Stochastic Volatility for Lévy Processes *

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Abstract

Three processes reflecting persistence of volatility are formulated by evaluating three Lévy processes at a time change given by the integral of a square root process. A positive stock price process is then obtained by exponentiating and mean correcting these processes, or alternatively by stochastically exponentiating the processes. The characteristic functions for the log price can be used to yield option prices via the fast Fourier transform. Our empirical results on index options and single name options suggest advantages to employing higher dimensional Lévy systems for index options and lower dimensional structures for single names. In general, mean corrected exponentiation performs better than employing the stochastic exponential. Martingale laws for the mean corrected exponential are also studied and two new concepts termed Lévy and martingale marginals are introduced.

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Three processes reflecting persistence of volatility are formulated by evaluating three Lévy processes at a time change given by the integral of a square root process. A positive stock price process is then obtained by exponentiating and mean correcting these processes, or alternatively by stochastically exponentiating the processes. The characteristic functions for the log price can be used to yield option prices via the fast Fourier transform. Our empirical results on index options and single name options suggest advantages to employing higher dimensional Lévy systems for index options and lower dimensional structures for single names. In general, mean corrected exponentiation performs better than employing the stochastic exponential. Martingale laws for the mean corrected exponential are also studied and two new concepts termed Lévy and martingale marginals are introduced.

1 Introduction

It has been clear that the standard option pricing model of Black-Scholes [8] and Merton [30] has been inconsistent with options data for at least a decade. Despite this result, no other model comes close to the popularity of the Black Merton Scholes (BMS) model in both theory and practice A partial explanation lies in the simplicity of the BMS model. Models which produce superior empirical performance almost always require more parameters, and consequently are usually significantly slower in terms of both calibration and computation.

The empirical performance of a potential successor to BMS is usually measured in terms of hedging and/or pricing performance. This performance can be analyzed in terms of at least four ubiquitous inputs, which are calendar time, the underlying's price, the option's term, and the options's strike. While capturing the effect of variation in these four inputs should improve the overall performance, capturing the effect of variation in the first pair is particularly important for reducing risk, while capturing the variation with respect to the latter pair is critical for reducing pricing error.

To improve on the pricing and hedging performance of the BMS model, the majority of the research has been directed towards modifying the continuous time stochastic process followed by the underlying asset. In particular, asset returns have been modeled as diffusions with stochastic volatility (eg. Hull and White[23] or Heston [22]), as jump-diffusions (eg. Merton [31] or Kou[25]), or both (eg. Bates [6] [7] or Duffie, Pan, and Singleton [14]). Empirical work on these models has generally supported the need for both stochastic volatility and jumps. Stochastic volatility appears to be needed to explain the variation in strike at longer terms, while jumps are needed to explain the variation in strike at shorter terms. Furthermore, jumps and the inability to trade continuously are usually the favored explanations for the existence of substantial hedging errors, whose persistence has been documented repeatedly.

On the theoretical side, arguments have been proposed by Geman, Madan, and Yor [20] which suggest that price processes for financial assets must have a jump component, while they need not have a diffusion component. Their argument rests on recognizing that all price processes of interest may be regarded as Brownian motion subordinated to a random clock. This clock may be regarded as a cumulative measure of economic activity, as conjectured by Clark [12], and as estimated by Ané and Geman [1]. As time must be increasing, the random clock can be modelled as a pure jump increasing process, or alternatively as a time integral of a positive diffusion process, and thus devoid of a continuous martingale component. If jumps are suppressed, then the clock is locally deterministic, which they rule out a priori. Thus, the required jumps in the clock induce jumps in the price process, while no argument similarly requires that prices have a diffusion component.

The explanation usually given for the use of jump diffusion models is that jumps are needed to capture the large moves that occasionally occur, while diffusions are needed to capture the small moves which occur much more frequently. However, since at least the pioneering work of Mandelbrot [26] on stable processes, it has been recognized that many pure jump models are able to capture both rare large moves and frequent small moves. Motivated by the possibility that price processes could be pure jump, several authors have focussed attention on pure jump models in the Lévy class. Technically, these processes can capture frequent small moves through the use of a Lévy density whose spatial integral is infinite

There are at least three examples of such pure jump infinite activity Lévy processes. First, we have the normal inverse Gaussian (NIG) model of Barndorff-Nielsen [4], and its generalization to the generalized hyperbolic class by Eberlein, Keller, and Prause [15]. Second, we have the symmetric variance gamma (VG) model studied by Madan and Seneta [29] and its asymmetric extension studied by Madan and Milne [28], Madan, Carr, and Chang [27]. Finally, we have the model developed by Carr, Geman, Madan, and Yor (CGMY) [10], which further generalizes the VG model. CGMY study the empirical adequacy of the VG and CGMY models in explaining equity option prices across the strike range. They find that these models can explain the so-called volatility smile, and that the empirical performance of these models is typically not improved by adding a diffusion component for returns. These results raise the disturbing question as to whether diffusion components are needed at all when modeling asset returns.

The empirical success of pure jump Lévy processes is not maintained when one considers the variation of option prices across maturity. It has been observed in Konikov and

Madan [24] that these homogeneous Lévy processes impose strict conditions on the term structure of the risk-neutral variance, skewness, and kurtosis. Specifically, the variance rate is constant over the term, skewness is inversely proportional to the square root of the term, while kurtosis is inversely proportional to the term. In contrast, the data suggests that these risk-neutral moments are often rising with term. Economically, the usual supposition that investor uncertainty is increasing with the term suggests that return distributions should spread out as the holding period is increased. On the other hand, risk premia in markets are likely to display mean reversion, which would impact the term structure of skewness and kurtosis implicit in the risk-neutral distribution. Collectively, these considerations suggest that it may be desirable to incorporate a richer behavior across maturity than is implied by homogeneous Lévy processes.

In a parallel development in the literature, it has been observed by several authors such as Engle [16], Bates [6], [7], Heston [22], Duan [13], and Barndorff-Nielsen and Shephard [5], that volatilities estimated from the time series are usually clustered, which is commonly referred to as volatility persistence. This persistence is inconsistent with homogeneous Lévy processes, and possibly explains the failure of such processes to explain option prices across the maturity dimension.

For these reasons, the objective of this paper is to extend the otherwise fairly successful Lévy process models cited above by incorporating stochastic and mean-reverting volatilities. We take three homogeneous Lévy processes, viz the NIG, VG, and the $CGMY$ models, and generate the desired volatility properties by subordinating them to the time integral of a Cox, Ingersoll, and Ross [11] (CIR) process. The randomness of the CIR process induces stochastic volatility, while the mean reversion in this process induces volatility clustering. We term the resulting processes $NIGSV$, $VGSV$, and $CGMYSV$ in recognition of their synthesis with stochastic volatility. These processes are tractable in that analytical expressions can be derived for their characteristic functions. On employing their exponentials to describe stock prices, European options can be priced via Fourier methods as described in Bakshi and Madan [3] and Duffie, Pan and Singleton [14]. In particular, the current paper applies the fast Fourier transform (FFT) method, which is developed in Carr and Madan [9].

The three new processes collectively provide us with a flexible family of option pricing models, capable of being calibrated to market option prices varying across both the strike and maturity dimensions. The calibrated process may then be used for pricing standard options not included in the calibration or for pricing exotic options. These processes can also be used in a simulation to evaluate the quality of hedging strategies, by for example pricing claims whose payoffs are functions (eg. squares) of the cumulative hedging error. These new stochastic processes can thereby be used to enhance our structural understanding of risk management issues. With the resulting models, one may assess the impact on market values of changes in reasonably intuitive parameters such as the speed of adjustment, the level of long run volatility, and the variance of volatility. Thus, the parameters of the models synthesize the information content of option prices in a concise manner, and open the door to interesting investigations into the economic determinants of asset pricing as inferred from an analysis of derivative markets.

In constructing risk-neutral price processes from the $NIGSV, VGSV$, and $CGMYSV$ processes, two approaches are followed. The approaches differ in terms of the filtration in which the martingale condition is based on. The first approach assumes that investors can only condition trades on the level of the stock price, while the second approach assumes that trades can also be conditioned on the level of the Lévy process and the time on the new clock. Thus, the first approach prohibits arbitrages based only on the stock price, while the second approach further precludes arbitrages based on the level of the driving Lévy process and the new clock. The reason that the two approaches were tried is that one can argue that the stock price is far more observable in practice than either of the variables used to model the stock price process.

To operationalize the first approach, we construct the risk-neutral distribution for the stock price at each future time as the exponential of $NIGSV, VGSV$, and $CGMYSV$ processes, normalized to reflect the initial term structure of forward prices. This procedure ensures that spot-forward arbitrage is not possible. We also exclude arbitrages involving calendar spreads of options as these also require knowledge of just the stock price (at the earlier maturity) The class of models generated by excluding price-based arbitrages are termed $NIGSA, VGSA$, and $CGMYSA$ respectively. The second approach is operationalized by compensating the pure jump processes $NIGSV, VGSV$, and $CGYMSV$ to form martingales. These martingales are then stochastically exponentiated to yield martingale candidates (in the enlarged filtration of the Lévy process and the integrated CIR time change) for forward prices. This class of models is termed $NIGSAM, VGSAM$, and $CGMYSAM$ respectively. Characteristic functions for the log of the stock price are formulated analytically in all 6 cases. These characteristic functions are used to generate model option prices numerically, which are then compared with the data.

We note that the $NIGSAM$, $VGSAM$,and $CGMYSAM$ models are martingales with respect to the enlarged filtration, which includes knowledge of the driving Lévy process and knowldege of the subordinator given by the time-integrated CIR process. To the extent that these two processes can not be separately ascertained from a time series of prices, serious issues arise as to the practical relevance of the associated martingale condition. Working with purely discontinuous price processes, Geman, Madan, and Yor [21] provide a precise formulation of conditions under which the two processes can be determined from the time series of underlying asset prices. Even if the two processes can be determined from a time series, it is unlikely that the rich dynamics of the option price matrix can be adequately captured by a martingale which reflects movements in only two processes. Hence, if the market is precluding arbitrage based on a richer filtration than the one generated by the two processes, one is again forced to confront the practical relevance of martingale conditions which are based on filtrations that are essentially unobservable.

The models $NIGSA$, $VGSA$, and $CGMYSA$ take a more conservative approach than the martingale models $NIGSAM, VGSAM$, and $CGMYSAM$. Relying only on the ability to observe stock prices, these models generate stock price processes whose risk-neutral expectation is consistent with the initial term structure of forward prices, but which do not require that these forward prices be martingales with respect to the filtration generated by the Levy process and the subordinator. We find that these more conservative models consistently provide substantially superior empirical performance over the models which prohibit arbitrage based on the richer and perhaps unobservable filtration. Given these results, we take up a deeper study of the properties of these more conservative models . In this regard, we introduce two important new concepts, which we term the martingale marginal property and the Lévy marginal property . We define a process as having the martingale marginal property if it has the same marginal distributions as some martingale process. We further define a process as having the Lévy marginal property, if it has the martingale marginal property and if the martingale is derived from normalizing the exponential of a time inhomogeneous Lévy process. We show first that if the CIR process is started at *zero*, then our conservative processes have this Lévy marginal property. When the starting value is not *zero*, we conjecture that these processes have the martingale marginal properties.[1] Although these questions may be investigated computationally by constructing Lévy densities associated with the characteristic functions of the processes, we pursue a richer understanding of the possibilities by structurally

[1] At this writing, necessary conditions for this property in some parametric special cases have been verified and we leave to future research the complete study of this issue.

reconstructing the one dimensional distributions in alternative ways. This leads to an attractive representation in the form of an inhomogeneous Lévy process perturbed by a process for conditional abnormal returns that are unconditionally absent and eventually zero. Whether trading strategies may be formulated to exploit this information is an open question.

We report the results of estimating all six models using $S\&P500$ option closing prices for the second Wednesday of each month of the year 2000. For other underliers, we report on just the dominating three models $NIGSA, VGSA$, and $CGMYSA$. In the interests of brevity, we provide here a sample of quarterly results. The models are observed to be capable of adequately fitting a wide range of strikes and maturities consistently across the year. A detailed study of the pricing errors shows that absolute errors are higher for out-of-the-money options and for shorter maturities. These results suggest directions for further model improvement, but they could at least partially be due to our experimental design, which minimized absolute errors as opposed to relative errors. The optimal design of a heteroskedasticity adjustment is an open question, to be pursued in future research.

Empirical work on options data suggests that there are very few models capable of explaining option prices across both the strike and maturity dimensions. The pioneering study of Bakshi, Cao, and Chen [2] implicitly demonstrates this point as the authors were forced to partition the data by term and moneyness in order to get adequate pricing quality. The only class of models with comparable effectiveness to the models discussed here appear to be the jump-diffusion models studied by Bates [7], and by Duffie, Pan, and Singleton [14]. These models also employ jump processes and stochastic volatility, but differ from the current paper in that the jump component has jumps occuring rarely (finite activity), thus requiring the use of a diffusion component to capture the frequent small moves of the underlying. This diffusion component must also have stochastic volatility in order to capture the observed strike variation in price of options with longer terms. By exploring the possibility that the frequent small moves can be captured by infinite activity pure jump processes, the class of processes studied in the current paper offers the possibility of reducing the number of parameters required to achieve adequate pricing and hedging performance

In general, the discovery of empirically adequate and yet parametrically parsimonious characterizations of economic data is especially instructive in understanding the underlying economic structure. More specifically, the development of processes which simultaneously explain both the statistical and risk-neutral dynamics is needed to understand the change of measure density process. The latter is critical to developing an understanding of how risks are priced in the financial markets. Knowledge of these mechanisms is critical to the successful development of new derivative markets, which expand the domain of price discovery and extend the benefits of risk allocation to other underlying assets. Thus, given the relative paucity of models which adequately fit the options data, we view the developments of this paper as a significant and important contribution.

The outline of the paper is as follows. In section 2, we briefly summarize the three homogeneous Lévy processes, NIG, VG, and $CGMY$. Section 3 introduces the time change using an integrated CIR process and presents the characteristic functions for the processes $NIGSV, VGSV$, and $CGMYSV$. In section 4, we develop the characteristic functions for the log of the stock price for the six models $NIGSA, VGSA$, $CGMYSA, NIGSAM, VGSAM$, and $CGMYSAM$. Section 5 studies the martingale properties of the models $NIGSA, VGSA$, and $CGMYSA$. Section 6 describes the data and briefly reviews the estimation methodology. The results for our class of models are presented in section 7, while a comparison with other models with stochastic volatility and jumps is presented in section 8. Section 9 summarizes the paper and provides suggestions for further research.

2 The Lévy Processes

The three homogeneous Lévy processes investigated are NIG, VG, and $CGMY$. All 3 processes are pure jump with infinite activity. The NIG process arises by subordinating an arithmetic Brownian motion (ABM) to an inverse Gaussian process, while the VG process arises by alternatively subordinating the ABM to a gamma process. The $CGMY$ process generalizes the VG process in order to parametrically investigate whether log price processes display finite activity (eg. a Poisson process) or infinite activity (eg. a VG process). The $CGMY$ model was also developed to investigate whether log price processes display infinite variation (eg any diffusion) or finite variation (eg. VG again). CGMY [10] conclude that price processes (in the class considered) generally display infinite activity and finite variation, both statistically and risk-neutrally. In the interests of notational parsimony, the reader is forewarned that "notational overloading" is employed in the discussion of the 3 models. Confusion is easily avoided by simply noting the context in which the notation is used.

Just as the specification of the instantaneous volatility differentiates models within the diffusion class, the specification of the Lévy density differentiates models within the pure jump class. Just as the instantaneous volatility describes the local uncertainty of a diffusion, the Lévy density describes the local uncertainty of a pure jump process. The Lévy density has the same mathematical requirements as a probability density, except that it need not be integrable and must have zero mass on the origin. Integration of the Lévy density over a particular spatial domain yields the arrival rates of jump sizes in this domain. If the Lévy density is risk-neutral, this arrival rate is interpreted as the (forward) price of a claim which has a positive payout if and only if a jump of the specified size occurs. An economic study of the determinants of the risk-neutral Lévy density would considerably enhance our understanding of market pricing. Needless to say, a robust estimation of the risk-neutral Lévy density is the important precursor accomplished here.

As the class of Lévy densities is quite large, the first step in selecting a statistical or risk-neutral Lévy density is to impose economic and tractability considerations which cut the size of the class down. Economically, the 3 pure jump Lévy processes explored in this paper all reflect a particular structural presumption concerning the arrival rate of price movements in the market. Fixing the size of a move, down moves are presumed to have an arrival rate and a risk-neutral price which is independent and usually higher than those of the corresponding up move. Similarly, fixing the direction of the move, large moves are reasonably assumed to have a lower frequency and price than those of any smaller move. The independence of down and up moves contained in the first restriction can be accomodated by the use of two non-negative functions, each of which has a single argument which is a positive real. One such function is used to determine the arrival rates associated with the absolute size of down moves, while a second such function is used to determine the arrival rates of up moves. If this second function has a lower mean than the first, then the negative directional premium mentioned in the first economic restriction is accomodated. The negative size premium implicit in the second structural restriction can be accomodated though the use of a pair of monotonically decreasing functions. Functions satisfying both restrictions arise in an analytically attractive way through the use of any non-negative linear combination of negative exponential functions, whose arguments are again restricted to be positive. This class is called the completely monotone class (see CGMY [10] for details) since it has the property that all derivatives are monotone and that successive derivatives alternate in sign.

Since the Lévy density is a function, it is an infinite dimensional object in general. However, by focusing attention on a parametric model of the Lévy density, a parsimonious synthesis of the local uncertainty can be obtained in much the same way as the instantanous volatility can be parametrically specified to be for example, a CEV process. Ideally, the Lévy density should be parametrized so that the resulting parameters describe the average level and the rate of decay of the Lévy density in both directions. Parametric specifications generally yield extremely smooth functions of the dimensions upon which an extrapolation is intended (eg option term or strike). In general, extrapolations based on a smooth and low dimensional parametric specification which fits the data reasonably well are more likely to succeed than a less smooth and high-dimensional specification, even if the latter fits the data perfectly (eg. splines)

2.1 The Normal Inverse Gaussian Model

The NIG process has a characteristic function defined by three parameters (see Barndorff-Nielsen [4])

$$\phi_{NIG}(u;\alpha,\beta,t\delta) = \exp\left(-t\delta\left(\sqrt{\alpha^2-(\beta+iu)^2}-\sqrt{\alpha^2-\beta^2}\right)\right) \tag{1}$$

From the linearity of the log characteristic function in the time variable, we observe that this is an infinitely divisible process with stationary independent increments. We can relate the NIG process to time-changed Brownian motion by introducing an independent inverse Gaussian process. Let T_t^ν be the first time that a Brownian motion with drift ν reaches the positive level t. It is well known that the Laplace transform of this random time is

$$E\left[\exp\left(-\lambda T_t^\nu\right)\right] = \exp\left(-t\left(\sqrt{2\lambda+\nu^2}-\nu\right)\right) \tag{2}$$

Now consider evaluating Brownian motion with drift θ and volatility σ at the inverse Gaussian process to define the new process

$$X_{NIG}(t;\sigma,\nu,\theta) = \theta T_t^\nu + \sigma W(T_t^\nu) \tag{3}$$

Suppressing the dependence of the process on its parameters, the characteristic function is

$$\begin{aligned}
E\left[e^{iuX_{NIG}(t)}\right] &= E\left[\exp\left(iu\theta T_t^\nu - \frac{\sigma^2u^2}{2}T_t^\nu\right)\right] \\
&= E\left[\exp\left(iu\theta - \frac{\sigma^2u^2}{2}\right)T_t^\nu\right] \\
&= \exp\left(-t\left(\sqrt{\nu^2-2iu\theta+\sigma^2u^2}-\nu\right)\right) \\
&= \exp\left(-t\sigma\left(\sqrt{\frac{\nu^2}{\sigma^2}-2iu\frac{\theta}{\sigma^2}+u^2}-\frac{\nu}{\sigma}\right)\right) \\
&= \exp\left(-t\sigma\left(\sqrt{\frac{\nu^2}{\sigma^2}+\frac{\theta^2}{\sigma^4}-\left(\frac{\theta}{\sigma^2}+iu\right)^2}-\frac{\nu}{\sigma^2}\right)\right)
\end{aligned}$$

Hence we may define

$$\begin{aligned}
\beta &= \frac{\theta}{\sigma^2} \\
\alpha^2 &= \frac{\nu^2}{\sigma^2}+\frac{\theta^2}{\sigma^4} \\
\delta &= \sigma
\end{aligned}$$

and observe that the NIG process is

$$X_{NIG}(t;\alpha,\beta,\delta) = \beta\delta^2 T_t^{\delta\sqrt{\alpha^2-\beta^2}} + \delta W\left(T_t^{\delta\sqrt{\alpha^2-\beta^2}}\right) \tag{4}$$

To obtain the NIG Lévy density, note that conditioning on a jump of g in the time change, the move is Gaussian with mean $\beta\delta^2 g$ and variance $\delta^2 g$. The arrival rate for the jumps is given by the Lévy density for inverse Gaussian time

$$k(g) = \frac{\exp\left(-\frac{\delta^2(\alpha^2-\beta^2)}{2}g\right)}{g^{3/2}}$$

It follows that the Lévy density for NIG is

$$\begin{aligned}
& \int_0^\infty \frac{1}{\delta\sqrt{2\pi g}}\exp\left(-\frac{(x-\beta\delta^2 g)^2}{2\delta^2 g}\right)\frac{1}{g^{3/2}}\exp\left(-\frac{\delta^2(\alpha^2-\beta^2)}{2}g\right)dg \\
= & \frac{1}{\delta}\int_0^\infty \frac{1}{\sqrt{2\pi}}g^{-2}\exp\left(-\frac{x^2}{2\delta^2 g} - \frac{\delta^2(\alpha^2-\beta^2)}{2}g + \beta x - \frac{\beta^2\delta^2}{2}g\right)dg \\
= & \frac{e^{\beta x}}{\delta}\int_0^\infty \frac{1}{\sqrt{2\pi}}t^{-2}\exp\left(-\frac{\delta^2\alpha^2}{2}t - \frac{x^2}{2\delta^2 t}\right)dt \\
= & \frac{e^{\beta x}}{\delta}\int_0^\infty \frac{1}{\sqrt{2\pi}}\exp\left(-s - \frac{x^2\alpha^2}{4s}\right)s^{-2}\frac{\delta^2\alpha^2}{2}ds
\end{aligned}$$

We now define

$$K_a(x) = \frac{1}{2}\left(\frac{x}{2}\right)^a \int_0^\infty \exp\left(-\left(t+\frac{x^2}{4t}\right)\right)t^{-a-1}dt$$

so we may write

$$\int_0^\infty \exp\left(-\left(t+\frac{x^2}{4t}\right)\right)t^{-a-1}dt = 2K_a(x)\left(\frac{2}{x}\right)^a$$

Hence, the NIG Lévy density is

$$k_{NIG}(x) = \sqrt{\frac{2}{\pi}}\delta\alpha^2\frac{e^{\beta x}K_1(|x|)}{|x|}. \tag{5}$$

We see from the structure of this density that direction premia are controlled via the parameter β, while the size premia are determined by the shape of the function K_1.

For later use, we record here the unit time log characteristic function expressed in terms of the parameters of the time-changed Brownian motion

$$\psi_{NIG}(u;\sigma,\nu,\theta) = \sigma\left(\frac{\nu}{\theta} - \sqrt{\frac{\nu^2}{\theta^2} - 2\frac{\theta i u}{\sigma^2} + u^2}\right)$$

2.2 The Variance Gamma Model

The variance gamma process is defined by evaluating Brownian motion with drift θ and volatility σ at a gamma time. Specifically, we have

$$X_{VG}(t;\sigma,\nu,\theta) = \theta G_t^\nu + \sigma W\left(G_t^\nu\right)$$

where G_t^ν is a gamma process with mean rate t and variance rate νt. The probability density of the gamma distributed random time g at time t is

$$f(g) = \frac{g^{(t/\nu)-1}e^{-g/\nu}}{\nu^{t/\nu}\Gamma(\frac{t}{\nu})} \tag{6}$$

and its Laplace transform is

$$E\left[\exp\left(-\lambda G_t^\nu\right)\right] = (1+\lambda\nu)^{-\frac{t}{\nu}} \tag{7}$$

The characteristic function of the VG process is easily evaluated as

$$E\left[e^{iuX_{VG}(t)}\right] = \left(1 - iu\theta\nu + \sigma^2\nu u^2/2\right)^{-\frac{t}{\nu}},$$

by conditioning on the time change and then employing (7) for $\lambda = \frac{\sigma^2 u^2}{2} - i\theta u$. Madan, Carr, and Chang [27] show that the variance gamma process may also be expressed as the difference of two independent gamma processes, with one describing the up moves and the other describing the down moves. This characterization allows the Lévy density to be determined, as shown in CGMY [10]:

$$k_{VG}(x) = \begin{cases} \frac{C\exp(Gx)}{|x|} & x < 0 \\ \frac{C\exp(-Mx)}{x} & x > 0 \end{cases}$$

where

$$C = \frac{1}{\nu} \tag{8}$$

$$G = \left(\sqrt{\frac{\theta^2\nu^2}{4} + \frac{\sigma^2\nu}{2}} - \frac{\theta\nu}{2}\right)^{-1} \tag{9}$$

$$M = \left(\sqrt{\frac{\theta^2\nu^2}{4} + \frac{\sigma^2\nu}{2}} + \frac{\theta\nu}{2}\right)^{-1}. \tag{10}$$

The parameter C controls the overall activity rate of the process, while the parameters G and M govern the rate at which arrival rates decline with the size of the move. Thus, the average of the parameters G and M can be regarded as a measure of the size premium, while their difference can be regarded as a directional premium. Alternatively, the parameter θ measures the directional premium since it primarily affects the skewness of the process. When $\theta = 0$ then $G = M$ and the distribution is symmetric. Negative values of θ lead to lower values for G resulting in negatively skewed processes, with the opposite holding for $\theta > 0$. Similarly, the parameter ν primarily controls the kurtosis of the process, since excess kurtosis arises whenever the time change is stochastic. It may shown that for $\theta = 0$, the kurtosis is $3(1+\nu)$, so that ν is the percentage excess kurtosis over that of a standard normal.

For later use, we will need the following unit time log characteristic function in the Lévy measure parametrization:

$$\psi_{VG}(u; C, G, M) = C\log\left(\frac{GM}{GM + (M-G)iu + u^2}\right) \tag{11}$$

2.3 The CGMY model

A compound Poisson process has finite activity and finite variation of the sample paths. The VG process also has finite variation, but it has infinite activity. The NIG process has both infinite activity and infinite variation. To capture all of these possibilities, CGMY[10] introduced the $CGMY$ process. They generalize the VG process by introducing a fourth parameter Y. Setting this parameter to a particular value results in the VG process. However, for lower values of Y, the Lévy density integrates to a finite value yielding a process of finite activity. At such levels, the integral of $|x|$ times the Lévy density is also finite, and thus the process has finite variation, as in a compound Poisson process. For higher values of Y, the process has infinite activity but finite variation, as in the VGprocess. For yet higher values of Y, the process has infinite activity and infinite variation as in the NIG process. The specific form for the $CGMY$ Lévy density is

$$k_{CGMY}(x) = \begin{cases} \frac{C\exp(Gx)}{(-x)^{1+Y}} & x < 0 \\ \frac{C\exp(-Mx)}{x^{1+Y}} & x > 0 \end{cases}$$

The characteristic function is

$$\begin{aligned} & E\left[\exp\left(iuX_{CGMY}(t)\right)\right] \\ = \; & \exp\left(tC\Gamma(-Y)\left[(M-iu)^Y + (G+iu)^Y - M^Y - G^Y\right]\right). \end{aligned} \tag{12}$$

In what follows, the C and Y parameters will be allowed to take different values for positive and negative outcomes in x. Letting C_p, Y_p denote the parameters for $x > 0$ and C_n, Y_n denote the parameters for $x < 0$,the generalized characteristic function is

$$\begin{aligned} & E\left[\exp(iuX_{CGMY}(t)\right] \\ = \; & \exp\left(tC_p\Gamma(-Y_p)((M-iu)^{Y_p} - M^{Y_p}\right) + C_n\Gamma(-Y_n)((G+iu)^{Y_n} - G^{Y_n})). \end{aligned}$$

For later use, we record here the unit time log characteristic function

$$\begin{aligned} & \psi_{CGMY}(u; C_p, G, M, Y_p, Y_n, \zeta) \\ = \; & C_p\left(\Gamma(-Y_p)((M-iu)^{Y_p} - M^{Y_p}\right) + \zeta\Gamma(-Y_n)((G+iu)^{Y_n} - G^{Y_n}) \end{aligned}$$

with ζ defined as the ratio of C_n to C_p.

The six parameter Lévy process is considerably richer than its predecessors as one may now independently calibrate the level, slope, and curvature of the arrival rate as a function of the size and sign of the move. In contrast, the continuity requirement of diffusion models forces the arrival rates of all jump sizes to zero, and thus forces the local variation of uncertainty in the price dimension to be explained with a single instantaneous volatility parameter. The many parameters governing the arrival rates and the single parameter governing the instantaneous volatility can all be generalized to depend on time, price or other random processes. However, this does not alter the fact that diffusion processes are severely restricted in terms of their ability to describe local behavior.

3 Clustering Time or Activity Persistence

The basic intuition underlying our approach to stochastic volatility arises from the Brownian scaling property This property relates changes in scale to changes in time and thus random changes in volatility can alternatively be captured by random changes in time. The instantaneous rate of time change must be positive if the new clock is to be increasing. Furthermore, this rate of time change must be mean-reverting if the random time changes are to persist. The classic example of a mean-reverting positive process is the "so-called" square root process of Cox, Ingersoll, and Ross (CIR). Hence, we define the process $y(t)$ as the solution to the stochastic differential equation

$$dy = \kappa(\eta - y)dt + \lambda\sqrt{y}dW \tag{13}$$

where $W(t)$ is a standard Brownian motion independent of any processes encountered thus far. The parameter η has the usual interpretation as the long run rate of time change, κ is the rate of mean reversion, and λ governs the volatility of the time change.

The process $y(t)$ is the instantaneous rate of time change and so the new clock is given by its integral

$$Y(t) = \int_0^t y(u)du. \tag{14}$$

The characteristic function for $Y(t)$ is well known from the work of CIR [11] and from the literature on Brownian motion, since it is closely associated with Lévy's stochastic area formula (see e.g. [33], [36]). The characteristic function for $Y(t)$ is explicitly given by

$$\begin{aligned}
E\left[\exp\left(iuY(t)\right)\right] &= \phi(u, t, y(0); \kappa, \eta, \lambda) \\
&= A(t,u)\exp\left(B(t,u)y(0)\right) \\
A(t,u) &= \frac{\exp\left(\frac{\kappa^2\eta t}{\lambda^2}\right)}{\left(\cosh(\frac{\gamma t}{2}) + \frac{\kappa}{\gamma}\sinh(\frac{\gamma t}{2})\right)^{\frac{2\kappa\eta}{\lambda^2}}} \\
B(t,u) &= \frac{2iu}{\kappa + \gamma\coth(\frac{\gamma t}{2})} \\
\gamma &= \sqrt{\kappa^2 - 2\lambda^2 iu}
\end{aligned}$$

3.1 The Generic Stochastic Volatility Lévy Process

Let $X(t)$ be a Lévy process, so that it has stationary independent increments. Its characteristic function is thus of the form

$$E\left[\exp\left(iuX(t)\right)\right] = \exp\left(t\psi_X(u)\right) \tag{15}$$

For simplicity, we assume a Lévy density exists and denote it by $k(x)$. When $X(t)$ is a process of finite variation, the log characteristic function at unit time $\psi_X(u)$ is related to $k(x)$ by

$$\psi_X(u) = \int_{-\infty}^{\infty}(e^{iux} - 1)k(x)dx \tag{16}$$

Explicit forms for $\psi_X(u)$ in the case of the NIG, the VG, and the $CGMY$ models were exhibited in section 2.

The class of stochastic volatility Lévy processes ($SVLP$) is defined by

$$Z(t) = X(Y(t)). \tag{17}$$

where Y is independent of X. Thus, Z is obtained by Bochner's procedure of subordinating X to Y. The characteristic functions for these processes are obtained simply as follows

$$\begin{aligned}
E\left[\exp\left(iuZ(t)\right)\right] &= E\left[\exp\left(Y(t)\psi_X(u)\right)\right] \\
&= \phi(-i\psi_X(u), t, y(0); \kappa, \eta, \lambda)
\end{aligned} \tag{18}$$

The specific parametrizations for the three Lévy processes are developed next.

3.1.1 The Process NIGSV

The stochastic volatility version of the NIG process is

$$Z_{NIG}(t) = X_{NIG}(Y(t); \sigma, \nu, \theta)$$

We note that $y(0) = \sigma$ and so we can write

$$E \exp(iuZ_{NIG}(t)) = \phi(-i\psi_{NIG}(1, \nu, \theta), t, \sigma; \kappa, \eta, \lambda)$$

This is a six parameter process with parameters

$$\sigma, \nu, \theta, \kappa, \eta, \lambda$$

3.1.2 The Process VGSV

The stochastic volatility version of the VG process is

$$\begin{aligned} Z_{VG}(t) &= X_{VG}(Y(t); \sigma, \nu, \theta) \\ &= X_{VG}(Y(t); C, G, M) \end{aligned}$$

where the second representation employs the parameters of the Lévy density as defined in equations ((8,9,10)). It is clear on considering the role of $Y(t)/t$ that the parameter C is identified with $y(0)$. Hence, we may write

$$E[\exp(iuZ_{VG}(t))] = \phi(-i\psi_{VG}(u; 1, G, M), t, C; \kappa, \eta, \lambda) \tag{19}$$

This is a six parameter process with parameters

$$C, G, M, \kappa, \eta, \lambda$$

3.1.3 The Process CGMYSV

The stochastic volatility version of the CGMY process is

$$Z_{CGMY}(t) = X_{CGMY}(Y(t); C_p, G, M, Y_p, Y_n, \zeta)$$

where we have replaced C_n by its ratio to C_p. The identification in this case is between C_p and $y(0)$ and we continue to use the notation C. We thus obtain that

$$E[\exp(iuZ_{CGMY}(t))] = \phi(-i\psi_{CGMY}(1, G, M, Y_p, Y_n, \zeta), t, C; \kappa, \eta, \lambda)$$

This is a nine parameter process with parameters

$$C, G, M, Y_p, Y_n, \zeta, \kappa, \eta, \lambda$$

4 The Stock Price Processes

This section considers two approaches for obtaining a positive stock price process. The first approach uses the ordinary exponential function, while the second uses the stochastic exponential. The second approach is a little more involved and has some desirable and possibly undesirable features from an economic point of view . The most desirable feature is that one easily obtains the martingale laws required by the exclusion of dynamic arbitrage. The most undesirable feature is that this representation increases the dimension of the filtration, which stresses the implicit assumption that the filtration is observable. The reason for the dimensional increase is that the price process is adapted to the joint process given by the Lévy process $X(t)$ and the time change $Y(t)$. As these processes are usually not observable from the price path, it could be argued that it is not observable at all. We next provide the details for the ordinary exponential and the stochastic exponential approach in separate subsections.

4.1 Ordinary Exponentials of SVLP Processes

Under this approach, the risk-neutral stock price process is given by mean correcting the exponential of a *svlp* process. Let $S(t)$ denote the stock price at time t and let r and q denote the constant continuously compounded interest rate and dividend yield respectively. Let $Z(t)$ be a generic *svlp* process as described in (17). We define the stock price at time t by the random variable

$$S(t) = S(0)\frac{\exp\left((r-q)t + Z(t)\right)}{E\left[\exp(Z(t)\right]}. \tag{20}$$

Noting that

$$E\left[\exp(Z(t))\right] = \phi(-i\psi_X(-i), t, y(0); \kappa, \eta, \lambda)$$

we get that the characteristic function for the log of the stock price at time t is given by

$$\begin{aligned} E\left[\exp\left(iu\log(S(t))\right)\right] &= \exp\left(iu(\log(S(0) + (r-q)t)\right) \\ &\quad \times \frac{\phi\left(-i\psi_X(u), t, y(0); \kappa, \eta, \lambda\right)}{\phi(-i\psi_X(-i), t, y(0); \kappa, \eta, \lambda)^{iu}} \end{aligned} \tag{21}$$

The three specific results for the $NIGSA$, $VGSA$, and $CGMYSA$ models are presented next.

4.1.1 NIGSA Characteristic Function for Log Stock Price

This characteristic function for the NIGSA process at time t is explicitly given as

$$\begin{aligned} NIGSACF(u) &= \exp\left(iu(\log(S(0) + (r-q)t)\right) \\ &\quad \times \frac{\phi(-i\psi_{NIG}(u; 1, \nu, \theta), t, \sigma; \kappa, \eta, \lambda)}{\phi(-i\psi_{NIG}(-i; 1, \nu, \theta), t, \sigma; \kappa, \eta, \lambda)^{iu}} \end{aligned} \tag{22}$$

4.1.2 VGSA Characteristic Function for Log Stock Price

This characteristic function for the VGSA process at time t is explicitly given as

$$\begin{aligned} VGSACF(u) &= \exp\left(iu(\log(S(0) + (r-q)t)\right) \\ &\quad \times \frac{\phi(-i\psi_{VG}(u; 1, G, M), t, C; \kappa, \eta, \lambda)}{\phi(-i\psi_{VG}(-i; 1, G, M), t, C; \kappa, \eta, \lambda)^{iu}} \end{aligned} \tag{23}$$

4.1.3 CGMYSA Characteristic Function for Log Stock Price

This characteristic function for the CGMYSA process at time t is explicitly given as

$$\begin{aligned} CGMYSACF(u) &= \exp\left(iu(\log(S(0) + (r-q)t)\right) \\ &\quad \times \frac{\phi(-i\psi_{CGMY}(u; 1, G, M, Y_p, Y_n, \zeta), t, C; \kappa, \eta, \lambda)}{\phi(-i\psi_{CGMY}(-i; 1, G, M, Y_p, Y_n, \zeta), t, C; \kappa, \eta, \lambda)^{iu}} \end{aligned} \tag{24}$$

4.2 Stochastic Exponentials of SVLP Processes

Under this approach, martingale models for the discounted stock price are obtained by stochastically exponentiating martingales. Let $Z(t)$ be a generic $SVLP$ process. The process $Z(t)$ is a pure jump process with a predictable compensator given by

$$\rho(dx, dt) = y(t)k(x)dxdt.$$

It follows that

$$n(t) = Z(t) - \int_0^t \int_{-\infty}^{\infty} x\rho(dx, dt)$$

is a martingale. Let $\mu(dx,dt)$ be the integer valued random measure associated with the jumps of the process $Z(t)$, so that

$$Z(t) = \int_0^t \int_{-\infty}^{\infty} x\mu(dx,dt)$$

Then $n(t)$ is the compensated jump martingale

$$n(t) = x * (\mu - \rho).$$

Now define the compensated jump martingale $m(t)$ by

$$m(t) = (e^x - 1) * (\mu - \rho)$$

and consider the stochastic exponential of $m(t)$ given by

$$M(t) = \exp\left(Z(t) - \int_0^t \int_{-\infty}^{\infty} (e^x - 1)k(x)y(s)dxds\right)$$

Employing (16), we have with $Y(t) = \int_0^t y(s)ds$ that

$$M(t) = \exp(Z(t) - Y(t)\psi_X(-i)) \tag{25}$$

We may also write $M(t)$ as

$$M(t) = \exp(X(Y(t)) - Y(t)\psi_X(-i))$$

which is the martingale

$$\exp(X(t) - t\psi_X(-i))$$

evaluated at an independent random time change $Y(t)$, and hence is also a martingale. The development of (25) shows that the relationship to the stochastic volatility process $Z(t)$ is precisely one of stochastically exponentiating $m(t)$.

This second approach to developing stock price processes adopts the formulation

$$S(t) = S(0)\exp\left((r-q)t\right)\exp\left(X(Y(t)) - Y(t)\psi_X(-i)\right) \tag{26}$$

In this case, the characteristic function for the log of the stock price is given by

$$\begin{aligned} E\left[\exp\left(iu\log(S(t))\right)\right] &= \exp\left(iu(\log(S(0) + (r-q)t)\right) \\ &\quad \times\phi(-i\psi_X(u) - u\psi_X(-i), t, y(0); \kappa, \eta, \lambda) \end{aligned} \tag{27}$$

The three special cases of interest are formulated next.

4.2.1 The NIGSAM characteristic function for log stock price

For the NIG process at time t, the characteristic function is

$$\begin{aligned} NIGSAMCF(u) &= \exp\left(iu(\log(S(0) + (r-q)t)\right) \\ &\quad \times\phi(-i\psi_{NIG}(u,1,\nu,\theta) - u\psi_{NIG}(-i,1,\nu,\theta), t, \sigma; \kappa, \eta, \lambda) \end{aligned}$$

4.2.2 The VGSAM characteristic function for log stock price

For the VG process at time t, the characteristic function is

$$\begin{aligned} VGSAMCF(u) &= \exp\left(iu(\log(S(0) + (r-q)t)\right) \\ &\quad \times\phi(-i\psi_{VG}(u,1,G,M) - u\psi_{VG}(-i,1,G,M), t, C; \kappa, \eta, \lambda) \end{aligned}$$

4.2.3 The CGMYSAM characteristic function for log stock price

For the CGMY process at time t, the characteristic function is

$$\begin{aligned} & CGMYSAMCF(u) \\ = \quad & \exp\left(iu(\log(S(0)+(r-q)t)\right) \\ & \times\phi(-i\psi_{CGMY}(u,1,G,M,Y_p,Y_n,\zeta)-u\psi_{CGMY}(-i,1,G,M,Y_p,Y_n,\zeta),t,C;\kappa,\eta,\lambda) \end{aligned}$$

5 The NIGSA, VGSA, and CGMYSA Martingale Laws

The $NIGSA, VGSA$, and $CGMYSA$ models are formulated by writing the discounted stock price relative as a process of unit unconditional expectation obtained on exponentiating the $NIGSV, VGSV$, and $CGMYSV$ processes and dividing the exponential by its mean. This formulation leads to prices free of static arbitrage since expectations are calculated with respect to a measure on the space of paths that respects spot forward arbitrage. If the log price processes had independent increments, then forward price processes would be (local) martingales since conditional expectations are now identified with unconditional expectations. However, the lack of independence in the increments of the SA processes implies that forward price processes need not be martingales, and hence these processes are subject to the possibility of dynamic arbitrage.

This section addresses the relatively deeper question of whether one dimensional distributions obtained from possibly non-martingale models, like the SA processes, are nonetheless consistent with alternative martingale dynamics. This question could be investigated from a computational perspective by constructing Lévy densities associated with characteristic functions for the processes, but a richer understanding of the possibilities is provided by the more structural representation of the processes pursued here.

We first ask whether there exist processes of independent increments, possibly inhomogeneous, with the same one dimensional densities as the processes $NIGSA$, $VGSA$,or $CGMYSA$. Since European option prices only determine the one dimensional density of the stock price at each maturity, it is possible that two or more probability measures are consistent with the same option prices. It is also possible that one of these measures is a martingale measure, while the other arises from the $NIGSA, VGSA$, or $CGMYSA$ models. We show that there exists a very large class of martingale measures for which this is indeed true. More generally we investigate the nature of the departures when such a representation is not available.

To assist the discussion, we focus on the generic case where $X(t)$ is a homogeneous Lévy process and $Z(t) = X(Y(t))$ with $Y(t)$ defined in accordance with (14) and (13). Let $V(t)$ be a generic representation of our constant unconditional expectation process

$$V(t) = \frac{\exp(Z(t))}{E\left[\exp(Z(t)\right]}.$$

It is clear that if one constructs a process of independent and possibly inhomogeneous increments $U(t)$ with the same one dimensional distributions as those of $Y(t)$, then the one-dimensional distributions of $V(t)$ are those of

$$\widetilde{V}(t) = \frac{\exp(X(U(t))}{E\left[\exp(X(U(t))\right]}$$

where now $\widetilde{V}(t)$ is a process of independent multiplicative increments and a martingale. This leads us to focus our attention on representing the one dimensional distributions of the process $Y(t)$.

The process $Y(t)$ has three parameters κ, η, λ and it is useful to employ scaling changes to relate the process to the case where $\lambda = 2$. In fact, if we let

$$h(t) = \frac{4}{\lambda^2} y(t)$$

then $H(t) = \int_0^t h(s)ds = \frac{4}{\lambda^2} Y(t)$ and an application of Ito's lemma shows that $h(t)$ satisfies the stochastic differential equation

$$dh = \left(\frac{4\kappa\eta}{\lambda^2} - \kappa h \right) dt + 2\sqrt{h} dW(t).$$

To simplify the notation and to better relate to the results in Pitman and Yor [33], we introduce the stochastic differential equation

$$dh = (\delta + 2\beta h)\, dt + 2\sqrt{h} dW(t) \tag{28}$$

where our case of interest is $\delta = \frac{4\kappa\eta}{\lambda^2}$ and $\beta = -\frac{\kappa}{2}$. We denote by ${}^{\beta}Q_x^{\delta}$ the law of the process $h(t)$ satisfying (28) and starting at $h(0) = x$. It is well known that for fixed β, this two parameter family enjoys the additivity property

$${}^{\beta}Q_x^{\delta} * {}^{\beta}Q_{x'}^{\delta'} = {}^{\beta}Q_{x+x'}^{\delta+\delta'} \tag{29}$$

Furthermore, as shown by Shiga-Watanabe [34], these diffusions are (up to a trivial homothetic change of variable), the only family of $\mathbb{R}_+$ valued diffusions to have this additivity property. Denoting the solution by $(h(u), u \geq 0)$, (29) implies that for every non-negative measure $\mu(ds)$ on $\mathbb{R}_+$ and every $t \geq 0$, the random variable

$$I_{\mu,t}(h) = \int_0^t \mu(ds) h(s)$$

is infinitely divisible under the law ${}^{\beta}Q_x^{\delta}$, with parameters of infinite divisibility x and δ. Its Lévy Khintchine representation is studied in Pitman and Yor [33]. In fact, Pitman and Yor [33] use classical Ray-Knight theorems on Brownian local times (among other arguments) to show the existence, for given β, of two σ-finite measures ${}^{\beta}M$, and ${}^{\beta}N$ on $C(\mathbb{R}_+, \mathbb{R}_+)$ such that

$${}^{\beta}Q_x^{\delta} \left(\exp\left(-\gamma I_{\mu,t}\right)\right) = \exp\left(- \int \left(x\, {}^{\beta}M + \delta\, {}^{\beta}N \right) (dh)(1 - e^{-\gamma I_{\mu,t}}) \right)$$

The Lévy measure associated to $I_{\mu,t}$ under ${}^{\beta}Q_x^{\delta}$ is

$$x\, {}^{\beta}m_{\mu,t} + \delta\, {}^{\beta}n_{\mu,t}$$

where ${}^{\beta}m_{\mu,t}, {}^{\beta}n_{\mu,t}$ are the images of ${}^{\beta}M, {}^{\beta}N$ by the mapping $h \longrightarrow I_{\mu,t}(h)$. A number of computations of these Lévy measures are found in Pitman and Yor [33].

We are interested here in yet another possible infinite divisibility property. Specifically, for a given "reasonable" μ,we wish to determine whether the marginals of the process $(I_{\mu,t}(h), t \geq 0)$ are those of a process with inhomogeneous independent increments. For simplicity, we take $\mu(ds) = ds$ as the Lebesgue measure and we say that the process $H(t) = \int_0^t h(u)du$ has the Lévy marginal (LM) property if there exists an inhomogeneous Lévy process $(\theta(t), t \geq 0)$ such that for any given t

$$H(t) \stackrel{(d)}{=} \theta(t).$$

Our main result is the following.

Theorem 1 *Let $\beta \in \mathbb{R}$, and p, q be two reals*

(i) The process $\left(Y_{p,q}(t) = py(t) + q\int_0^t y(s)ds, t \geq 0\right)$, *under* ${}^{\beta}Q_0^{\delta}$ *enjoys the (LM) property*

(ii) Let $x \neq 0$. The process $\left(Y_{0,1}(t) = \int_0^t y(s)ds, t \geq 0\right)$ *considered under* ${}^{\beta}Q_x^{\delta}$ *does not enjoy the (LM) property.*

We first deal with the case $\delta = 2$ and for this case, the theorem is a consequence of the following theorem.

Theorem 2 *Let* $\left({}^{(-\mu)}y(a), a \geq 0\right)$ *denote a process distributed as* ${}^{(-\mu)}Q_0^2$. *Then one has*

a)

$$\left({}^{-(\mu)}y(b); \int_0^b da\, {}^{(-\mu)}y(a)\right) \stackrel{(d)}{=} \left(\ell^0_{T_b}; \int_0^{T_b(X^{(\mu)})} ds \mathbf{1}_{\left(X_s^{(\mu)}>0\right)}\right), \text{ for every } b \geq 0, \quad (30)$$

where $\left(X_t^{(\mu)}, t \geq 0\right)$ *is the solution of*

$$X_t = B_t + \mu \int_0^t ds \mathbf{1}_{(X_s>0)}. \quad (31)$$

and $T_b(X^{\mu}) = \inf\{t \geq 0 : X_t^{\mu} = b\}$

b) There is the identity:

$$\left(\int_0^{T_b(X^{(\mu)})} ds \mathbf{1}_{\left(X_s^{(\mu)}>0\right)}, b \geq 0\right) \stackrel{(d)}{=} \left(T_b(\left|Z^{(\mu)}\right|), b \geq 0\right),$$

where $\left(Z_t^{(\mu)}, t \geq 0\right)$ *is the solution to*

$$Z_t = \gamma_t + \mu \int_0^t sgn(Z_s) ds, \quad (32)$$

with $(\gamma_t, t \geq 0)$ a Brownian motion.

c) The identity in law:

$$\left(\left|Z_t^{(\mu)}\right|, t \geq 0\right) \stackrel{(d)}{=} \left(S_t(\beta^{(-\mu)}) - \beta_t^{(-\mu)}, t \geq 0\right),$$

holds, where on the right hand side $\beta_t^{(-\mu)} = \beta_t - \mu t$ for a Brownian motion β_t, and $S_t(\theta) = \sup_{s \leq t} \theta_s$

Proof. (of Theorem 2):

a) This is a consequence of a slight generalization of Theorem 3.1 in Yor [36]. Let P denote the measure induced by the standard Wiener process and define $X_t^{(\mu)}$ by (31). By Girsanov's theorem, the law of this process has density with respect to P given by

$$\begin{aligned} P^{\mu,+}_{|\Im_t} &= \exp\left\{\mu \int_0^t \mathbf{1}_{(X_s>0)} dX_s - \frac{\mu^2}{2} \int_0^t ds \mathbf{1}_{(X_s>0)}\right\} \cdot P_{|\Im_t} \\ &= \exp\left\{\mu\left(X_t^+ - \frac{1}{2}\ell_t^0\right) - \frac{\mu^2}{2}\int_0^t ds \mathbf{1}_{(X_s>0)}\right\} \cdot P_{|\Im_t} \end{aligned}$$

We denote simply by $T_b : \inf \left\{ t : X_t^{(\mu)} = b \right\}$ and consider a functional F of the local time at $b-a$ of $X^{(\mu)}$ up to time T_b, $(\ell_{T_b}^{b-a}(X^{(\mu)}); 0 \le a \le b)$. We have that

$$\begin{aligned} & E^{\mu,+}\left[F(\ell_{T_b}^{b-a}(X^{(\mu)}); 0 \le a \le b)\right] \\ = \quad & E\left[F(\ell_{T_b}^{b-a}(X); 0 \le a \le b) \exp\left(-\frac{\mu}{2}\left(\ell_{T_b}^{0} - 2b\right) - \frac{\mu^2}{2}\int_0^b da \ell_{T_b}^{a}\right)\right] \\ = \quad & Q_0^2\left(F(Z_a, 0 \le a \le b) \exp\left(-\frac{\mu}{2}(Z_b - 2b)\right) - \frac{\mu^2}{2}\int_0^b da Z_a\right) \\ = \quad & {}^{(-\mu)}Q_0^2\left(F(Z_a, 0 \le a \le b)\right) \end{aligned}$$

In particular, we have that

$$\mathcal{L}\left(Z_{b-a}; 0 \le a \le b; {}^{-\mu} Q_0^2\right) = \mathcal{L}\left(\ell_{T_b}^{a}(X^{(\mu)}; 0 \le a \le b; P^{\mu,+}\right)$$

where $\mathcal{L}(H;P)$ denotes the law of H under P. As a consequence, there is the identity in law between the pairs of 2 dimensional variables:

$$\begin{aligned} \mathcal{L}\left\{Z_b, \int_0^b Z_a da; {}^{(-\mu)} Q_0^2\right\} \quad &= \quad \mathcal{L}\left\{\ell_{T_b}^{0}(X), \int_0^b \ell_{T_b}^{b-a}(X^{(\mu)}) da; P^{\mu,+}\right\} \\ &= \quad \mathcal{L}\left\{\ell_{T_b}^{0}(X), \int_0^{T_b} \mathbf{1}_{(X_s>0)} ds; P^{\mu,+}\right\} \end{aligned}$$

so that (30) holds.

b) From the equation (32) and Tanaka's formula we deduce:

$$\begin{aligned} X_t^+ \quad &= \quad \int_0^t \mathbf{1}_{(X_s>0)}\left(dB_s + \mu ds\right) + \frac{1}{2}\ell_t^0(X) \\ &= \quad \beta^{(\mu)}_{\int_0^t ds \mathbf{1}_{(X_s>0)}} + L_{\int_0^t ds \mathbf{1}_{(X_s>0)}} \end{aligned} \tag{33}$$

where $L_{\int_0^t ds \mathbf{1}_{(X_s>0)}}$ is defined by $\sup_{s \le t}\left(-\beta^{(\mu)}_{\int_0^s du \mathbf{1}_{(X_u>0)}}\right)$ in accordance with Skorohod's lemma.

On the other hand, we have from (32) and Tanaka's formula:

$$|Z_t| = \int_0^t sgn(Z_s) d\gamma_s + \mu t + L_t(Z) \tag{34}$$

Comparing (33) and (34), we note that

$$X_t^+ = |Z_u|_{|u=\int_0^t ds \mathbf{1}_{(X_s>0)}} \tag{35}$$

for some process $(Z_u, u \ge 0)$.

Now the identity in law proposed in b) follows immediately from (35).

c) is an immediate consequence of (32) and Skorohod's lemma. ■

Proof. of Theorem 1 continued. We come next to the general case for $\delta > 0$. Some important references for this development are [17], [35] and [19]. Here we note that with $x = \delta b/2$

$$
\begin{aligned}
& {}^{(-\mu)}Q_0^\delta\left(F(Z_a, 0\le a\le b)\right. \\
= \;& Q_0^\delta\left(F(Z_a, 0\le a\le b)\exp\left\{-\frac{\mu}{2}(Z_b-\delta b)-\frac{\mu^2}{2}\int_0^b Z_a da\right\}\right) \\
= \;& E\left[\begin{array}{c} F\left(\ell_{\tau_x}^{a-b}\left(|B|-\frac{2}{\delta}\ell\right); a\le b\right)\times \\ \exp\left(-\frac{\mu}{2}\left(\ell_{\tau_x}^0\left(|B|-\frac{2}{\delta}\ell\right)-\delta b\right)-\frac{\mu^2}{2}\int_0^{\tau_x} ds\mathbf{1}_{\left(|B_s|-\frac{2}{\delta}\ell_s\le 0\right)}\right)\end{array}\right]
\end{aligned}
$$

where $\tau_x = \inf\left\{t\ge 0 : \ell_t^0 = x\right\}$.
Define

$$
H_s = sgn(B_s)\mathbf{1}_{\left(|B_s|-\frac{2}{\delta}\ell_s\le 0\right)}
$$

so that by Tanaka's formula, we may write

$$
\begin{aligned}
& {}^{(-\mu)}Q_0^\delta\left(F(Z_a, 0\le a\le b)\right. \\
= \;& E\left[F\left(\ell_{\tau_x}^{a-b}\left(|B|-\frac{2}{\delta}\ell\right); a\le b\right)\exp -\mu\int_0^{\tau_x} H_s dB_s-\frac{\mu^2}{2}\int_0^{\tau_x} dsH_s^2\right]
\end{aligned}
$$

and it follows by Girsanov's theorem that this is

$$
= E^{\mu,\delta}\left[F\left(\ell_{\tau_x}^{a-b}\left(|X|-\frac{2}{\delta}\ell\right); a\le b\right)\right]
$$

where, under $P^{\mu,\delta}$, X solves

$$
X_t = \beta_t - \mu\int_0^t ds\; sgn(X_s)\mathbf{1}_{\left(|X_s|-\frac{2}{\delta}\ell_s\le 0\right)} \tag{36}
$$

It follows in particular that the law of $\left(\int_0^b da^{(-\mu)}y^\delta(a), b\ge 0\right)$ [2] has the same one dimensional marginals as the inhomogeneous Lévy process

$$
\left(\int_0^{\tau_x} ds\mathbf{1}_{\left(|X_s|-\frac{2}{\delta}\ell_s\le 0\right)}, b\ge 0\right),\ \text{under } P^{\mu,\delta}.
$$

That this process is an inhomogeneous Lévy process in b follows from the fact that, when we apply the Markov property in τ_x we obtain that $X_{\tau_x} = 0$ and $\ell_{\tau_x} = x$, hence the process $(X_{\tau_x+u,} u\ge 0)$ is independent from $(X_v, v\le \tau_x)$.

For part (ii) of theorem 1 we note that by arguments similar to the ones used in the proof of part (i) we may show that for $x\neq 0$ we have that

$$
\int_0^b da^{(-\mu)}y_x^\delta(a) \stackrel{(d)}{=} \int_0^{\tau_x^b(X)} ds\mathbf{1}_{(0\le X_s\le b)}
$$

where X solves (36) but now $\tau_x^b(X) = \inf\left\{t\ge 0 : \ell_t^b(X) > x\right\}$. That the LM property fails may be explained on account of $\tau_x^b(X)$ no longer being an increasing process in b. In fact, it can be shown that the associated Lévy measures are not increasing in b. ■

From these results, we may write the law

$$
{}^\beta Q_x^\delta = {}^\beta Q_0^\delta * {}^\beta Q_x^0
$$

[2] In accordance with Theorem 1, we ought to consider jointly ${}^{(-\mu)}y(b)$, but we do not write this down, for the sake of simplicity.

and hence, one may write

$$^{\beta}y_x^{\delta}(t) = {}^{\beta}y_0^{\delta}(t) + {}^{\beta}y_x^{0}(t)$$

where the processes ${}^{\beta}y_0^{\delta}$, ${}^{\beta}y_x^{0}$ are independent. On integrating, we obtain

$$\begin{aligned} {}^{\beta}Y_0^{\delta}(t) &= \int_0^t {}^{\beta}y_0^{\delta}(u)du \\ {}^{\beta}Y_x^{0}(t) &= \int_0^t {}^{\beta}y_x^{0}(u)du \end{aligned}$$

The marginals of the process $X(Y(t))$ now agree with the marginals of $X({}^{\beta}Y_0^{\delta}(t)) + X({}^{\beta}Y_x^{0}(t))$ and hence we may write

$$\begin{aligned} \frac{\exp\left(X(Y(t)\right)}{E\left[\exp(X(Y(t)))\right]} &\stackrel{(d)}{=} \frac{\exp\left(X({}^{\beta}Y_0^{\delta}(t))\right)}{E\left[\exp\left(X({}^{\beta}Y_0^{\delta}(t))\right)\right]} \frac{\exp\left(X({}^{\beta}Y_x^{0}(t))\right)}{E\left[\exp\left(X({}^{\beta}Y_x^{0}(t))\right)\right]} \\ &\stackrel{def}{=} M(t)U(t) \end{aligned}$$

The process $M(t)$ has the multiplicative LM property and there exists an inhomogeneous Lévy process of independent multiplicative increments with unit unconditional expectations and the same marginal distributions. In particular, this inhomogeneous Lévy process is also a martingale. On the other hand, the process $U(t)$ does not have the LM property. Hence, its one dimensional distributions may not be consistent with a martingale process by such an argument.

Some properties of the process $U(t)$ are worthy of note. First, we observe that ${}^{\beta}y_x^{0}(t)$ starts at x, but is eventually absorbed at 0. The distribution of the first hitting time of 0 by the process $y_x^0(t)$ is (see Yor [37], Getoor [18])

$$T^0\left(y_x^0\right) \stackrel{(d)}{=} \frac{x}{2\mathfrak{e}}$$

where $\mathfrak{e}$ is a standard exponential random variable. More generally, for general β we have that

$$P\left[T^0 \leq s\right] = \exp\left(-\frac{\kappa x/2}{\exp(\kappa s) - 1}\right)$$

It follows that the numerator in the expression for $U(t)$ is eventually constant. The process is a smooth differentiable process that may be viewed as a random drift component that is unconditionally absent that is used to perturb the martingale $M(t)$ by adding a conditional unobservable drift term. We may interpret this conditional drift as a conditional abnormal return that is unconditionally absent and eventually zero.

Leaving aside these considerations, we now introduce the property of martingale marginals (MM). We say that a process $H(t)$ of constant expectation has the property of martingale marginals (MM) just if there exists a martingale $N(t)$ with the same marginal distributions as those for $H(t)$ for each t. The process $U(t)$ may possess the property of martingale marginals and by such a decomposition, we could write martingale laws for the class of SA processes defined here.

The (LM) and (MM) properties introduced here are related in that if $\left(\widetilde{L}_t\right)$ satisfies the (LM) property, then $\left(\widetilde{M}_t\right) = \frac{\exp(\widetilde{L}_t)}{E\left[\exp(\widetilde{L}_t)\right]}$ satisfies the (MM) property. A priori, the converse does not hold, i.e. if $\left(\widetilde{M}_t\right)$ satisfies the (MM) property, there does not necessarily exist $(\widetilde{L}_t)$ satisfying the (LM) property.

6 The Data and Estimation Procedure

We obtained data on out-of-the-money S&P 500 closing option prices for maturities between a month and a year for each second Wednesday of each month for the year 2000. This provides us with a monthly time series of option prices on a single but important underlying asset. The dates employed were $Jan.12$, $Feb.9$, $Mar.8$, $Apr.12$, $May10$, $Jun.14$, $Jul.12$, $Aug.9$, $Sept.13$, $Oct.11$, $Nov.8$, and $Dec.13$. Similar data was obtained for some 20 other underliers. By ticker symbol, they are BA, BKX, CSCO, DRG, GE, HWP, IBM, INTC, JNJ, KO, MCD, MSFT, ORCL, PFE, RUT, SUNW, WMT, XAU, XOI, and XOM.

For each model and each underlier, we follow a uniform procedure for constructing the option price. In particular, we use the fast Fourier transform (FFT) to invert the generalized Fourier transform of the call price, as developed in Carr and Madan [9]. This generalized Fourier transform is analytic whenever the characteristic function for the log of the stock price is analytic. More precisely, let $C(k,t)$ be the price of a call option with strike $\exp(k)$ and maturity t. Let a be a positive constant such that the a^{th} moment of the stock price exists. Carr and Madan [9] show that

$$\begin{aligned}\gamma(u,t) &= \int_{-\infty}^{\infty} e^{iuk} e^{\alpha k} C(k,t)dk \\ &= \frac{e^{-rt}\zeta(u-i(\alpha+1),t)}{\alpha^2+\alpha-u^2+i(2\alpha+1)u}\end{aligned}$$

where $\zeta(u,t)$ denotes the characteristic function for the log of the stock price. The call prices follow on performing the (FFT) integration

$$C(k,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iuk-\alpha k}\gamma(u,t)du$$

Put option prices are obtained using put-call parity.

One advantage of this procedure is that all models may be handled with a single code since a model change only involves changing the specific characteristic function which is called. Furthermore, as the FFT works equally well on matrix structures, all strikes and maturities may be simultaneously computed in a very efficient manner. This is a very desirable property when we consider that parameters have to estimated within an optimization algorithm.

The model parameters in each case are estimated by minimizing the root mean square error between market close prices and model option prices. The root mean square error is taken here over all strikes and maturities. We also compute the average absolute error as a percentage of the mean price. For comparative purposes, we report this statistic as an overall measure of the quality of the fit.

7 Results of Estimations for the Year 2000

The results are presented in two categories. First, we present monthly results on the $S\&P$ 500 index for all six models. In the interests of brevity, we present a sample of quarterly results on the other underliers for just the three dominating models of the $NIGSA$, $VGSA$, and $CGMYSA$.

7.1 S&P 500 Estimations

Each of the six models was estimated on S&P 500 Index options using one day from each of the twelve months for the year 2000. The estimation results for the six models

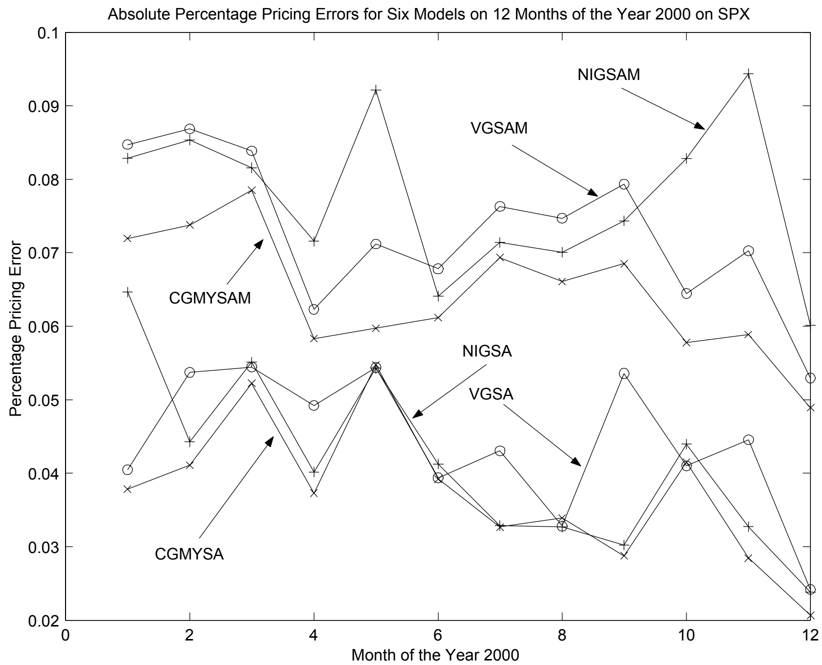

Figure 1: Graphs of absolute percentage errors for the six models across the twelve months

are presented in six tables, one for each model. We present the three tables associated with the exponential methods $NIGSA, VGSA$, and $CGMYSA$. These are followed by the stochastic exponential martingale models $NIGSAM, VGSAM$, and $CGMYSAM$.

A comparison of the average percentage pricing errors shows that the exponential method dominates the stochastic exponential method in all cases. The average improvement of the exponential over the stochastic exponential in the three cases of NIG, VG, and $CGMY$ is respectively 3.62%, 2.87%, and 2.70% respectively. We present in figure (1) a graph of the percentage absolute pricing errors for each of the six models over the twelve months.

The domination of the mean corrected exponential over the martingale stochastic exponential is markedly evident. From a practical standpoint, the use of the theoretically superior stochastic exponential is associated with a high price in terms of the quality of the model's fit. Thus, for the rest of this study, we restrict attention to the mean corrected exponential models (which are still consistent with no spot forward arbitrage)

The strong negative skew is captured by all 3 SA models. This is reflected by consistently strong negative estimates of θ for $NIGSA$, with an average value of -9.84. For the $VGSA$ and $CGMYSA$ models, this is reflected by a consistently lower value for G than for M. For the $VGSA$ model, the average markup of M over G is 20.83, while for the $CGMYSA$ model, it is 68.03.

The rates of mean reversion in volatility or activity are comparable for $NIGSA$, $VGSA$,and $CGMYSA$, averaging to $6.79, 4.27$, and 3.34 respectively. These are associated with half lives of around 7.5 weeks.

All three models indicate a comparable long term level, relative to the initial value of the time change process. For the $NIGSA, VGSA$, and $CGMYSA$ models, this ratio averages to .5076 .4681,and .5118 respectively. The models are quite consistent in this

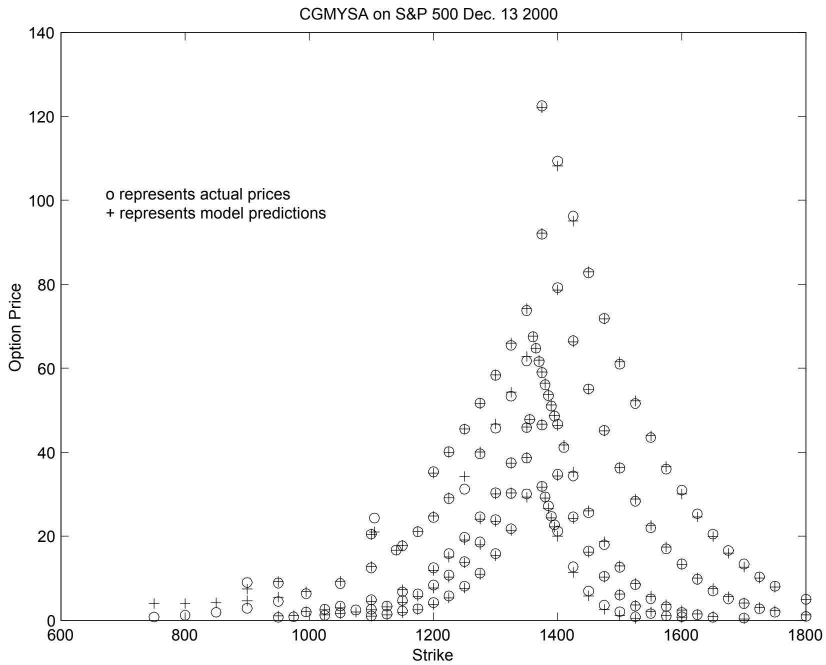

Figure 2: The fit of the model $CGMYSA$ to option price data for all atrikes and maturities on December 13 2000

regard.

The estimates for the volatility of the time change are consistent between $NIGSA$ and $CGMYSA$ with mean values of 5.34 and 6.97 respectively. The values are somewhat higher for $VGSA$ at 16.45.

The best performing model by far is the $CGMYSA$ model, as it consistently has the lowest pricing errors. The parameters of this model are also more stable across time. The mean pricing errors for the models $NIGSA, VGSA$, and $CGMYSA$ are $4.1346\%, 4.4251\%$, and 3.3738% respectively. Among the six models studied here, the tentatively best model is $CGMYSA$. The best overall fit was for December 13 2000 for $CGMYSA$, and we present a graph (see figure (2)) of the actual and predicted prices for this day. However, it should be noted that lower dimensional calibrations are faster, and the improvements in fit may not warrant the extra time required for some applications.

7.2 Absolute Percentage Errors for the Year

For the three exponential models estimated for each of the twelve days in the year 2000, we stacked all of the absolute percentage pricing errors across strikes and maturities. The pricing errors are themselves orthogonal to strike and maturity, but the absolute pricing errors tend to be larger for shorter maturities and options which are further out-of-the-money. This is confirmed by regression results of the absolute errors on moneyness and maturity, where we employ moneyness and its square to capture the fact that we have out-of-the-money options on both sides of the forward. Table 7 presents the results.

We note that the quadratic in moneyness is significant in both its linear and quadratic terms. The shape is consistent with absolute errors rising as an option gets further out-of-the-money. The coefficient for maturity is also negative and significant, which is indicative of higher absolute errors for shorter maturity options. The R^2 coefficients are around 50% with values of $51.62, 49.07$, and 47.12 for the models $NIGSA, VGSA$, and $CGMYSA$ respectively.

7.3 Estimation Results for Other Underliers

For other underliers, we present a sample of quarterly estimates on the three dominating models $NIGSA, VGSA$, and $CGMYSA$. For each model, we present results on five selected names. For NIGSA, results are presented for BKX, MCD, PFE, INTC and MSFT. For VGSA, we present result from RUT, GE, JNJ, IBM, and SUNW. Finally, from CGMYSA, results are given for DRG, HWP, BA, CSCO and ORCL. These results are presented in tables 8 through 10, one for each model.

The presence of skewness is once again consistently observed in the form of consistently negative values for θ in the case of $NIGSA$, and consistently lower values for G than M for the $VGSA$ and $CGMYSA$ models. Generally, the long term activity levels are lower than current activity levels, as reflected in comparing the parameters C with the values η. This is consistent with the view that volatilities over the year 2000 were on the high side. The performance of $NIGSA$ and $VGSA$ is quite good for options on single names. This may be a consequence of the fact that fewer options are available for calibration, with the result that the lower dimensional models are sufficient to capture their variation.

8 Comparative Results on Jump-Diffusion Models

In this section we outline some recent jump-diffusion models proposed by Bates [7] and Duffie, Pan, and Singleton [14](henceforth DPS). We also present some estimation results for these models on our data set. Bates' model is similar to our SV models in that the jump intensity is proportional to the level of a CIR process. In Bates' model, this process is also the volatility of the diffusion component in the stock. In addition, DPS allow for jumps in the volatility with an exponential density for the jump magnitude in volatility. Conditioning on a jump occurring, the Bates and DPS models both have a lognormally distributed jump in the stock price. Thus the resulting jump process for the log of the price has finite activity and has a Levy density which is not completely monotone.

We focus on the following special case of the model by Bates, which is also studied by Pan [32]:

$$\begin{aligned} dS(t) &= (r - q - \lambda_y \mu) S_t dt + \sqrt{V_t} S_t dW_S(t) + J_y S_t dq_y(t) \\ dV_t &= (\theta_v - \kappa_v V_t) dt + \sigma_v \sqrt{V_t} dW_V(t) \\ dW_S dW_V &= \rho dt \\ & \quad (1 + J_y) \text{ is lognormally distributed with mean } \mu_y \text{ and variance } \sigma_y^2 \\ & \quad q_y(t) \text{ is a Poisson process with arrival rate } \lambda_y V_t \\ \mu &= (\exp(\mu_y + \sigma_y^2/2) - 1). \end{aligned}$$

This model has eight parameters

$$V(0), \rho, \sigma_v, \kappa_v, \theta_v, \lambda_y, \mu_y, \sigma_y.$$

We also focus on the following subset of the class studied by DPS [14]:

$$\begin{aligned}
dS_t &= (r - q - \lambda_y \mu) S_t dt + \sqrt{V_t} S_t dW_S(t) + J_y S_t dq_y(t) \\
dV_t &= \kappa_v(\theta_v - V_t)dt + \sigma_v \sqrt{V_t} dW_V(t) + J_V dq_v(t) \\
dW_S dW_V &= \rho dt \\
& (1 + J_y) \text{ is lognormally distributed with mean } \mu_y \text{ and variance } \sigma_y^2 \\
& J_V \text{ has an exponential distribution with mean } \mu_V \\
& q_y(t), q_v(t) \text{ are independent Poisson processes with arrival rates } \lambda_y, \lambda_v
\end{aligned}$$

The parameter μ in the stochastic differential equation for the stock is defined as in Bates. The model has ten parameters

$$V(0), \rho, \sigma_v, \kappa_v, \theta_v, \lambda_y, \mu_y, \sigma_y, \mu_v, \lambda_v$$

Table 11 presents results on the Bates model estimated for every second Wednesday of each month, of the year 2000. Table 12 provides the results for the DPS models. We note that these models provide a competitive performance in summarizing the surface of option prices at a point of time.

9 Conclusion

Six stochastic volatility models were formulated by time changing three homogeneous Lévy processes. The Lévy processes employed were the normal inverse Gaussian model of Barndorff-Nielsen [4], the variance gamma of Madan, Carr, and Chang [27], and the $CGMY$ model of Carr, Geman, Madan, and Yor [10]. The time change used to induce stochastic volatility was the integral of the CIR (Cox, Ingersoll and Ross [11]) process. This resulted in three process respectively termed $NIGSV, VGSV$, and $CGMYSV$. Models for the stock price were built by exponentiating these processes and correcting the mean in accordance with spot forward arbitrage considerations, leading to the $NIGSA, VGSA$, and $CGMYSA$ models. A second class of discounted stock price models was obtained using stochastic exponentials, resulting in the $NIGSAM$, $VGSAM$,and $CGMYSAM$ models. These models imply that forward prices are martingales in the expanded filtration, which includes a knowledge of the integrated CIR time change process.

The paper also introduces two properties of stochastic processes, termed respectively the property of Lévy marginals and the property of martingale marginals. The $NIGSV, VGSV$, and $CGMYSV$ processes satisfy the Lévy marginal property when the underlying CIR is started at zero. In this case, the resulting $NIGSA$, $VGSA$,and $CGMYSA$ processes have the property of martingale marginals. More generally, however, one may have the martingale marginal property for the SA processes, without having the Lévy marginal property for the SV processes. We hope to devote some future research on these notions. In our view, the property of the existence of martingale marginal processes is fundamental at each time point for the risk neutral dynamics. Its application delivers a parametric model consistent with observed prices at each date, with parameters that will vary from day to day. The arbitrage-free risk-neutral dynamics is to be sought in the higher dimensional filtration of the asset price and the parameters of the synthesizing martingale marginal processes.

The six models were estimated for every second Wednesday of the month for each month of the year 2000 on data for S&P 500 options and 20 other underlying assets. For the S&P 500 options, the exponential models were significantly better than their stochastic exponential counterparts in all 3 cases, suggesting that this may be the general preferred direction. The results for the S&P 500 index options consistently reflected market skews, and stochastic volatility, with mean reversion rates of around seven weeks. Similar patterns were observed for other underliers.

The best model by far was the $CGMYSA$ model, with percentage errors across all strikes and maturities reaching as low as 2% for the S&P 500 index options. For options on single names, the performance of the lower dimensional $NIGSA$ and $VGSA$ models was adequate. The class of models proposed here is for the first time providing us with a relatively parsimonious representation of the surface of option prices, with some stability over time in the parameter estimates. Results on competing models in the jump-diffusion class are also provided. These structures lead to interesting applications on pricing exotic products and analysing risk management strategies in empirically realistic, yet tractable contexts. We expect continuing research to shed further light on these interesting questions. Of particular interest is the study of the statistical dynamics in the same parametric class, which would lead to explicit representations of the measure change process and its representation of risk pricing in financial markets.

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TABLE 1							
NIGSA on S&P 500 for the year 2000							
Month	σ	ν	θ	κ	η	λ	ape
Jan	1.68	31.06	-6.24	12.46	1.014	11.27	.0647
Feb	.9345	22.70	-9.23	3.997	.5127	2.92	.0443
Mar	1.14	24.24	-8.18	4.23	.6925	3.51	.0551
Apr	2.64	38.94	-8.15	9.28	1.27	7.73	.0402
May	5.19	70.21	-7.54	9.85	2.47	12.17	.0543
Jun	2.50	63.52	-11.28	8.71	1.63	6.53	.0412
July	.7256	21.71	-9.51	2.62	.5187	1.57	.0328
Aug	.9455	33.59	-18.56	2.47	.4826	1.20	.0327
Sept	.6038	20.62	-10.65	.9225	.1734	.92	.0303
Oct	1.58	24.15	-8.73	13.61	.7238	7.06	.0439
Nov	2.66	43.43	-9.34	11.38	1.117	8.11	.0327
Dec	.7878	18.93	-10.63	1.91	.2246	1.124	.0238

TABLE 2							
VGSA on S&P 500 for the year 2000							
Month	C	G	M	κ	η	λ	ape
Jan	14.96	19.38	34.45	3.09	7.90	10.23	.0405
Feb	25.18	27.47	44.21	5.22	16.92	19.01	.0537
Mar	26.23	26.34	42.47	4.08	15.99	16.52	.0545
Apr	13.04	14.50	33.34	1.55	0	4.80	.0492
May	86.92	42.40	57.21	8.40	42.05	43.57	.0544
Jun	48.98	40.62	62.14	7.24	32.15	24.81	.0394
July	25.86	28.26	60.32	2.53	10.47	7.59	.0431
Aug	31.50	32.97	66.83	2.67	18.18	7.63	.0328
Sept	12.19	21.26	42.95	.25	0	3.76	.0536
Oct	10.74	14.15	33.46	1.09	3.35	2.90	.0410
Nov	102.67	48.78	69.20	13.00	44.39	50.52	.0445
Dec	15.75	19.14	38.69	2.18	5.71	5.67	.0242

TABLE 3										
CGMYSA on S&P 500 for the year 2000										
Month	C	G	M	Y_p	Y_n	ζ	κ	η	λ	ape
Jan	5.50	15.34	54.36	.4026	.3084	.9138	2.24	3.59	6.04	.0378
Feb	6.61	17.14	66.59	.3990	.2909	.9197	2.45	4.54	7.01	.0411
Mar	3.99	13.75	52.97	.5323	.5764	.5280	2.67	2.93	5.72	.0523
Apr	13.82	20.50	86.83	.4503	.4572	.5208	5.54	7.88	13.64	.0373
May	5.76	17.25	38.57	.5266	.6086	.5527	6.67	2.94	9.67	.0548
Jun	5.48	21.87	66.41	.6423	.8376	.1827	7.64	3.57	8.47	.0392
July	2.18	14.20	48.88	.6459	.7522	.4051	2.77	1.69	2.97	.0326
Aug	22.81	22.17	43.93	.1270	.3109	.2748	1.89	4.47	4.54	.0339
Sept	1.69	12.24	48.72	.6359	.7152	.4707	.4675	.6772	1.60	.0287
Oct	2.44	4.68	64.07	.7804	.9441	.1165	1.06	.0058	1.44	.0415
Nov	13.75	20.32	252.02	.5729	.6084	.3143	5.06	7.12	11.77	.0285
Dec	37.66	19.78	192.20	.2893	.3291	.2022	1.63	22.89	10.83	.0206

TABLE 4							
NIGSAM on S&P 500 for the year 2000							
Month	σ	ν	θ	κ	η	λ	ape
Jan	.2983	13.2215	-17.71	0	6.17	.3417	.0829
Feb	.3041	10.62	-11.52	0	5.49	.2231	.0854
Mar	.3214	9.28	-9.50	0	2.61	.1044	.0816
Apr	.3786	24.76	-47.58	0	.0106	.6091	.0716
May	.4577	12.33	-12.59	0	7.55	.5795	.0922
Jun	.4009	12.72	-9.98	0	11.52	.0006	.0641
July	.3261	11.61	-9.55	.0267	2.46	.0007	.0714
Aug	.4141	17.48	-14.71	.0092	15.67	0	.0701
Sept	.3566	14.56	-11.58	.0043	11.74	.0005	.0744
Oct	.4129	13.81	-13.98	0	7.27	.2729	.0828
Nov	.4263	31.11	-53.81	.0007	0	.6566	.0944
Dec	.4399	31.34	-51.35	0	19.43	.3861	.0602

TABLE 5							
VGSAM on S&P 500 for the year 2000							
Month	C	G	M	κ	η	λ	ape
Jan	4.12	7.45	47.99	7.60	3.11	6.10	.0847
Feb	4.85	6.92	24.71	33.15	2.53	10.76	.0869
Mar	3.71	7.33	32.40	12.66	3.38	7.35	.0839
Apr	8.17	8.09	36.90	21.36	3.99	11.33	.0623
May	10.89	7.21	24.86	38.17	3.19	16.66	.0712
Jun	5.30	11.02	30.11	0	4.15	1.14	.0678
July	3.56	9.48	25.75	.0289	29.56	.2564	.0763
Aug	7.06	14.18	44.61	.4404	14.31	.0016	.0747
Sept	5.17	12.35	33.61	.0762	16.94	.6939	.0794
Oct	9.92	8.54	33.28	26.96	3.74	14.09	.0645
Nov	12.19	10.49	80.13	23.17	5.28	18.70	.0703
Dec	16.43	11.72	42.67	58.51	6.17	24.00	.0529

TABLE 6									
CGMYSAM on S&P 500 for the year 2000									
Month	C	G	M	Y_p	Y_n	ζ	κ	η	λ	ape
Jan	.3509	.8878	20.81	-3.10	1.41	.1577	5.46	.2748	1.65e-5	.0720
Feb	.2733	.6692	21.51	-3.57	1.46	.1609	6.44	.2244	3.96e-6	.0738
Mar	.1635	.6965	21.97	-3.65	1.45	.2883	8.51	.1497	.00022	.0785
Apr	.1448	.6548	25.83	-2.86	1.52	.4005	9.59	.0895	.00164	.0583
May	.7414	.1839	44.38	.1789	1.58	.1002	30.73	.2795	.0008	.0597
Jun	.3587	.4231	24.64	-4.51	1.67	.0526	6.65	.3469	.0006	.0612
July	.6423	2.96	21.18	-1.27	1.28	.1256	.2331	1.31	.0115	.0694
Aug	.5258	2.12	16.17	-1.62	1.53	.0607	.9059	.7831	.0133	.0661
Sept	.4041	1.64	16.91	-2.90	1.54	.0676	4.85	.4474	2.78e-5	.0685
Oct	1.36	1.71	13.05	-2.31	1.34	.0869	13.13	.7234	.5735	.0577
Nov	.8461	.6404	12.88	-2.71	1.57	.0496	6.47	.4668	.001	.0588
Dec	2.044	3.68	52.86	-2.12	1.22	.0855	15.91	1.37	1.70	.0489

TABLE 7

Model	Constant	Moneyness	Moneyness^2	Maturity	R^2
VGSA	7.7628	-15.6155	7.8784	-.1220	49.07
	(43.62)	(-42.01)	(40.40)	(-4.93)	
NIGSA	8.6899	-17.3529	8.7147	-.1813	51.62
	(45.99)	(-43.98)	(42.10)	(-6.89)	
CGMYSA	8.043	-16.2076	8.2097	-.1307	47.12
	(42.20)	(-40.72)	(39.31)	(-4.93)	

TABLE 7

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NIGSA	8.6899	-17.3529	8.7147	-.1813	51.62
	(45.99)	(-43.98)	(42.10)	(-6.89)	
VGSA	7.7628	-15.6155	7.8784	-.1220	49.07
	(43.62)	(-42.01)	(40.40)	(-4.93)	
CGMYSA	8.043	-16.2076	8.2097	-.1307	47.12
	(42.20)	(-40.72)	(39.31)	(-4.93)	

TABLE 8

NIGSA on BKX quarterly for the year 2000							
Month	σ	ν	θ	κ	η	λ	ape
Mar	1.16	7.01	-3.41	2.46	.636	1.10	.0196
Jun	2.53	14.94	-2.76	7.64	1.77	6.17	.0146
Sep	2.64	38.31	-22.81	11.22	1.11	3.81	.0239
Dec	1.86	14.68	-6.41	14.57	1.07	2.65	.0132
NIGSA on MCD quarterly for the year 2000							
Mar	19.47	108.1	-19.33	6.00	11.03	1.70	.0251
Jun	1.39	8.89	-2.92	1.02	.351	.065	.0248
Sep	34.46	171.91	-37.21	43.58	13.52	9.19	.0119
Dec	29.06	165.37	-1.21	12.41	22.35	42.54	.0080
NIGSA on PFE quarterly for the year 2000							
Mar	3.80	22.27	-1.89	5.59	3.48	4.64	.0229
Jun	2.09	18.95	-7.52	.0058	2.03	.4205	.0098
Sep	2.64	24.79	-12.58	.0103	2.47	.2957	.0125
Dec	1.90	16.45	-11.10	6.14	1.47	.6101	.0211
NIGSA on INTC quarterly for the year 2000							
Mar	7.67	23.82	-4.37	8.21	3.51	6.32	.0177
Jun	2.59	10.69	-5.61	2.55	1.32	1.12	.0171
Sep	7.61	23.87	-9.23	9.99	2.90	4.12	.0128
Dec	3.35	6.27	-2.61	1.59	.0584	.8037	.0191
NIGSA on MSFT quarterly for the year 2000							
Mar	5.07	9.37	-1.21	10.69	1.59	.0005	.0189
Jun	1.11	5.74	-.9985	2.09	.7534	.1159	.0179
Sep	1.67	11.27	-1.74	1.26	1.42	1.44	.0123
Dec	1.97	3.31	-1.96	4.34	.2491	2.15	.0130

TABLE 9							
VGSA on RUT quarterly for the year 2000							
Month	C	G	M	κ	η	λ	ape
Mar	4.15	7.54	10.34	.5327	.8934	4.79	.0264
Jun	9.19	8.07	16.65	3.00	.0368	3.75	.0221
Sep	23.19	20.51	50.21	.4040	63.03	3.42	.0253
Dec	260.9	26.71	454.6	121.55	54.03	13.37	.0826
VGSA on GE quarterly for the year 2000							
Mar	98.43	27.44	46.75	1.07	6.92	5.03	.0236
Jun	21.92	12.82	28.21	1.94	9.88	2.48	.0217
Sep	21.56	18.67	27.90	.3856	5.54	5.91	.0367
Dec	19.75	9.09	18.14	12.27	6.85	18.04	.0391
VGSA on JNJ quarterly for the year 2000							
Mar	134.0	33.96	42.71	2.50	43.01	12.07	.0199
Jun	19.29	16.37	24.28	.1951	8.67	4.39	.0254
Sep	11.45	11.82	23.39	2.30	9.28	1.25	.0392
Dec	13.55	11.45	23.28	3.44	8.23	.1042	.0395
VGSA on IBM quarterly for the year 2000							
Mar	25.61	10.70	14.81	9.37	11.27	9.41	.0138
Jun	66.88	24.87	35.05	11.05	60.17	22.14	.0216
Sep	45.27	19.29	31.40	22.85	31.78	32.35	.0295
Dec	17.42	5.54	10.82	10.98	3.96	6.68	.0161
VGSA on SUNW quarterly for the year 2000							
Mar	73.66	14.69	21.96	6.18	35.76	9.97	.0187
Jun	20.79	8.21	15.42	.5104	8.69	1.99	.0309
Sep	20.94	7.51	11.69	2.46	3.29	2.87	.1707
Dec	41.24	5.45	10.77	6.39	6.31	10.91	.0299

TABLE 10

Month	C	G	M	Y_p	Y_n	ζ	κ	η	λ	ape
CGMYSA on DRG quarterly for the year 2000										
Mar	36.52	29.08	47.77	.1861	.0929	1.83	13.50	18.50	11.91	.0147
Jun	37.91	24.56	42.60	-.1548	.4957	.2021	2.99	.0005	3.15	.0146
Sep	33.43	25.57	42.96	.071	.3656	.3578	5.41	15.51	4.35	.0061
Dec	33.52	19.75	42.84	.1214	-.1962	1.87	5.59	17.08	15.26	.0188
CGMYSA on HWP quarterly for the year 2000										
Mar	6.91	9.86	24.91	.7334	.6002	1.07	1.65	2.13	.009	.0401
Jun	6.62	11.15	24.21	.7432	.3263	2.20	4.26	5.30	.0025	.0174
Sep	6.31	9.83	24.89	.7785	.5428	.9529	3.87	5.77	3e-6	.0218
Dec	17.69	18.16	23.40	.5293	.7693	.7905	13.33	3.24	2.86	.0186
CGMYSA on BA quarterly for the year 2000										
Mar	24.27	14.75	18.23	.3552	-.5478	3.59	8.07	7.58	.0004	.0240
Jun	10.98	11.32	20.86	.2024	.5589	.3315	1.40	1.76	1.87	.0192
Sep	11.59	12.67	21.23	.1431	.6869	.2505	2.45	2e-5	1.34	.0242
Dec	12.19	12.91	19.76	.3055	.5124	.6895	3.60	.3216	2.49	.0159
CGMYSA on CSCO quarterly for the year 2000										
Mar	8.02	8.05	13.73	.6386	.8214	.3391	11.28	2.76	9.89	.0214
Jun	5.27	6.03	18.06	.4163	.1313	1.56	4.67	6.27	9.24	.0180
Sep	7.55	7.76	19.97	.4882	.3761	1.28	6.68	4.07	7.47	.0136
Dec	8.37	6.33	6.50	-.4403	-.4323	5.51	7.45	2.53	6.65	.0165
CGMYSA on ORCL quarterly for the year 2000										
Mar	9.68	5.99	17.01	.6804	-.0542	2.17	4.73	4.56	.004	.0315
Jun	5.53	5.42	21.41	.6800	.6164	1.08	5.01	3.56	2.90	.0229
Sep	13.22	6.78	19.98	.6087	-.5269	5.10	13.19	3.56	.1504	.0178
Dec	8.71	5.05	18.77	.6902	.5689	1.38	4.83	2.99	4.77	.0195

TABLE 11

Bates submodel for S&P 500 options for the year 2000

Month	$V(0)$	ρ	σ_v	κ_v	θ_v	λ_y	μ_y	σ_y	*ape*
Jan	.0426	-.7216	.5654	1.572	.1108	.2789	-.7295	.00063	.0418
Feb	.0403	-.7401	.5691	1.879	.1268	.3033	-.7514	.00042	.0458
Mar	.0451	-.6972	.5909	1.257	.1178	.2716	-.7217	1.17e-5	.0545
Apr	.0684	-.6994	.9558	6.052	.3544	.2034	-.9164	1.98e-5	.0430
May	.0659	-.8292	.5843	6.834	.3379	.7631	-.6698	9.68e-6	.0496
Jun	.0357	-.9998	.3152	5.798	.2445	.2117	-1.4729	.0994	.0472
July	.0315	-.7812	.2617	1.683	.0876	.5468	-.6040	1.98e-5	.0349
Aug	.0291	-1.0	.1821	1.849	.0954	.4906	-.5709	5.19e-5	.0341
Sept	.0281	-.7799	.2596	1.678	.0784	.4574	-.6581	5.26e-4	.0376
Oct	.0609	-.7712	.7767	9.827	.4177	1.159	-.4742	2.57e-5	.0414
Nov	.0608	-.7053	.8947	5.927	.2840	.0581	-1.6741	.7609	.0301
Dec	.0458	-.6641	.5909	2.886	.1591	.0673	-1.148	.0703	.0240

TABLE 12											
Duffie, Pan and Singleton submodel on S&P 500 for the year 2000											
Month	$V(0)$	ρ	σ_v	κ_v	θ_v	λ_y	μ_y	σ_y	μ_v	λ_v	ape
Jan	.0408	-.7037	.5246	1.614	.0599	.0303	-.8451	.5735	.0036	.0707	.0413
Feb	.0395	-.7174	.5477	1.710	.0650	.0165	-1.357	.1299	.66e-3	.0334	.0459
Mar	.0438	-.9288	.5632	1.116	.0904	.0206	-1.082	.0181	.0552	4.25e-6	.0552
Apr	.0710	-.7035	.9859	7.575	.0531	.0143	-2.904	.6764	5.12e-4	.4141	.0427
May	.0681	-.9974	.4208	8.413	.0335	.0549	-1.037	.1644	.0086	6.603	.0489
Jun	.0352	-1	.2806	4.919	.0402	.0150	-5.485	2.482	.0244	.0015	.0471
July	.0303	-.9948	.1174	.1274	.0129	.0343	-.8005	8.09e–5	.0104	.0104	.0361
Aug	.0285	-.9976	.1204	.4169	.0858	.0292	-.7381	8.43e-5	1.06e-4	.0015	.0346
Sept	.0276	-.9871	.1800	1.206	.0293	.0154	-1.1789	.0559	.0204	1.231	.0373
Oct	.0634	-1	.3573	15.393	.0240	.2001	-.4025	.1786	.0085	2.678	.0385
Nov	.0620	-.7089	.8819	7.256	.0429	.0106	-2.212	.6922	6.4e-4	.8725	.0296
Dec	.0449	-.6785	.5431	2.956	.0506	.0141	-.8147	.6445	.5763	.0036	.0240