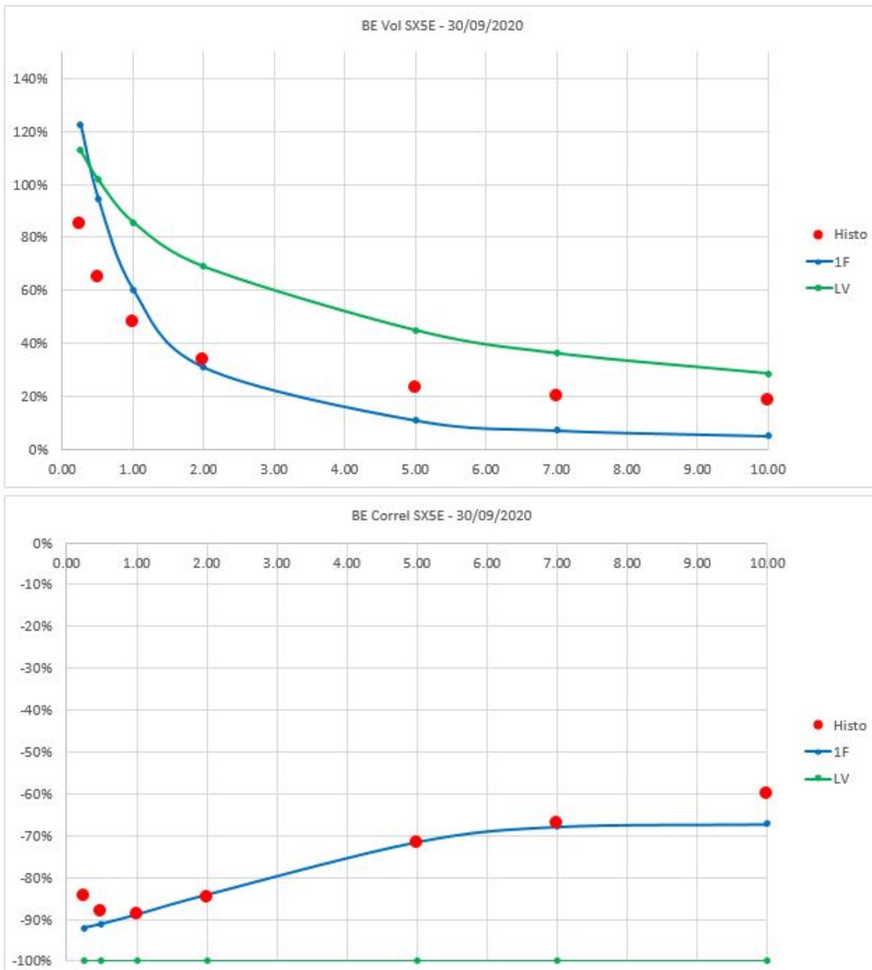


Figure 1: Market BE versus LSV1F BE

$$\pi_{LV} \longleftarrow \pi_{LV} - \int_0^T \int_0^\infty q_{K,T}(C_{LV}^{K,T} - C_{Market}^{K,T})dTdK$$

where $q_{K,T}$ is defined in (1.7).

The objective of this paper is to combine this idea and interpret the adjustment as an exotic greek as well as the design of a well-chosen scenario which will permit the precise computation of the LSV impact at a very low computational cost.

4 Singular Exotic Perturbation

In this section, we present the recipe behind the singular exotic perturbation. The objective is to solve the pricing dynamic as a function of ϵ and see how if converges when $\epsilon \longrightarrow 0$ for equation (1.1).

f is the conditional expectation in such a way that we fit the vanilla. It satisfies the equation (1.3).

Making the expansion in the presence of f_ϵ is hard. Instead, we use the following recipe that we name the **singular exotic perturbation**:

- We perform the singular perturbation without calibrating the vanilla, i.e. $\lim_{\epsilon\to 0} f_\epsilon^2(t,S) = \mathbb{E}(h^2(Y_t^{\epsilon\to 0}))$. The zero order gives back the Dupire local volatility model. We identify the higher orders as volga and vanna contributions.
- We apply the modified Newton lemma in order to adjust the expansion and recover an exact calibration,
- We explicitly compute zero, first and second order order adjustment due to the singular perturbation and correct them in order to maintain the vanilla fit.
- We finally design well chosen scenari in order to ease the previous computations.

4.1 Singular perturbation on the non calibrated process

The non calibrated dynamic has the following form :

$$\frac{dS_t}{S_t} = \sigma_D\,(t,S_t)e^{Y_t^\epsilon - \sigma_y^2}dW_t$$

Let $u(t,x,y)$ be the price of the derivative. It satisfies the PDE:

$$u_t + \frac{1}{2}\sigma_D^2(t,x)x^2e^{2(y-\sigma_y^2)}u_{xx} + \frac{1}{\epsilon}\mathcal{L}_y u + \frac{1}{\sqrt{\epsilon}}\rho x\sigma_D(t,x)e^{y-\sigma_y^2}\nu u_{xy} = 0$$

$\mathcal{L}_y = -\kappa u_y + \frac{1}{2}\nu^2 u_{yy}$ We search for u as follows: $u = u_0 + \sqrt{\epsilon}u_1 + \epsilon u_2$

We then apply Poisson in u_2 (centring condition):

$$(\partial_t + \frac{1}{2}\sigma_D^2(t,x)x^2 < e^{2(y-\sigma_y^2)} > \partial_{xx})u_0 = 0$$

Where $< . >$ is the integration over the invariant distribution of Y.
which becomes:

$$\mathcal{L}_D u_0 = 0$$

where L_D is the Dupire operator:

$$\mathcal{L}_D = (\partial_t + \frac{1}{2}\sigma_D^2(t,x)x^2\partial_{xx})$$

We apply Poisson condition on u_3

$$\mathcal{L}_D u_1 + \rho\nu x\sigma_D(t,x) < e^{y-\sigma_y^2}\partial_{xy}u_2 >= 0$$

But $\mathcal{L}_y u_2 = -\frac{1}{2}x^2\sigma_D^2(t,x)(e^{2y-\sigma_y^2} - < e^{2(y-\sigma_y^2)}) >)\partial_{xx}u_0$, with $\mathcal{L}_y\phi = e^{2(y-\sigma_y^2)}$
This implies $u_2 = -\frac{1}{2}x^2\sigma_D^2(t,x)\phi(y)\partial_{xx}u_0 + C(t,x,\not{y})$.

$\mathcal{L}_D u_1 + \rho\nu x\sigma_D(t,x)\partial_x(-\frac{1}{2}x^2\sigma_D(t,x)\partial_{xx}u_0) = 0$ with $u_1(T,x) = 0$. with $V_3 =< e^{(y-\sigma_y)}\phi'(y) > (-\frac{1}{2}\frac{\rho\nu}{\kappa})$. (cf [6])

Therefore, applying Feynman-Kac to the previous formula the order 1 can be computed using only the local volatility model and its derivatives:

$$\begin{aligned}
u_1 &= V_3\mathbb{E}\int_0^T S_t\sigma_D(t,S_t)\partial_x(S_t^2\sigma_D^2(t,S_t)\partial_{xx}u_0)dt \\
&= -2V_3\mathbb{E}\int_0^T S_t\sigma_D(t,S_t)\partial_x(\partial_{\sigma_D(t,x)}u_0)dt \\
&= -2V_3\mathbb{E}\int_0^T\int_0^T\int_0^\infty S_t\sigma(t,S_t)\partial_x(\partial_{\sigma_{K,T}}u_0\partial_{\sigma_D(t,x)}\sigma(K,T))dKdTdt \\
&= -2V_3\mathbb{E}\int_0^T\int_0^T\int_0^\infty S_t\sigma_D(t,S_t)\partial_{\sigma_D(t,x)}\sigma(K,T)\partial_x\partial_{\sigma_{K,T}}u_0 dKdTdt \\
&- 2V_3\mathbb{E}\int_0^T\int_0^T\int_0^\infty S_t\sigma_D(t,S_t)\partial_{\sigma_{K,T}}u_0\partial_x(\partial_{\sigma_D(t,x)}\sigma(K,T))dKdTdt
\end{aligned}$$

In the last equality, the first term (I) shows the full vanna of the product summing up all contribution for a co movement of $S_t\sigma_D(t,S_t)$. The second term (II) has a contribution coming only from vanilla options weighted by the vega KT of the product. Therefore, when the product is vega KT hedged two consequences follow, the first term which is a vanna becomes an exotic vanna and the second term which is a combination of european contribution just vanishes. More precisely, [2],

$$
\begin{aligned}
(I) &= \mathbb{E}\int_0^T\int_0^T\int_0^\infty \sigma_D(t,S_t)\partial_{\sigma_D(t,x)}\sigma(K,T)\partial_{\ln x}\partial_{\sigma_{K,T}}u_0 dKdTdt \\
&= \int_0^T\int_0^\infty \partial_{\ln x}\partial_{\sigma_{K,T}}u_0 dKdT \\
&= \partial^2_{\ln x,\sigma}u_0
\end{aligned}
$$

$$
\begin{aligned}
(II) &= \mathbb{E}\int_0^T\int_0^T\int_0^\infty \sigma_D(t,S_t)\partial_{\sigma_{K,T}}u_0\partial_{\ln x}(\partial_{\sigma_D(t,x)}\sigma(K,T))dKdTdt \\
&= \int_0^T\int_0^\infty q_{K,T}\partial_{\sigma_{K,T}}C_{K,T}\partial_{\ln x}\sigma(K,T)dKdT
\end{aligned}
$$

4.2 Probabilistic interpretation

In the computation of the order 2 impact, the stochastic volatility is important. We start as if there is no recalibration.

$$
\frac{dS_t}{S_t} = \sigma_D\,(t,S_t)e^{Y-\sigma_y^2}dW_t
$$

where, Y is the invariant distribution of the previous process. Let π_{LV}^y as in equation (4.1). Note that $\pi_{LV}^{\sigma_y^2} = \pi_{LV}$. [3]

$$
\begin{aligned}
u_0 + u_2 &\approx \pi_{LV}^{\sigma_y^2} + \frac{1}{2}Var(Y)\frac{\pi_{LV}^{+\beta+\sigma_y^2} + \pi_{LV}^{-\beta+\sigma_y^2} - 2\pi_{LV}^{+\sigma_y^2}}{\beta^2} \\
&= u_0 + \frac{1}{2}\sigma_y^2\frac{\partial^2\pi_{LV}^\beta}{\partial\beta^2}|_{\beta=+\sigma_y^2}
\end{aligned}
$$

4.3 Adjusting the prices to recover the vanilla fit

At this stage, we can apply the modified Newton lemma in order to compensate for the non calibration. Let us note P^{NC} the price obtained using u_0+u_2 which is non calibrated. Let us note P the calibrated price to the vanilla. We see that this pricing does not match vanilla options due to the extra term $u_2 = \frac{1}{2}\sigma_y^2\frac{\partial^2\pi_{LV}^\beta}{\partial\beta^2}|_{\beta=+\sigma_y^2}$. We can build the adjusted price P by compensating at first order.

[2] We use the following equation $\mathbb{E}\int_0^T \sigma_D(t,S_t)\partial_{\sigma_D(t,x)}\sigma(K,T)dt = 1.0$

[3] $\mathbb{E}(f(Y)) \approx f(\mathbb{E}(Y)) + \frac{1}{2}Var(Y)f''(\mathbb{E}(Y))$

$$
\begin{aligned}
P &= P^{NC} - \int_0^T \int_0^\infty q_{K,T}(P^{NC}(C_{K,T}) - C^{K,T}_{Market})dTdK \\
&= \pi_{LV} + \frac{1}{2}\sigma_y^2 \frac{\partial^2 \pi^\beta_{LV}}{\partial \beta^2} - \int_0^T \int_0^\infty q_{K,T}(C^{K,T}_{LV} + \frac{1}{2}\sigma_y^2 \frac{\partial^2 C^{\beta,K,T}_{LV}}{\partial \beta^2} - C^{K,T}_{Market})dTdK \\
&= \pi_{LV} + \frac{1}{2}\sigma_y^2 (\frac{\partial^2 \pi^\beta_{LV}}{\partial \beta^2} - \int_0^T \int_0^\infty q_{K,T} \frac{\partial^2 C^{\beta,K,T}_{LV}}{\partial \beta^2} dTdK)
\end{aligned}
$$

Where we have used, $C^{K,T}_{LV} = \pi^{K,T}_{Market}$.

We introduce the exotic volga greek:

$$
\frac{\partial^2_E \pi_{LV}}{\partial \beta^2} = (\frac{\partial^2 \pi^\beta_{LV}}{\partial \beta^2} - \int_0^T \int_0^\infty q_{K,T} \frac{\partial^2 C^{\beta,K,T}_{LV}}{\partial \beta^2} dTdK)
$$

The exotic volga appears naturally for perturbation greeks once we adjust for the calibration of vanilla.

Similarly, we adjust the term I by introducing the effect of the calibration to the vanilla options and calculate I_C.

$$
\begin{aligned}
I_C &= I - \int_0^T \int_0^\infty q_{K,T} \partial^2_{\ln x, \sigma_{K,T}} C_{K,T} dKdT \\
&= \frac{\partial^2_E \pi_{LV}}{\partial \ln S_0 \partial \sigma}
\end{aligned}
$$

We end up with the final result described in (1.4).

We have obtained the desired result. In particular we have identified the functions that intervene in the expansion and their exotic nature gives an important property of the formula: it makes that the vanilla impact of the LSV is exactly zero. This formula is very interesting in order to understand the effect of stochastic volatility on top of the local volatility. However, a brute force implementation will need to provide all the q_{KT}. We can for example use an offline computation as proposed in [7]. We can also use the AAD technology (Algorithmic Automatic differentiation) to compute all the $q_{K,T}$ at a cost that does not exceed 4 price computations. We shall instead propose the design of particular scenarii in order to compute these exotic greeks based on two building blocks: implied Black & Scholes calculator and local volatility pricer. We approach the problem as the computation of the exotic theta as presented in [1].

5 Designing Exotic scenarii

In order to compute an exotic version of a sensitivity we proceed as follows:

- Compute the sensitivity using a classical scenarii approach,

- Adjust the scenarii in such a way that the effect on vanilla options disappear.

We use the perturbed local volatility 4.1.

We therefore obtain the exact definition of the asymmetric bump that makes the volga vanilla disappear:

$$\sigma_{K,T} + x_{K,T} = C^{-1}_{K,T}\{C^{K,T}_{LV} + (C^{K,T}_{LV} - C^{-\beta,K,T}_{LV})\} \tag{5.1}$$

The **exotic volga** greek is then computed as follows: [4]

$$\frac{\partial^2_E \pi_{LV}}{\partial \beta^2} \overset{\beta \to 0}{=} \frac{\pi_{LV}(\sigma + x) - 2\pi_{LV} + \pi^{-\beta}_{LV}}{\beta^2}$$

This greek is the result of 3 local volatility prices. One of them is already computed as it is the central price with no deformation of the volatility surface. The two others are computed by firstly generating a scenario of volatility deformation using a bump with a value of β and secondly creating an implied volatility bump $x(K,T)$ constructed point by point by implying the volatility from equation (5.1). This construction guarantees that the exotic term is mechanically zero on vanilla options. This property gives it its name : exotic.

It is only non zero if and only if the product is a non vanilla option, i.e. non replicable using vanilla option.

At this stage we have shown an analytic formula for the (LSV) impact and precised a low complexity that permit the computation of the exotic in order to implement the formula.

In the next section we shall apply the previous formula on autocalls and show how precise it is for these types of products.

6 Numerical examples

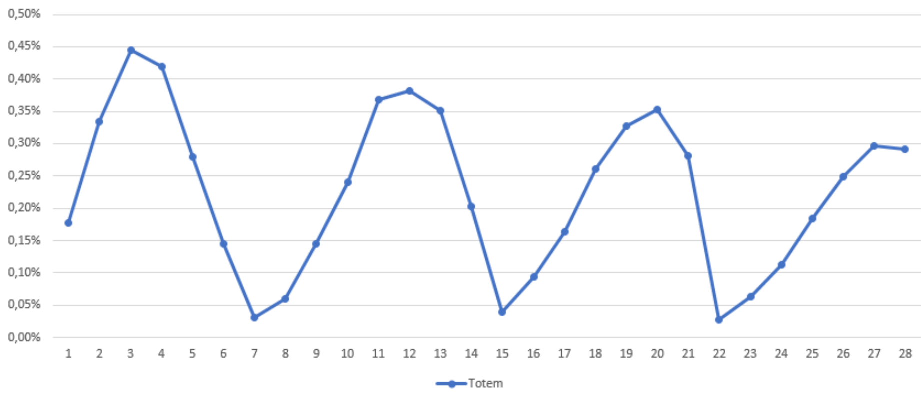
28 different structures are exchanged at Totem:

- 7 Barrier levels: $80\%, 90\%, 95\%, 100\%, 105\%, 110\%, 120\%$,
- 4 maturities : $T_1 = 1y$, $T_2 = 3y$, $T_3 = 5y$, $T_4 = 8y$,
- Quarterly coupon equal to 1.25%.

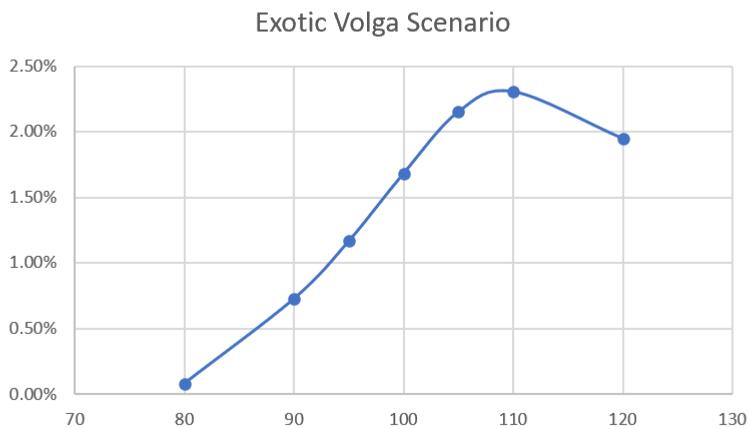
Totem provides the running cost of these 28 = 4 x 7 structures:

$$\pi_{Totem}(B,T) = \frac{\pi_{LSV}(B,T) - \pi_{LV}(B,T)}{T}$$

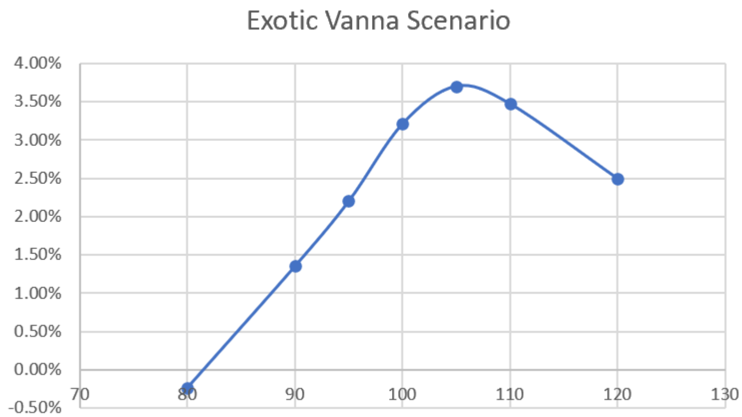
[4] Note that $\pi^{-\beta,K,T}_{LV}$ is the price of vanilla options obtained with the local vol process. This is done using a forward PDE sweep.



We compute the exotic greeks for different payoffs and then simulate hundreds of random stochastic volatility model parameters (ρ, ν, κ). We compare the full LSV impact computed using a full implementation with the formula for the different strikes. We show the results in the following graphs.



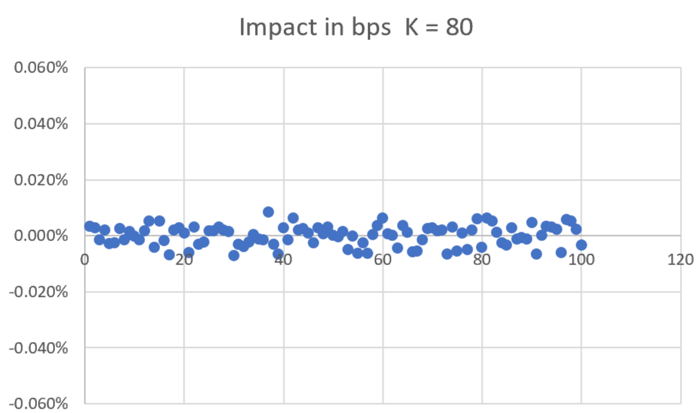
Exotic Volga 8y for the different Barriers

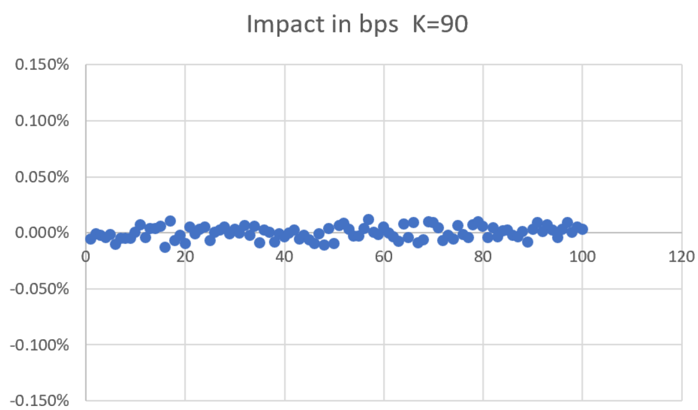


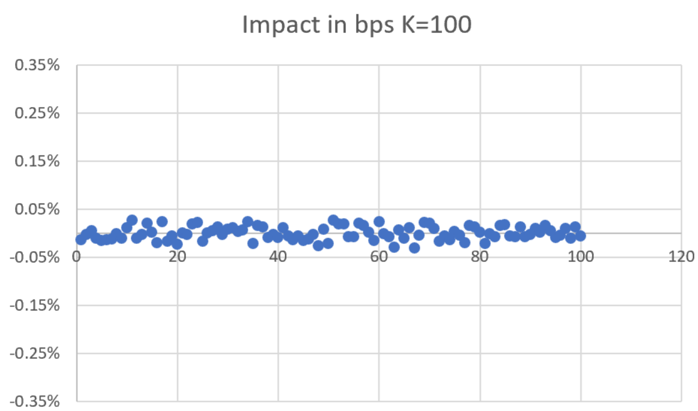
Exotic Vanna 8y for the different Barriers

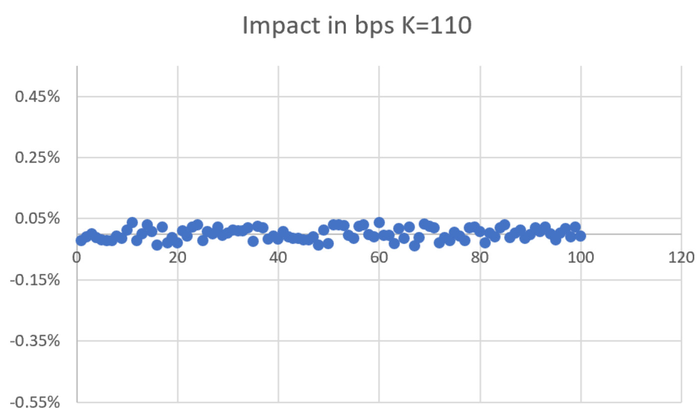
These exotic greeks depend only on the products and the volatility surface. They do not depend on the stochastic volatility parameters.

We then use our formula for randomly simulated stochastic volatility parameters and compare the full LSV with the proxy formula. Needless to say that the formula is instantaneous whereas the full LSV takes a non negligible time to run.









7 Conclusion

In this paper, we introduced a new methodology, singular exotic perturbation. It gives an efficient approach to compute the impact of smile dynamic without running a costly (LSV) model. Instead, it builds on well chosen scenario priced all under a simpler model, local volatility model (LV). The proposed formula ensures zero impact on vanilla and performs very well on more complex products such as Autocalls. The methodology proposed can be used in different contexts as it is the combination of three building blocks, singular perturbation, first order price adjustment and computability through the introduction of exotic greeks. We suggest to use this methodology in other cases such stochastic rates or multi asset case or correlation skew. Further work is needed to detail these practical applications.

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