Modeling VXX

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Abstract

We study the VXX Exchange Traded Note (ETN), that has been actively traded in the New York Stock Exchange in recent years. We propose a simple model for the VXX and derive an analytical expression for the VXX roll yield. The roll yield of any futures position is the return not due to movements of the underlying, in commodity futures it is often called the cost of carry. Using our model we confirm that the phenomena of the large negative returns of the VXX, as first documented by Whaley (2013), which we call the VXX return puzzle, is due to the predominantly negative roll yield as proposed but never quantified in the literature. We provide a simple and robust estimation of the market price of variance risk which uses historical VXX returns. Our VXX price model can be used to study the price of options written on the VXX.

1 Introduction

There are three major risk factors which are traded in financial markets: market risk which is traded in the stock market, interest rate risk which is traded in the bond markets and interest rate derivative markets, and volatility risk which up until recently was only traded indirectly in the options market.

It is well accepted in the literature that both equity returns and variance are random (French, Schwert, and Stambaugh, 1987). It is also well understood that the variance risk premium is significant and negative (Carr and Wu, 2009). Investors trade volatility either to take advantage of the opportunity in the variance risk premium or to hedge against volatility risk. One way investors can trade volatility would be to buy at-the-money (ATM) options, but these do not necessarily stay at-the-money. When the options are Out-of-the-money (OTM) and in-the-money (ITM) they have smaller volatility sensitivity (Vega) and are therefore less effective for trading volatility. Options contracts will not always be able to meet investors need for volatility risk management as there will not be enough liquidity in the options markets when the market goes down. Also when investors trade volatility in the options market they also trade market risk and possibly interest rate risk (for longer term options), therefore trading volatility through options is often contaminated by these other risk factors making it inefficient for risk management. Developing a financial market to trade volatility directly is very important for researchers and practitioners. (Zhu and Zhang, 2007)

Chen, Chung, and Ho (2011) show that VIX (if tradeable) and VIX options expand investor opportunity set and are useful for diversification, in a mean-variance optimizing markowitz framework.

In 2003, the methodology for calculating the VIX index changed and the index using the old methodology was renamed to VXO. The VIX is now calculated using all out-of-the-money options on the S&P 500 which have a bid price. Following this change in 2004, the CBOE launched the much anticipated VIX futures and in 2006 VIX options also started trading on the CBOE. Both VIX futures and options have consistently grown in daily dollar trading volume.

In 2009, S&P Dow Jones Indices started reporting several different VIX futures indices which represent the returns of different VIX futures positions. One example of a VIX futures index is the S&P 500 VIX Short-Term Futures Index (SPVXSTR) which tracks the performance of a position in the nearest and second nearest maturing VIX futures. The SPVXSTR is rebalanced daily to create a constant one month maturity VIX futures position. Shortly after the VIX futures indices started were developed Barclays Capital iPath launched the first ever VIX futures index Exchange Traded Product (ETP), the VXX Exchange Traded Note (ETN). An ETN is unsecured senior debt that pays no coupons (interest) and does not have a fixed redemption at maturity but rather its redemption value is linked to the performance of some underlying (Bao, Li, and Gong, 2012).The VXX's redemption value, for example, depends on the value of the SPVXSTR at maturity less an annual management fee of 0.89%.

There are now many different VIX futures ETNs with different underlying indices, all of these combined make the VIX futures ETN market. The VIX futures ETN market has become vastly popular, all of the ETNs combined have a market capitalization of nearly 4 billion US dollars and average daily trading volume in excess of 800 million US dollars (Whaley, 2013). One of the main drivers of VIX futures ETNs growing popularity may be that Mutual funds and Hedge funds are often restricted from trading futures and options but they still have a need to hedge volatility risk therefore they trade in the VIX futures ETN market.

The VXX is the most popular of the VIX futures ETNs and is now the third most traded ETP, amongst all ETPs, based on average daily trading volume. The VXX is only just behind the iShares MSCI Emerging Markets ETF (EEM), much further behind the S&P 500 ETF (SPY) and in front of the iShares Russell 2000 ETF in terms of daily trading volume (IWM)[1].

Whaley (2013) is the first to document the phenomena of the highly negative returns of the VXX, which we will refer to as the VXX return puzzle. Eraker and Wu (2013) also show the significant negative performance of VIX futures and VIX futures index ETPs (including the VXX). Deng, McCann, and Wang (2012) show that ETNs on VIX futures indices, such as the VXX, are not very effective hedging/diversification tools for equity and mixed equity and bond portfolios. Hancock (2013) tests the performance of VIX futures ETNs and compares them to three benchmarks. Hancock (2013) shows that the VXX and other VIX futures ETNs never consistently outperform benchmarks even when used to diversify equity portfolios. These findings hold even when different holding periods and portfolio weighting methods are used. Hancock (2013) suggests that the poor performance is unique to VIX futures ETNs and is not a property of volatility.

We document the VXX returns in table 1 which shows the summary statistics of the VXX, SPX (S&P 500 index ETP) and VIX returns from 30$^{\text{th}}$ January 2009 to the 27$^{\text{th}}$ June 2014. Note the abysmal performance of the VXX as can be seen firstly by the -0.32% average daily discrete return of the VXX and the average daily continously compounded return of -0.39% as opposed to the average daily continuously compouned and discrete returns of the VIX which were -0.09% and 0.15% respectively. Secondly, the Holding Period Return (HPR) shows that within our sample period the VXX has lost 99.59% of its value, the VIX has only lost 71.66%. The Compound Annual Growth Rate (CAGR) of -63.49% of the VXX compared to a CAGR of -20.41% of the VIX, further displays the underperformance of the VXX. We will later show that the main reason why the VXX does not follow the

[1]ETP database website: www.etfdb.com/compare/volume as of the 10$^{\text{th}}$ October 2014. The average daily trading volume is computed as an average of the daily number of shares of that ETP traded over the previous 3 months.

VIX, as the constant 30-day maturity VIX futures does, is due to the roll yield.In figure 2 we plot the VIX index, the VXX price and the constant 30-day-maturity VIX futures price, as in Zhang, Shu, and Brenner (2010), so the difference is visually observable. Even with the well documented and easily observed underperformance the VXX market has made great strides in popularity, figure 3 shows us the upward trend in the daily dollar trading volume and the initial increase in and then levelling off in market capitalization of the VXX since inception.

Whaley (2013), Deng, McCann, and Wang (2012), Husson and McCann (2011) and Bao, Li, and Gong (2012) all suggest that the VXX is subject to the roll yield of VIX futures and that this is the cause for the underperformance of the VXX. None of the aforementioned articles quantify the roll yield or attempt to measure it, therefore we will create a model for the VXX which allows the quantification of the roll yield and proves the hypothesis that the roll yield drives the significant negative returns of the VXX.

The roll yield of any futures position is the return that a futures investor captures when the futures price converges to the spot price, it is the part of the return which is not due to changes in the price of the underlying asset or index. When the market is in backwardation (i.e. downward sloping term-structure) the price rolls up to the spot price, therefore the roll yield will be positive. When the market is in contango (i.e. upward sloping term-structure) the price rolls down to the spot price, therefore the roll yield will be negative. The VIX futures term structure is in contango during normal times and therefore the roll yield is for example negative. The VIX futures term structure can be in backwardation, usually during large economic downturns, and the roll yield will become positive which can make ETPs on VIX futures indices profitable (Whaley, 2013).

We study the VXX price by using the VIX futures price approximation from Zhang, Shu, and Brenner (2010), which we review in section 3.1, to propose the first stochastic volatility model of the VXX which accounts for the underlying dynamics of the S&P 500 index (SPX) and the VIX index. We believe the relationship between the VXX, the VIX and the S&P 500 is essential in building a comprehensive model. We show that the difference between the 30-day-maturity VIX futures price change and the VXX price change in figure 2 is in fact due to the roll yield. We then go further and show that the roll yields sign is driven, on aggregate, by the negative market price of variance risk, λ. Eraker and Wu (2013) use an equilibrium model approach to show that the Variance Risk Premium (VRP) is the driver of the VXX's negative returns. This is consistent with our finding as the market price of variance risk, λ, and the VRP are almost proportional as shown by Zhang and Huang (2010).

In the next section we will explain the methodology for how the SPVXSTR index is calculated. Then in Section 3 we will review the theory behind pricing the VIX and VIX futures from Zhang and Zhu (2006) and Zhang, Shu, and Brenner (2010) and use this to create a stochastic model for the VXX price and examine the roll yield of the VXX. In section 4 we will use the VXX model to develop a simple way of estimating the market price of variance risk. In section 5 we will examine the effect of the rebalancing frequency of the SPVXSTR which will also be a robustness test of our continuous time VXX model. Finally in section 6 we will conclude and discuss on our findings.

2 The SPVXSTR index

To model the VXX we must first understand the SPVXSTR. In this section we will present the methodology for calculating the SPVXSTR index as interpreted from S&P Dow Jones Indices (2012).

The SPVXSTR index seeks to model the outcome of holding a long position in short-term VIX futures, specifically holding positions in the nearest and second nearest maturing VIX futures. The position is rebalanced daily to create a constant rolling one-month maturity VIX futures position (Barclays, 2013). The index is calculated by

$$SPVXSTR_t = SPVXSTR_{t-1}(1 + CDR_t + TBR_t), \tag{1}$$

where $SPVXSTR_t$ is the index level at time t, $SPVXSTR_{t-1}$ is the index level at time $t-1$, CDR_t is the Contract Daily Return of the VIX futures position and TBR_t is the Treasury Bill Return earned on the notional value of the position. The TBR_t is given by

$$TBR_t = \left[\frac{1}{1 - \frac{91}{360}TBAR_{t-1}}\right]^{\frac{Delta_t}{91}}, \tag{2}$$

where $Delta_t$ is the number of calendar days between the current and previous business days. $TBAR_{t-1}$ is the Treasury Bill Annual Return, which is equal to the most recent weekly high discount rate for 91-day US Treasury bills effective on the preceding business day. Usually the rates are announced by the US Treasury on each Monday, but if the Monday is a holiday then Fridays rates will apply. The CDR_t is calculated by

$$CDR_t = \frac{w_{1,t-1}F_t^{T_1} + w_{2,t-1}F_t^{T_2}}{w_{1,t-1}F_{t-1}^{T_1} + w_{2,t-1}F_{t-1}^{T_2}}, \tag{3}$$

where $w_{i,t-1}$ is the weight in the i^{th} nearest maturing VIX futures at time $t-1$, $F_t^{T_i}$ is the market price of the i^{th} nearest maturing VIX futures contract at time t and $F_{t-1}^{T_i}$ is the market price of the i^{th} nearest maturing VIX futures contract at time $t-1$.[2] The weights are adjusted daily to be

$$w_{1,t} = \frac{dr}{dt},$$

and

$$w_{2,t} = 1 - \frac{dr}{dt},$$

[2] In Equation (3) we use we use $w_{1,t-1}$ and $w_{2,t-1}$ in the numerator. Deng, McCann, and Wang (2012)CDR_t use $w_{1,t}$ and $w_{2,t}$ which is inconsistent with the methodology from S&P Dow Jones Indices (2012). When calculating discrete returns of any position the weights should stay constant over the period you are calculating the return for and only the prices should change.

where S&P Dow Jones Indices (2012) defines "dr =The total number of business days within a Roll Period beginning with, and including, the following business day and ending with, but excluding, the following CBOE VIX Futures Settlement Date. The number of business days includes a new holiday introduced intra-month up to the business day preceding such a holiday." and "dt =The total number of business days in the current Roll Period beginning with, and including, the starting CBOE VIX Futures Settlement Date and ending with, but excluding, the following CBOE VIX Futures Settlement Date. The number of business days stays constant in cases of a new holiday introduced intra-month or an unscheduled market closure" (S&P Dow Jones Indices, 2012, p. 7) Figure 1 shows the determination of dr and dt in a diagram for convenience of understanding.

3 Modeling VXX

3.1 Review of VIX and VIX futures model

To model the VXX we need a model for the VIX index and VIX futures. Zhang and Zhu (2006) and Zhang, Shu, and Brenner (2010) have developed a model for the VIX and VIX futures, for completeness we review and combine the results from both in this section.

The SPX (S&P 500 index) can be modeled by the following diffusion process with a stochastic process of instantaneous volatility as described by Heston (1993),

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dB^P_{1,t}, \tag{4}$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma_v \sqrt{V_t} dB^P_{2,t}, \tag{5}$$

where S_t is the SPX, V_t is the instantaneous variance of the SPX, μ is the expected return from investing in the SPX, θ is the physical measure for the long run mean level of the instantaneous variance, κ is the physical measure for the speed of mean reversion of instantaneous variance and σ_V measures the the variance of variance. $B^P_{1,t}$ and $B^P_{2,t}$ are two standard Brownian motions that describe the random noise in the SPX return and variance, respectively, they are correlated by a constant correlation coefficient ρ.

The transformations between physical and risk-neutral parameters are given by

$$\theta = \frac{\theta^* \kappa^*}{\kappa} \tag{6}$$

and

$$\kappa^* = \kappa + \lambda, \tag{7}$$

where κ^* is the risk-neutral speed of mean reversion of volatility, θ^* is the risk-neutral long run mean level of instantaneous variance and λ is the market price of variance risk. We can then describe the risk-neutral dynamics of the SPX as follows

$$dS_t = rS_t dt + \sqrt{V_t} S_t dB^*_{1,t}, \tag{8}$$

$$dV_t = \kappa^*(\theta^* - V_t)dt + \sigma_v \sqrt{V_t} dB^*_{2,t}, \tag{9}$$

where r is the risk free rate, and $dB^*_{1,t}$ and $dB^*_{2,t}$ are two new standard Brownian motions which are correlated by the constant correlation coefficient, ρ. The VIX is equal to the variance swap rate (Carr and Wu, 2009), which is equivalent to the conditional expectation in the risk-neutral measure

$$VIX_t^2 = E_t^* \left[\frac{1}{\tau_0} \int_t^{t+\tau_0} V_s ds \right] = (1 - B)\theta^* + BV_t, \tag{10}$$

where $\tau_0 = \frac{30}{365}$ and $B = \frac{1-e^{-\kappa^* \tau_0}}{\kappa^* \tau_0}$. Then the VIX futures price formula is given by

$$\begin{aligned} \frac{F_t^T}{100} = E_t^*(VIX_T) =& E_t^*(\sqrt{(1-B)\theta^* + BV_T}) \\ =& \int_0^{+\infty} \sqrt{(1-B)\theta^* + BV_T} f^*(V_T|V_t) dV_T, \end{aligned} \tag{11}$$

where the transition probability density as given by Cox et al. (1985) is

$$f^*(V_T|V_t) = ce^{-u-v}\left(\frac{v}{u}\right)^{q/2} I_q(2\sqrt{uv}), \tag{12}$$

where

$$c = \frac{2\kappa^*}{\sigma_V^2(1-e^{-\kappa^*(T-t)})}, \quad u = cV_t e^{-\kappa^*(T-t)}, \quad v = cV_T, \quad q = \frac{2\kappa^*\theta^*}{\sigma_V^2} - 1,$$

where $I_q(.)$ is the modified Bessel function of the first kind and of order q. The distribution function is the non-central chi-square, $\chi^2(2v; 2q+2, 2u)$ with $2q+2$ degrees of freedom and parameter of non-centrality $2u$ proportional to V_t. Note that $(T-t)$ is the time to maturity of the VIX futures contract. (Zhang and Zhu, 2006)

Equation (11) is the accurate formula for the VIX futures price from Zhang and Zhu (2006) using our own notation. Zhang, Shu, and Brenner (2010) provide us with a very good closed form approximation of equation (11) given by [3]

$$\frac{F_t^T}{100} = F_0 + F_1 + F_2, \tag{13}$$

where

$$F_0 = [\theta^*(1 - Be^{-\kappa^*(T-t)}) + V_t Be^{-\kappa^*(T-t)}]^{\frac{1}{2}},$$

$$\begin{aligned} F_1 = -\frac{\sigma_v^2}{8}&[\theta^*(1 - Be^{-\kappa^*(T-t)}) + V_t Be^{-\kappa^*(T-t)}]^{\frac{-3}{2}} \\ &\times B^2\left[V_t e^{-\kappa^*(T-t)}\frac{1-e^{-\kappa^*(T-t)}}{\kappa^*} + \theta^*\frac{(1-e^{-\kappa^*(T-t)})^2}{2\kappa^*}\right], \end{aligned}$$

[3] In Zhang, Shu, and Brenner (2010) θ is assumed to be time dependant, θ_t, but we stick with the simpler version of the model from Zhang and Zhu (2006) and assume that θ is constant.

$$
\begin{aligned}
F_2 =& \frac{\sigma_v^4}{16}[\theta^*(1 - Be^{-\kappa^*(T-t)}) + V_t Be^{-\kappa^*(T-t)}]^{\frac{-5}{2}} \\
& \times B^3 \left[\frac{3}{2} V_t e^{-\kappa^*(T-t)} \frac{(1 - e^{-\kappa^*(T-t)})^2}{\kappa^{*2}} + \frac{1}{2}\theta^* \frac{(1 - e^{-\kappa^*(T-t)})^3}{\kappa^{*2}} \right],
\end{aligned}
$$

where $F_1 + F_2$ can be thought of as a convexity adjustment from the Taylor series expansion of equation (11).

3.2 Nearly 30-day VIX futures

Table 2 presents the values of estimated VIX futures prices using the full formula from Zhang and Zhu (2006), the closed form approximation of the full formula from Zhang, Shu, and Brenner (2010), equation (13), and two simplifications of the closed form approximation, $F_0 + F_1$ and just F_0. From table 2 we can see that for 30-day VIX futures prices using just F_0 creates a very small error from the accurate formula, equation (11). The table shows that the error from using just F_0 instead of the accurate formula, , equation (11), is always within 3% when $\theta^* = 0.1$ and V_t ranges from 0.04 to 0.2, κ^* ranges from 4 to 7 and σ_v ranges from 0.1 to 0.7. There is one outlier when $V_t = 0.04$, $\kappa^* = 4$ and $\sigma_V = 0.7$, but the error is only just outside 3% at 3.20%. The Root Mean Squared Error (RMSE) is 1.29% which is very acceptable. The results of the numerical exercise presented in table 2 lead us to proposition 1 below.

Proposition 1 *The price of nearly 30-day-to-maturity VIX futures can be given by*

$$
\frac{F_t^T}{100} = [\theta^*(1 - Be^{-\kappa^*(T-t)}) + V_t Be^{-\kappa^*(T-t)}]^{\frac{1}{2}}, \tag{14}
$$

with some small error when compared to the accurate VIX futures price formula from Zhang and Zhu (2006), as demonstrated in Table 2. For example for the range of parameters $\sigma_V = 0.1$ *to* 0.7*,* $V_t = 0.04$ *to* 0.20*,* $\kappa^* = 4$ *to* 7*, constant* $\theta^* = 0.1$ *and maturity of 30 days, the RMSE is only* 1.29%.

We take the natural log of equation (14) to get an expression for the natural log price of VIX futures given by

$$\ln\left(\frac{F_t^T}{100}\right) = \frac{1}{2}\ln[\theta^*(1 - Be^{-\kappa^*(T-t)}) + V_t Be^{-\kappa^*(T-t)}], \tag{15}$$

where $\ln(\frac{F_t^T}{100})$ is the natural log the price of nearly 30-day to maturity VIX futures contract.

In Figure 4 we can see the theoretical term structure of VIX futures using equation (14), the full approximation of VIX futures prices, equation (13) and only the $F_0 + F_1$ segment of the full approximation. We use parameter estimates of $\theta^* = 0.1$, $\kappa^* = 5$, $\sigma_v = 0.1425$ and $V_t = 0.06$ to create an upward sloping VIX futures term structure, as is normal for the VIX futures market. The difference between the points at $t + 30$ and $t + 29$ is equal to the average one-day roll yield of a 30-day to maturity VIX futures contract. The spot return is zero when the underlying instantaneous variance is constant which means that any return that can be seen is due to the roll yield of VIX futures. It can be seen in the diagram that as you step through time from t+30 to t+29 the return will be negative therefore the one-day roll yield will be negative when the term structure is upward sloping.

3.3 Model of Contract Daily Return

We can model the change of nearly 30-day log VIX futures price by taking the Taylor series expansion of our simple log VIX futures price formula, equation (15), this gives us

$$d\ln F_t^T = \frac{\partial \ln F_t^T}{\partial V_t}dV_t + \frac{1}{2}\frac{\partial^2 \ln F_t^T}{\partial V_t^2}(dV_t)^2 + \frac{\partial \ln F_t^T}{\partial t}dt, \tag{16}$$

next we substitute in the partial derivatives to get[4]

[4]Seen the Appendix, section A.

$$
\begin{aligned}
d\ln F_t^T = \quad & \frac{1}{2}\left[\frac{\theta^*}{Be^{-\kappa^*(T-t)}} - \theta^* + V_t\right]^{-1} dV_t \\
& -\frac{1}{4}\left[\frac{\theta^*}{Be^{-\kappa^*(T-t)}} - \theta^* + V_t\right]^{-2} (dV_t)^2 \\
& +\frac{1}{2}\left[\frac{\kappa^*(V_t-\theta^*)Be^{-\kappa^*(T-t)}}{\theta^* + (V_t-\theta^*)Be^{-\kappa^*(T-t)}}\right] dt.
\end{aligned} \tag{17}
$$

Proposition 2 *The SPVXSTR index is rebalanced daily to maintain a VIX futures position with one month maturity, therefore we can model the contract daily return (CDR_t) of the SPVXSTR as the log return of a 30-day to maturity VIX futures position. From this and equation* (17) *we get*

$$
CDR_t = d\ln F_t^T \Big|_{T=t+\tau_0} = d\ln F_t^{t+\tau_0} + RY_t, \tag{18}
$$

where

$$
\begin{aligned}
d\ln F_t^{t+\tau_0} =& \frac{1}{2}\left[\frac{\theta^*}{Be^{-\kappa^*\tau_0}} - \theta^* + V_t\right]^{-1} dV_t \\
& -\frac{1}{4}\left[\frac{\theta^*}{Be^{-\kappa^*\tau_0}} - \theta^* + V_t\right]^{-2} (dV_t)^2
\end{aligned} \tag{19}
$$

$$
RY_t = \frac{1}{2}\left[\frac{\kappa^*(V_t-\theta^*)Be^{-\kappa^*\tau_0}}{\theta^* + (V_t-\theta^*)Be^{-\kappa^*\tau_0}}\right] dt, \tag{20}
$$

where $\tau_0 = 30/365$, $d\ln F_t^{t+\tau_0}$ *is the change in the log price of a constant 30-day to maturity VIX futures contract and* RY_t *is the roll yield of the SPVXSTR. The roll yield of the SPVXSTR is the return of the underlying VIX futures position due to the maturity of the position changing from 30 days to 29 days, from one rebalancing of the position to just before the next rebalancing.*

3.4 VXX model

Proposition 3 *We know that the change in the SPVXSTR index, and therefore the VXX, is composed of the return of the futures position, the CDR_t, and a risk-free return on the notional of the futures position, TBR_t. Therefore we can model the VXX using CDR_t combined with a risk free return r given by*

$$\begin{aligned} d\ln VXX_t = CDR_t + rdt = d\ln F_t^T\Big|_{T=t+\tau_0} + rdt \\ = d\ln F_t^{t+\tau_0} + RY_t + rdt \end{aligned} \tag{21}$$

where RY_t is the one day roll yield of the VXX going from 30-day maturity to 29-day maturity and r is the risk-free return on the notional value of the futures position.

This model of the log VXX price is to our knowledge the first attempt in the literature to model the VXX using the underlying dynamics of the SPX. The model can be used to derive the market price of variance risk, λ, from VXX returns, as is described in Section 4. We could also use this model to price VXX options, which are essentially Asian options on the underlying instantaneous variance, V_t. In the next section we use our VXX model to quantify the roll yield and show that it drives the VXX's returns.

3.5 VXX roll yield

Whaley (2013), Deng, McCann, and Wang (2012) and Husson and McCann (2011) all suggest the roll yield as the reason the VXX's returns are so negative. Figure 2 shows us a comparison between the performance of the VIX, the VXX and a constant 30-day to maturity VIX futures contract. We can see an obvious difference between the 30-day to maturity VIX futures contract and VXX. From equation (21) we know that the difference between the 30-day VIX futures price and the VXX can be explained by the roll yield, RY_t.

To examine what drives the roll yield of the VXX we assume that the instantaneous variance, V_t, is constant at the physical measure long run mean level of instantaneous variance, θ, to produce the aggregate upward sloping term structure of the VXX. If V_t is constant then $dV_t = 0$, and therefore equation (18) simplifies to

$$CDR = RY_t^* = \frac{1}{2}\left[\frac{\kappa^*(\theta - \theta^*)Be^{-\kappa^*\tau_0}}{\theta^* + (\theta - \theta^*)Be^{-\kappa^*\tau_0}}\right]\Delta t, \tag{22}$$

where $\frac{RY_t^*}{\Delta t}$ is the one day roll yield of the aggregate VXX. To examine what drives the roll yield to be negative, during normal times, we can use the transformation from the risk-neutral measure to the physical measure long-run mean level of instantaneous variance, equation (6) and substituting this into equation (22) we get

$$\frac{RY_t^*}{\Delta t} = \frac{1}{2}\frac{\frac{\lambda\kappa^*}{\kappa}Be^{-\kappa^*\tau_0}}{1 + \frac{\lambda}{\kappa}Be^{-\kappa^*\tau_0}}. \tag{23}$$

As all parameters apart from λ are always positive and $\kappa^* = \kappa + \lambda > 0$ (Zhang, Shu, and Brenner, 2010), from equation (23) we can see that λ, the market price of variance risk, is the driver of sign of the one day roll yield of the VXX, on aggregate. We conclude that the negative roll yield of the VXX is driven by the usually negative, as shown in table 3, market price of variance risk.

4 The Market Price of Variance Risk, λ

When modelling instantaneous volatility which is not directly tradable it is important to incorporate the market price of risk. Under the Cohen et al. (1972) model you can always perfectly hedge your position and therefore you do not need the market price of variance for your model. When you are modeling something that is not traded this situation changes as you will not be able to create a perfect risk free portfolio and therefore an investor will require a premium to compensate for the risk, this is the market price of variance risk.

When implementing a stochastic volatility model, such as in the Heston (1993) framework, estimating the market price of variance risk, λ, is essential. There is no clear consensus on the estimation of the market price of variance risk,λ. Table 3 shows some different author's recent estimates for the market price of variance risk λ, the risk neutral measure of the mean reverting speed of variance, κ^*, and the sample period used. We can see from table 3 that the estimation of λ can vastly vary depending on the estimation methodology used.

Our model lets us develop a simple method of estimating λ. If we substitute $V_t = \theta$ in equation (21) we get

$$d \ln VXX_t = \frac{1}{2}\left[\frac{\kappa^*(\theta - \theta^*)Be^{-\kappa^*\tau_0}}{\theta^* + (\theta - \theta^*)Be^{-\kappa^*\tau_0}}\right] dt + rdt, \tag{24}$$

where $dV_t = 0$, so we are isolating the aggregate effect of the roll yield.

We can now take the integral of equation (24) and substitute in the transformation from risk-neutral to physical measure long run mean level of variance, from equation (6) to get

$$R = \ln\left(\frac{VXX_T}{VXX_0}\right) = \frac{1}{2}\frac{\bar{\lambda}Be^{-\kappa^*\tau_0}}{(1 + \frac{\bar{\lambda}}{\kappa^*}Be^{-\kappa^*\tau_0})}T + rT, \tag{25}$$

where R is the continuously compounded return on the VXX over the sample. VXX_T is the last VXX price and VXX_0 is the starting VXX price, in the sample period, rT is the risk free return and $\bar{\lambda}$ is given by

$$\bar{\lambda} = \frac{\lambda\kappa^*}{\kappa} = \frac{\lambda\kappa *}{\kappa^* - \lambda}. \tag{26}$$

Proposition 4 *We can use the VXX return and a estimate of κ^* to measure λ, the market price of variance risk by solving equation* (26) *for λ, which gives us*

$$\lambda = \frac{\bar{\lambda}\kappa^*}{\kappa^* + \bar{\lambda}}, \tag{27}$$

and solving equation (25) *for $\bar{\lambda}$ we get*

$$\bar{\lambda} = \frac{2R_E}{(1 - \frac{2R_E}{\kappa^*})Be^{-\kappa^*\tau_0}}.$$

where R_E is the annualized excess return of the VXX over the sample period, given by

$$R_E = \frac{1}{T}\ln\frac{VXX_T}{VXX_0} - r \tag{28}$$

We use the parameter estimate of $\kappa^* = 5.4642$ from Luo and Zhang (2012) ,as their estimate of κ^* is the most recent available one in the literature and the closest to our sample period, to demonstrate our new methodology of calculating λ. We then use the VXX prices from inception $VXX_0 = 6693.12$ on 30 Jan 2009 and the VXX price at the end of our sample $VXX_T = 28.86$, on 27 Jun 2014.[5] $T = 5.4082$ in years and rT is the cumulative treasury bill return over the same time period, $rT = TBR_{0,T} = 0.558\%$ as defined in equation (2) from section 2 but cumulated over the entire sample. The cumulated TBR is very small but this is expected as Treasury bill rates have been almost zero since the financial crisis. We input these parameter estimates into equation (27) and (4) from proposition 4 to calculate that $\lambda = -6.0211$ with very little need for computing power. This estimate coincides with other authors as it is negative and of similar magnitude, refer to table 3 for comparison.

This method for estimating the market price of variance risk, λ, makes the calibration of of any Heston (1993) model much simpler, as λ is now a function of VXX prices and κ^*.

4.1 The Market Price of Variance Risk and the Variance Risk Premium

Eraker and Wu (2013) use an economic equilibrium model to show that the abysmal performance of VIX futures and VIX futures index ETPs can be explained by the negative Variance Risk Premium (VRP). The VRP and the market price of variance risk, λ, are

[5]VXX price data from NASDAQ website: www.nasdaq.com/symbol/vxx/historical.

similar concepts as they both measure the amount of compensation that risk adverse investors require for taking on the variance risk. Variance is negatively related with equity returns and therefore the Variance Risk Premium and market price of variance are both negative. Investors accept the negative returns during normal times, when taking a long position in volatility, in order to hedge against times of high volatility where they will receive a positive return from this long position, such as the 2008 financial crisis. Zhang and Huang (2010) show that the market price of variance risk, λ, from the Heston (1993) framework is almost proportional to the Variance Risk Premium, VRP, as defined by Carr and Wu (2009) as long as $\lambda\tau_0$ is small. Their result is shown by

$$VRP = \left[\left(\frac{1}{6}\kappa\tau_0 + O(\kappa^2\tau_0)\right)\theta + \left(\frac{1}{2} - \frac{1}{3}\kappa\tau_0 + O(\kappa^2\tau_0^2)\right)V_t\right]\lambda\tau_0 + O(\lambda^2\tau_0^2), \tag{29}$$

where $O(\cdot)$ is a function of order $\lambda^2\tau_0^2$ (Zhang and Huang, 2010). The first part of the equation is obviously proportional to λ as it is multiplied by $\lambda\tau_0$. The reason the relationship between VRP and λ is almost proportional is because of the $O(\lambda^2\tau_0^2)$ part of equation (29) which is not proportional to λ but aslong as $\lambda\tau_0$ is small (relative to 1) then $\lambda^2\tau_0^2$ will be very small.

Our findings are consistent with those from Eraker and Wu (2013) as we find a negative market price of variance risk drives the returns of the VXX to be so negative, through the negative roll yield, and they find a negative Variance Risk Premium as the cause of the negative returns of VIX futures positions and VIX futures ETNs.

5 Rebalancing Frequency of SPVXSTR

In this section we explore the effect of rebalancing frequency of the SPVXSTR. We start by replicating the SPVXSTR index using VIX futures prices from the 20$^{\text{th}}$ of December

2005 until the 28$^{\text{th}}$ of March 2014[6], using the methodology from S&P Dow Jones Indices (2012). This replicated SPVXSTR time series is displayed in figure 5 along with the actual SPVXSTR time series, the lines are almost exactly identical showing that our replication is accurate.

Figure 6 shows four time series of the replicated SPVXSTR index with different rebalancing frequencies of daily, weekly, bi-weekly and monthly rebalancing. The figure shows that as the rebalancing frequency is decreased from daily to weekly, biweekly and monthly, the SPVXSTR's value decreases. If this effect exists going from daily to more frequent rebalancing, for example hourly, then this would be a problem for our continuous time model. To examine the effect of the rebalancing frequency on the price of the VXX for smaller time steps than daily we needed a VIX futures price time series that was intraday, but real data for this is only available to us for the last 50 days, therefore we chose to simulate a five year long hourly VIX futures price time series.

To simulate the hourly time series of VIX futures prices we first need a time series of instantaneous volatility, which we get from the physical measure stochastic process of instantaneous variance Heston (1993), given by

$$dV_t = \kappa(\theta - V_t)dt + \sigma_v \sqrt{V_t} dB. \tag{30}$$

We then use the simple VIX futures price approximation, F_0, from equation (13) to find a time series of nearest and second nearest maturing VIX futures prices. We use $\kappa^* = 5.4642$ as this is the most recent estimation, we $\lambda = -6.0211$ as calculated in section 2. We propose $\theta = 0.1$ and $\sigma_v = 0.4$ as reasonable value. The results of this section are not sensitive to what parameters are used, aslong as they are reasonable. For simplicity we assume that VIX futures mature every 28 days, that there are no non-trading days, trading hours are 24 hours of the day and that the risk free rate is zero.

[6]Available at http://cfe.cboe.com/Data/HistoricalData.aspx#VX, accessed on the 20$^{\text{th}}$ of April 2014.

We then use the methodology from section 2 to calculate the SPVXSTR index for five years with different rebalancing frequencies and a starting value of one.

Figure 7 shows the resulting SPVXSTR hourly time series for different rebalancing frequencies from hourly to monthly. We can see in figure 7 that the simulated SPVXSTR time series for hourly and daily rebalancing are almost identical. The rebalancing effect going from daily to hourly rebalancing is therefore very very small and not a problem for our model. There is however a rebalancing effect if the index is rebalanced less often than dail, this is consistent with our findings using market VIX futures prices. To show that our conclusion on the rebalancing frequency is robust to the term structure of VIX futures we repeated the above exercise but holding V_t constant at different levels. This allows us to create a time series of SPVXSTR with a upward sloping (in contango) VIX futures term structure, as shown in figure 8, and downward sloping (in backwardation) VIX futures term structure, as shown in figure 9.

From figures 8 and 9 we can see that the rebalancing frequency does not significantly impact the SPVXSTR for hourly rebalancing. However there is a significcant effect when going to less frequent rebalancing. These results are robust to the term structure shape of VIX futures. Therefore we can conclude that shifting from a daily rebalancing to more frequent rebalancing does not affect the returns of the SPVXSTR significantly. Also in both figures the VXX model time series estimated using our model is the continuous limit of the rebalancing time series. The VXX model line is almost identical to the daily rebalancing time series, showing that our continuous time VXX model is adequate to model the discrete time VXX.

Figures 8 and 9 also show the importance of the roll yield as a driver of the SPVXSTR and subsequently the VXX. The two figures isolate the effect of the term-structure on the returns of the VXX SPVXSTR, by holding V_t constant and we know that the roll yield is a result of the term structure of VIX futures. When the term structure is upward sloping, causing a negative roll yield, the simulated SPVXSTR will tend to 0 as in figure 8, and when the term structure is downward sloping, causing positive roll yield, the simulated SPVXSTR is exponentially increasing as in figure 9.

6 Conclusions and Discussions

We study the VXX ETP which has been traded very actively on the New York Stock Exchange in recent years. We use the VIX futures price approximation from Zhang, Shu, and Brenner (2010) and simplify it for the nearly 30-day VIX futures contract. From this simplified formula for VIX futures prices we develop a model for the VXX. Our model is, to our knowledge, the first ever model of the VXX which encompasses the dynamics of the SPX index and the VIX index. Our model is the simplest way to model the VXX while capturing the relationship between the SPX, VIX and the VXX.

Our model explains the large negative returns of the VXX very well and is in line with the methodology from S&P Dow Jones Indices (2012). Our VXX model allows us to show that the difference in returns of the constant 30-day maturity VIX futures contract, as in Zhang, Shu, and Brenner (2010), and the VXX is due to the roll yield as suggested in the literature. We then examine the roll yield and show that λ, the market price of variance risk is the main driver of the roll yield. Therefore we have provided an explanation of the VXX return puzzle as the constant 30-day maturity VIX futures contract does not exhibit these negative price movements and is closely related to the VIX, therefore the roll yield drives the negative returns of the VXX.

We have also provided a simple and robust way of measuring the market price of variance risk, λ using our model and VXX prices. To understand the economic explanation for this we suggest examining the economic model for VIX Exchange Traded Notes from Eraker and Wu (2013). Their model finds that the negative return premium (Variance Risk Premium), which is almost proportional to λ (Zhang and Zhu, 2006), is an equilibrium outcome because long VIX futures positions allow investors to hedge against high volatility and low return states, such as exhibited in a financial crisis.

Our continuously rebalanced VXX model is adequate for modeling the daily rebalanced VXX as the effect of the rebalancing frequency is only significant at less frequent than daily rebalancing.

Our model for the VXX is the first of its kind, as it is the only one that includes the relationship between the SPX, the VIX and the VXX which we believe is fundamental in understanding the VXX. Our model could also be used by practitioners to price options written on the VXX, which can be regarded as Asian options written on the underlying instantaneous variance. Bao, Li, and Gong (2012) have created a model for pricing VXX options but they do not account for the dynamics of the S&P 500 or the VIX, which is essential in modeling the VXX.

Our research shows that the roll yield is the main cause for the negative performance of the VXX, as suggested in the literature. It would be interesting to see whether the roll yield also plays a large part in the returns of other VIX futures ETPs, we expect that it would. Our model could be expanded by using the full approximation formula of VIX futures from Zhang, Shu, and Brenner (2010) and letting θ be time dependant. One could use a similar approach to ours to explore the effect of the roll yield on other VIX futures ETNs but we advise caution in using the simplified VIX futures price formula F_0 as it will be prone to more error at longer maturities. Further research is also needed into the calibration technique best used for our model and its accuracy although it is theoretically sound. Exploring similar approaches to the one in this article to create models of other VIX futures Exchange Traded Products could help further develop the literature around these popular yet mysterious investment products.

Appendix

A. Solving for CDR_t model

From equation (15) and Ito's lemma we get

$$d\ln F_t^T = \frac{\partial \ln F_t^T}{\partial V_t} dV_t + \frac{1}{2}\frac{\partial^2 \ln F_t^T}{\partial V_t^2}(dV_t)^2 + \frac{\partial \ln F_t^T}{\partial t} dt, \tag{31}$$

therefore we need to find each of the partial derivatives $\frac{\partial \ln F_t^T}{\partial V_t}$, $\frac{\partial^2 \ln F_t^T}{\partial V_t^2}$ and $\frac{\partial \ln F_t^T}{\partial t}$. These are given by

$$\begin{aligned} \frac{\partial \ln F_t^T}{\partial V_t} &= \frac{1}{2}\frac{Be^{-\kappa(T-t)}}{\left[\theta^*(1-Be^{-\kappa(T-t)}) + V_t Be^{-\kappa(T-t)}\right]} \\ &= \frac{1}{2}\left[\frac{\theta^*}{Be^{-\kappa^*(T-t)}} - \theta^* + V_t\right]^{-1} \end{aligned}, \tag{32}$$

$$\frac{\partial^2 \ln F_t^T}{\partial V_t^2} = -\frac{1}{2}\left[\frac{\theta^*}{Be^{-\kappa^*(T-t)}} - \theta^* + V_t\right]^{-2} \tag{33}$$

and

$$\frac{\partial \ln F_t^T}{\partial t} = \frac{1}{2}\left[\frac{\kappa^*(V_t - \theta^*)Be^{-\kappa^*(T-t)}}{\theta^* + (V_t-\theta^*)Be^{-\kappa^*(T-t)}}\right]. \tag{34}$$

We then substitute all the partial derivatives into equation (31) giving us the full function of the log futures return as in equation (17) from section 3.3.

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Table 1: **Summary statistics of the daily returns for the SPY, VIX and VXX.** This table shows the summary statistics and correlations of the VXX, SPX (S&P 500 index ETP) and the VIX index returns from the 2nd February 2009 to the 13th August 2014. R_D represents estimates using discrete daily returns and R_C represents estimates using continuously compounded daily returns. The annualised standard deviation is calculated by multiplying the standard deviation by $\sqrt{252}$. The Holding Period Return (HPR) is the discrete return from the first price to the last price of the sample. The Compound Annual Growth Rate (CAGR) is the constant yearly growth rate that would lead to the change from the first price to the last price in the sample, it is calculated by $CAGR = (HPR+1)^{\frac{1}{T}} - 1$, where T is the length of the sample in years.

	SPX		VIX		VXX	
	R_D	R_C	R_D	R_C	R_D	R_C
Mean	0.08%	0.07%	0.15%	−0.09%	−0.32%	−0.39%
significance p-value	(0.0121)	(0.0217)	(0.4323)	(0.6226)	(0.0020)	(0.0001)
Standard Deviation (σ)	1.13%	1.13%	7.10%	6.86%	3.81%	3.78%
Annualised σ	18.00%	18.00%	112.70%	108.97%	60.55%	60.06%
Skew	−0.1280	−0.2377	1.3281	0.7659	0.7811	0.5213
significance p-value	(0.0596)	(0.0006)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
Excess Kurtosis	4.5210	4.4989	5.6978	3.8815	3.2867	2.8287
significance p-value	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
Holding Period Return	164.48%	97.26%	−71.66%	−126.09%	−99.55%	−540.82%
CAGR	19.25%	–	−20.41%	–	−62.43%	–

Correlations		R_D			R_C	
	SPY	VIX	VXX	SPY	VIX	VXX
SPY	1	−0.7659	−0.7834	1	−0.7710	−0.7846
significance p-value		(0.0000)	(0.0000)		(0.0000)	(0.0000)
VIX	–	1	0.8660	–	1	0.8651
significance p-value			(0.0000)			(0.0000)
VXX	–	–	1	–	–	1

Table 2: **30-day VIX futures price estimation.** This table shows the different VIX futures price estimates using four different formulae and range of parameter estimates for V_t, σ_V and κ^*. For this exercise we keep the time to maturity constant at 30 days, $\tau = \tau_0 = \frac{30}{365}$ and θ^* constant at $\theta^* = 0.10$. The first four columns show the hypothetical θ^*, V_t, σ_V and κ^* parameters used in the futures price estimates. The first column of estimated futures prices, labelled by F_0, uses the simple approximation for VIX futures prices, the F_0 part of equation (13). The next column of VIX futures prices, labelled by $F_0 + F_1$, uses the simple formula of VIX futures prices and the first half of the convexity adjustment, $F_0 + F_1$ from equation (13). The $F_0 + F_1 + F_2$ column of VIX futures prices uses the full approximation formula, equation (13), from Zhang, Shu, and Brenner (2010). The final column of VIX futures prices uses the accurate formula, equation (11), from Zhang and Zhu (2006). The columns labelled % error, are the percentage difference of the preceding column of prices from the prices estimated by the accurate formula.

Parameters				VIX Futures Price estimates						
θ^*	V_t	κ^*	σ_V	F_0	% error	$F_0 + F_1$	% error	$F_0 + F_1 + F_2$	% error	Accurate F
0.1	0.04	4	0.1	25.14	0.07%	25.12	0.00%	25.12	0.00%	25.12
0.1	0.04	4	0.4	25.14	1.08%	24.86	−0.02%	24.89	0.09%	24.87
0.1	0.04	4	0.7	25.14	3.20%	24.30	−0.25%	24.56	0.82%	24.36
0.1	0.04	5.5	0.1	26.32	0.05%	26.31	0.00%	26.31	0.00%	26.31
0.1	0.04	5.5	0.4	26.32	0.77%	26.12	−0.02%	26.13	0.05%	26.12
0.1	0.04	5.5	0.7	26.32	2.27%	25.69	−0.17%	25.85	0.42%	25.74
0.1	0.04	7	0.1	27.26	0.04%	27.25	0.00%	27.25	0.00%	27.25
0.1	0.04	7	0.4	27.26	0.57%	27.11	−0.01%	27.12	0.03%	27.11
0.1	0.04	7	0.7	27.26	1.69%	26.78	−0.12%	26.88	0.24%	26.81
0.1	0.12	4	0.1	33.51	0.05%	33.49	0.00%	33.49	0.00%	33.49
0.1	0.12	4	0.4	33.51	0.82%	33.24	0.00%	33.25	0.05%	33.24
0.1	0.12	4	0.7	33.51	2.53%	32.67	−0.02%	32.83	0.46%	32.68
0.1	0.12	5.5	0.1	33.20	0.04%	33.19	0.00%	33.19	0.00%	33.19
0.1	0.12	5.5	0.4	33.20	0.67%	32.98	0.00%	32.99	0.03%	32.98
0.1	0.12	5.5	0.7	33.20	2.04%	32.53	−0.04%	32.64	0.31%	32.54
0.1	0.12	7	0.1	32.95	0.03%	32.94	0.00%	32.94	0.00%	32.94
0.1	0.12	7	0.4	32.95	0.55%	32.77	0.00%	32.78	0.02%	32.77
0.1	0.12	7	0.7	32.95	1.66%	32.39	−0.04%	32.48	0.21%	32.41
0.1	0.2	4	0.1	40.17	0.04%	40.15	0.00%	40.15	0.00%	40.15
0.1	0.2	4	0.4	40.17	0.62%	39.92	0.00%	39.93	0.03%	39.92
0.1	0.2	4	0.7	40.17	1.94%	39.41	0.01%	39.51	0.27%	39.40
0.1	0.2	5.5	0.1	38.88	0.03%	38.87	0.00%	38.87	0.00%	38.87
0.1	0.2	5.5	0.4	38.88	0.55%	38.67	0.00%	38.68	0.02%	38.67
0.1	0.2	5.5	0.7	38.88	1.69%	38.23	0.00%	38.32	0.21%	38.24
0.1	0.2	7	0.1	37.79	0.03%	37.77	0.00%	37.77	0.00%	37.77
0.1	0.2	7	0.4	37.79	0.48%	37.61	0.00%	37.61	0.02%	37.61
0.1	0.2	7	0.7	37.79	1.47%	37.23	−0.01%	37.30	0.16%	37.24

Table 3: **$\boldsymbol{\lambda}$ and $\boldsymbol{\kappa^*}$ estimates by various authors.** This table shows the estimated value of λ, the market price of variance risk, and κ^*, the risk neutral speed of mean reversion of variance, from different authors using different sample periods and estimation methods.

Author	Data period	κ^*	λ
Lin (2007)	21 Apr 2004 - 18 Apr 2006	5.3500	-0.3528
Duan and Yeh (2010)	2 Jan 2001 - 29 Dec 2006	-1.7956	-7.5697
Zhang and Huang (2010)	18 May 2004 - 17 Aug 2007	1.2989	-19.1184
Luo and Zhang (2012)	2 Jan 1992 - 31 Aug 2009	5.4642	†
Our estimation	30 Jan 2009 - 27 Jun 2014	5.4642	-6.0211‡

† Luo and Zhang (2012) do not give the estimate for lambda, but their article is important here as we use their κ^* estimate.
‡ We use the $\kappa^* = 5.4642$ estimate from Luo and Zhang (2012) and assume that it is accurate for our sample period.

Figure 1: **Understanding the SPVXSTR Roll Period.** This diagram shows how *dr* and *dt* are determined for the calculation of the weights in each VIX futures contract of the SPVXSTR. T_i is the settlement date of the i$^{\text{th}}$ nearest maturing VIX futures, which is 30 days before S&P 500 options maturity date (3rd Friday of every month) and is usually on a Wednesday. $T_i - 1$ is the day before i$^{\text{th}}$ nearest maturing VIX futures settlement and the last day of the roll period. On the last day of the roll period the nearest settling VIX futures is eliminated and the second nearest settling VIX futures becomes the nearest. The *dr* and *dt* are the factors used in the calculation of the weights of each of the VIX futures contracts in the SPVXSTR, as shown in section 2. The roll period represents the time during which the weight in the nearest settling VIX futures contract is gradually replaced by a position in the second nearest VIX futures contract. At the end of the roll period all the weight will be in the second nearest VIX futures contract which then becomes the nearest as the old nearest matures, then the next roll period starts, and the process is repeated.

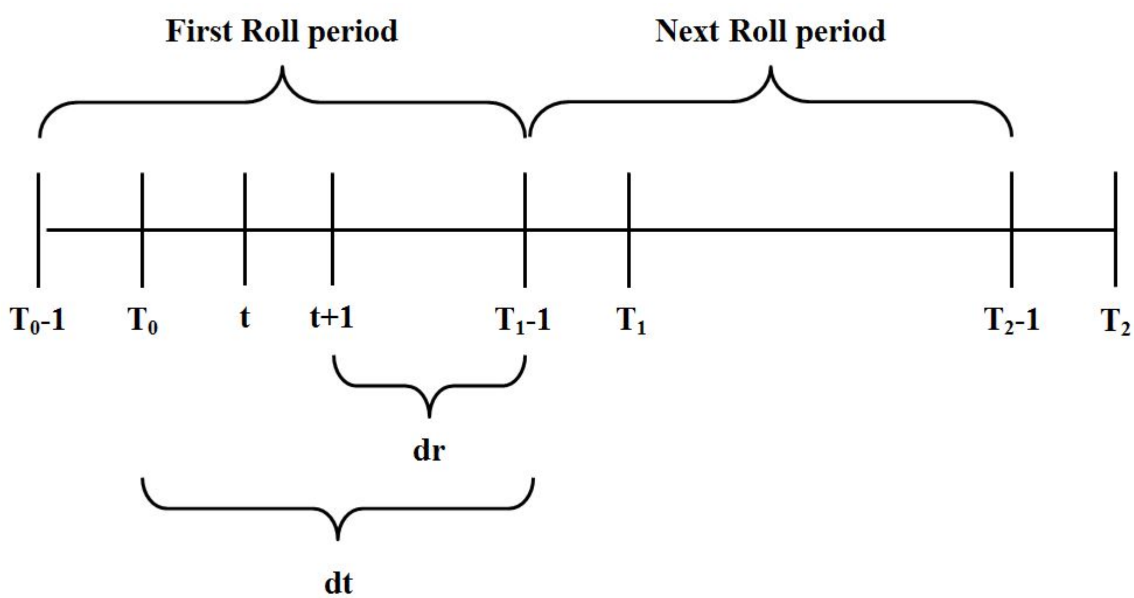

Figure 2: **Historical VIX, 30-day VIX futures price and VXX price.** This figure shows the level of the VIX and the price of 30-day VIX futures on the primary vertical axis and the VXX price on the secondary vertical axis. The 30-day VIX futures contract is the linearly interpolated price of a constant 30-day maturity VIX futures contract, as in Zhang, Shu, and Brenner (2010).

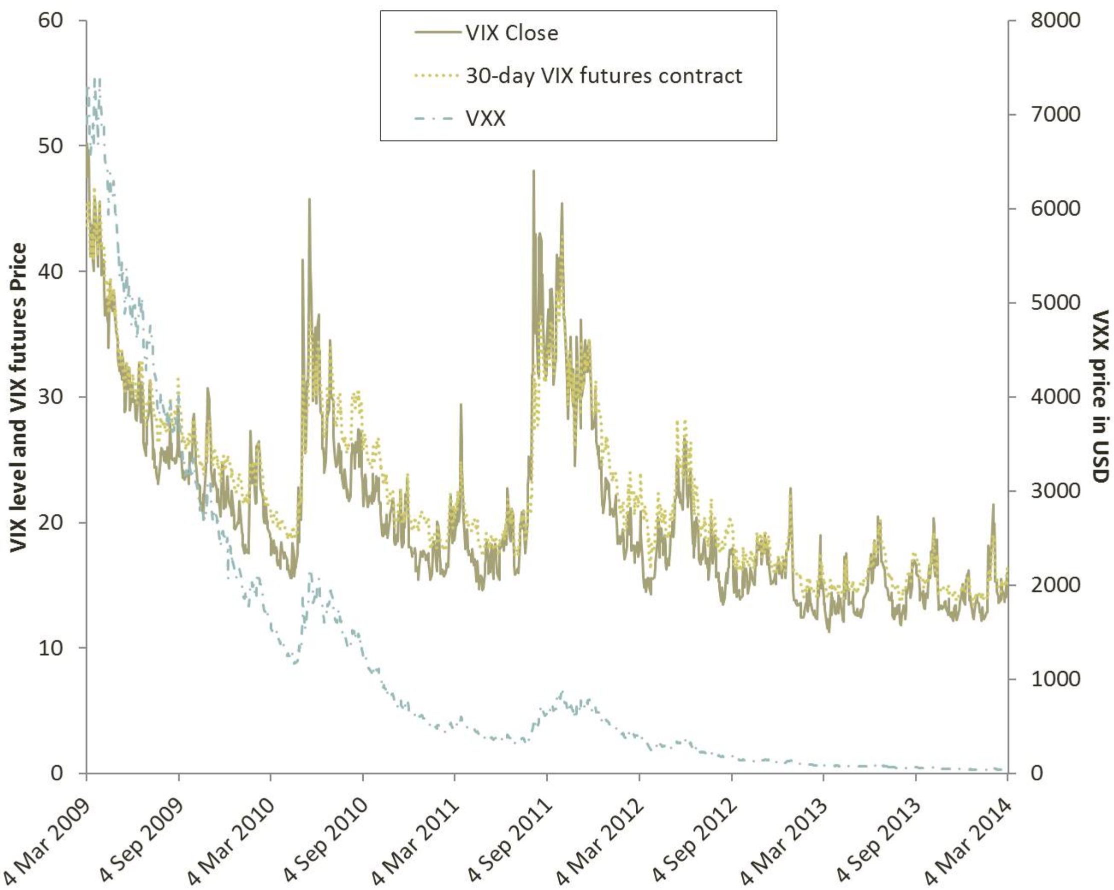

Figure 3: **Market Capitalization and Trading Value of VXX.** This figure shows the daily dollar trading volume and market capitalization of the VXX from the 30[th] January 2009 to the 27[th] June 2014 in billion US dollars.

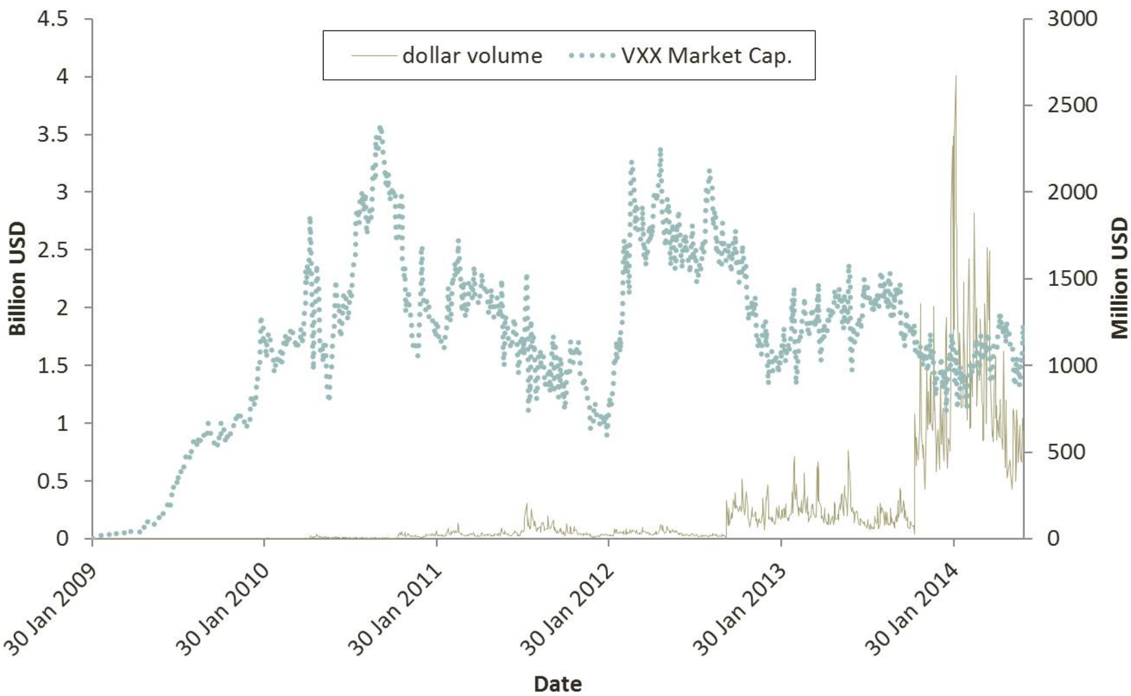

Figure 4: **VIX Term Structure.** This figure shows the term structure of VIX futures prices from 1 day to 50 day maturity calculated using our simple approximation ,F_0, the approximation with the first part of the convexity adjustment, $F_0 + F_1$ and the full approximation from Zhang, Shu, and Brenner (2010) , $F_0 + F_1 + F_2$. These estimated VIX futures prices are calculated using constant parameter estimates of $\theta^* = 0.1$, $\kappa^* = 5$, $\sigma_V = 0.1425$ and $V_t = 0.06$ but the time to maturity varies from 1 day to 50 days.

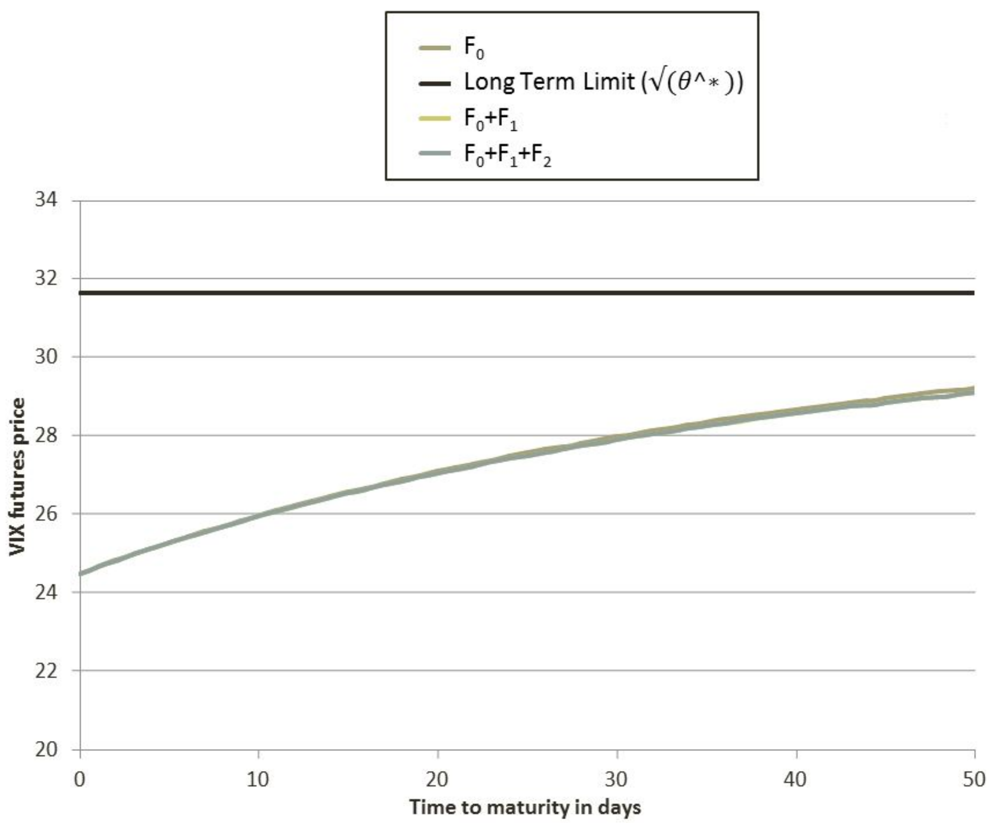

Figure 5: **Replicated vs. Actual SPVXSTR.** This figure shows the actual SPVXSTR time series and our replicated SPVXSTR time series using the methodology from S&P Dow Jones Indices (2012) from the 20th December 2005 until the 28th March 2014.

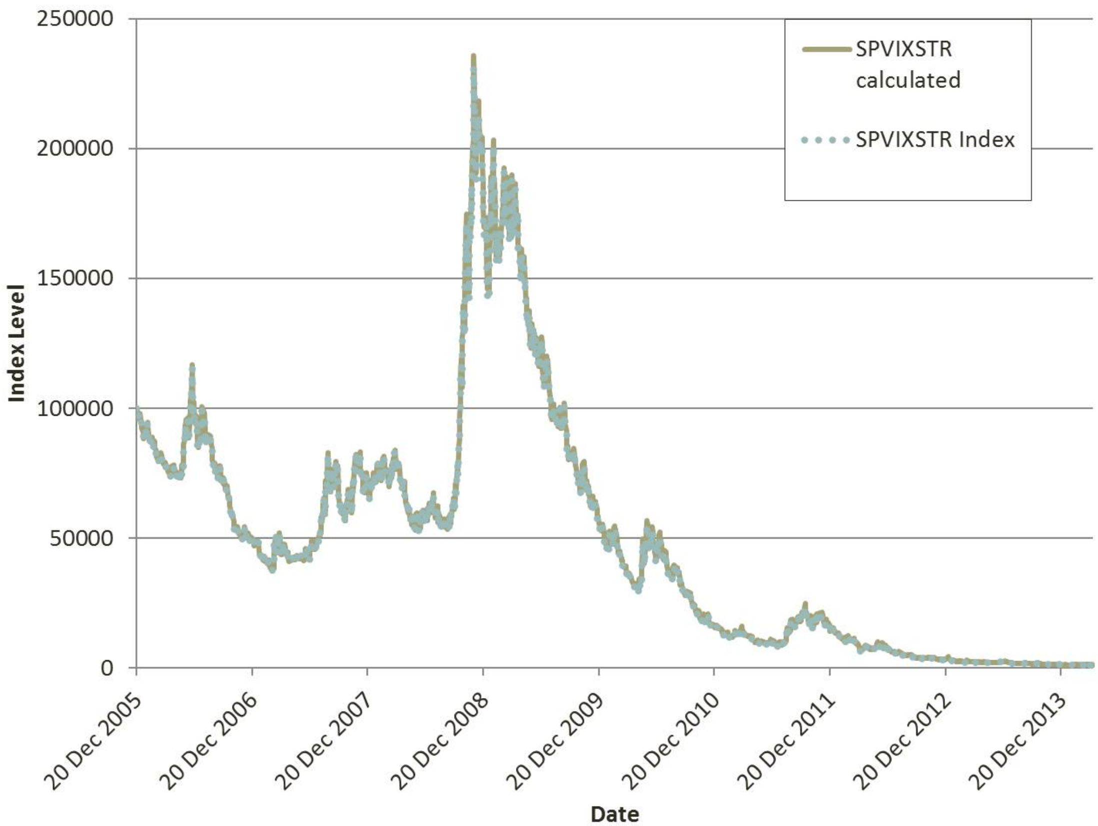

Figure 6: **Replicated SPVXSTR, different Rebalancing frequencies.** This figure shows four different time series of our replication of SPVXSTR. SPVXSTR daily corresponds to daily, SPVXSTR weekly to weekly, SPVXSTR bi-weekly to two weekly and SPVXSTR monthly to monthly rebalancing. The final values of the indices are 1178.63 for daily, 1088.38 for weekly, 842.24 for bi-weekly and 264.60 for monthly rebalancing.

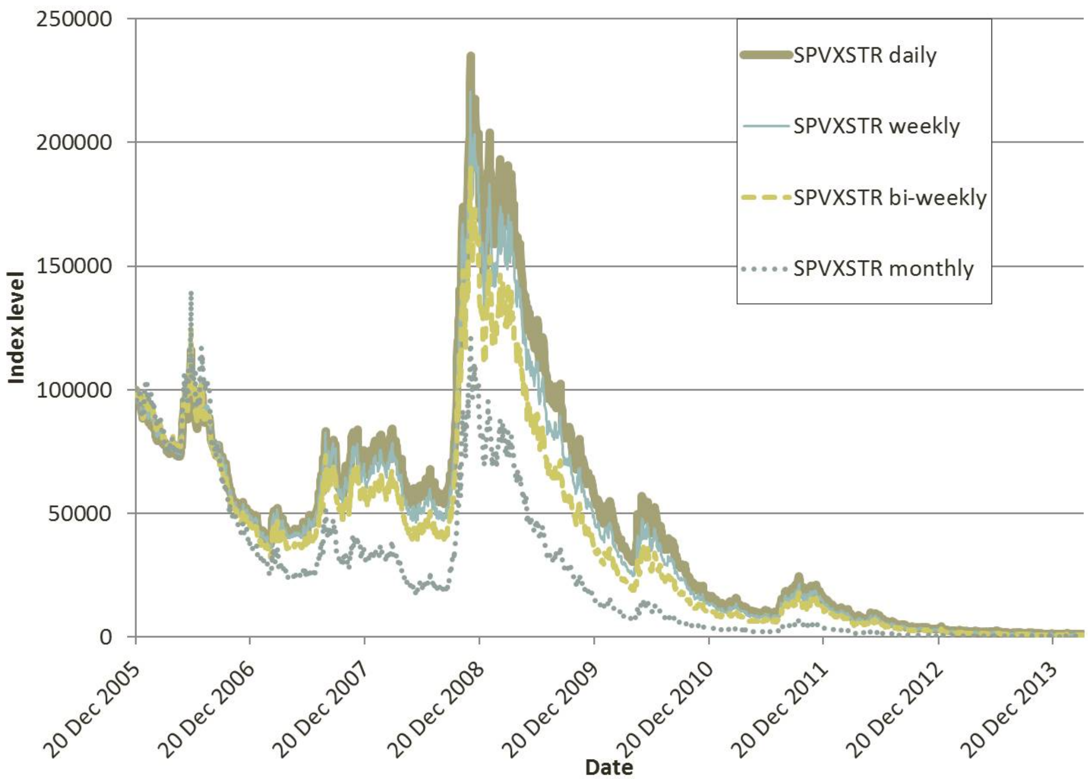

Figure 7: **Simulated index using physical process for V_t.** This figure shows the simulated SPVXSTR index over our 4 year simulation period using $V_0 = 0.02$, $\sigma_V = 0.4$, the risk-neutral parameter estimates $\kappa^* = 5.4642$ and $\theta^* = 0.1$, the physical process of dV_t as described in equation (30) and the simple VIX futures price formula, F_0, from equation eqrefapproxVIXfuture from Zhang, Shu, and Brenner (2010). The label of each time series corresponds to the rebalancing frequency used.

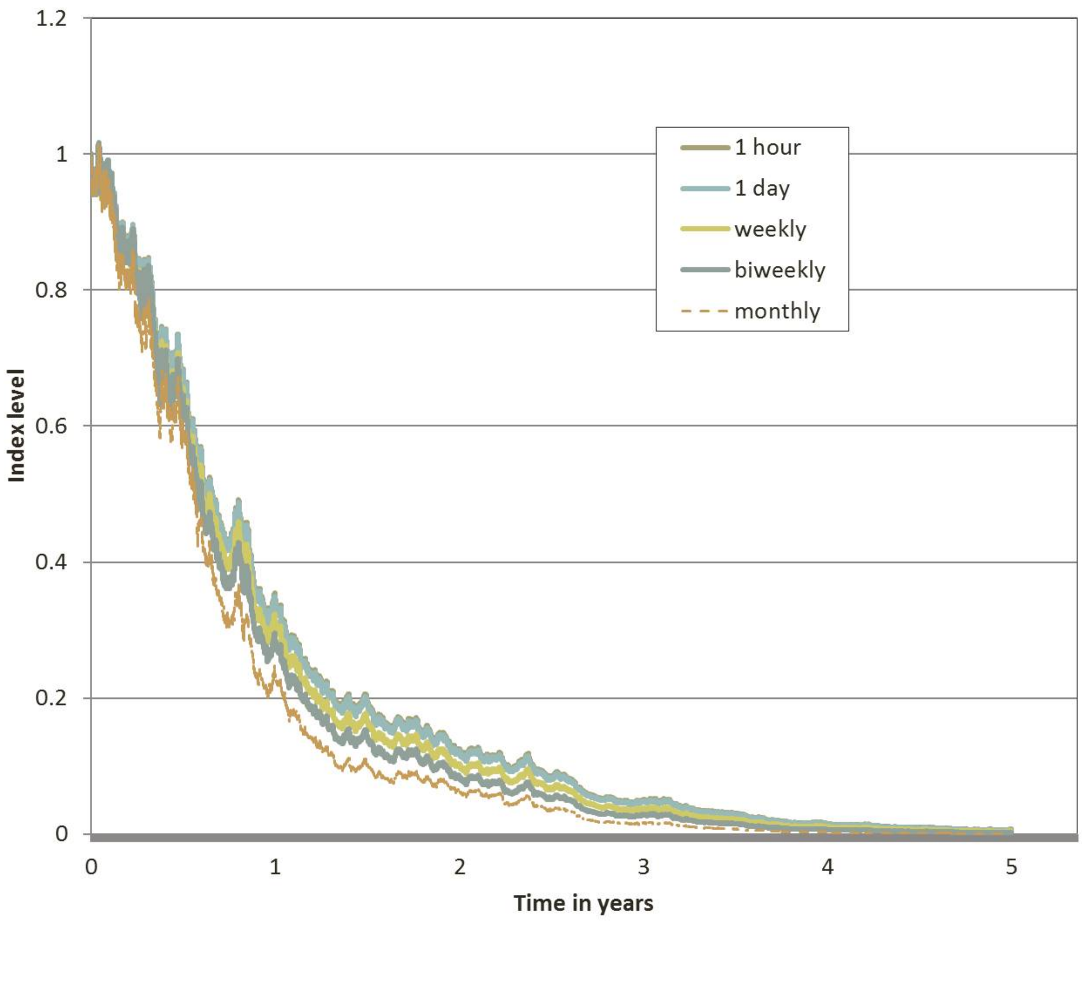

Figure 8: **Simulated index using $\boldsymbol{V_t = \theta < \theta^*}$.** This figure shows the time series of the simulated SPVXSTR, when the instantaneous variance is set constant at $V_t = \theta = 0.0476 < \theta^* = 0.1$ forcing a upward sloping VIX futures term structure. To calculate the futures prices we use the volatility of volatility $\sigma_V = 0.4$ and the risk-neutral parameter estimates $\kappa^* = 5.4642$ and $\theta^* = 0.1$ are used. The label of each time series corresponds to the rebalancing frequency used.

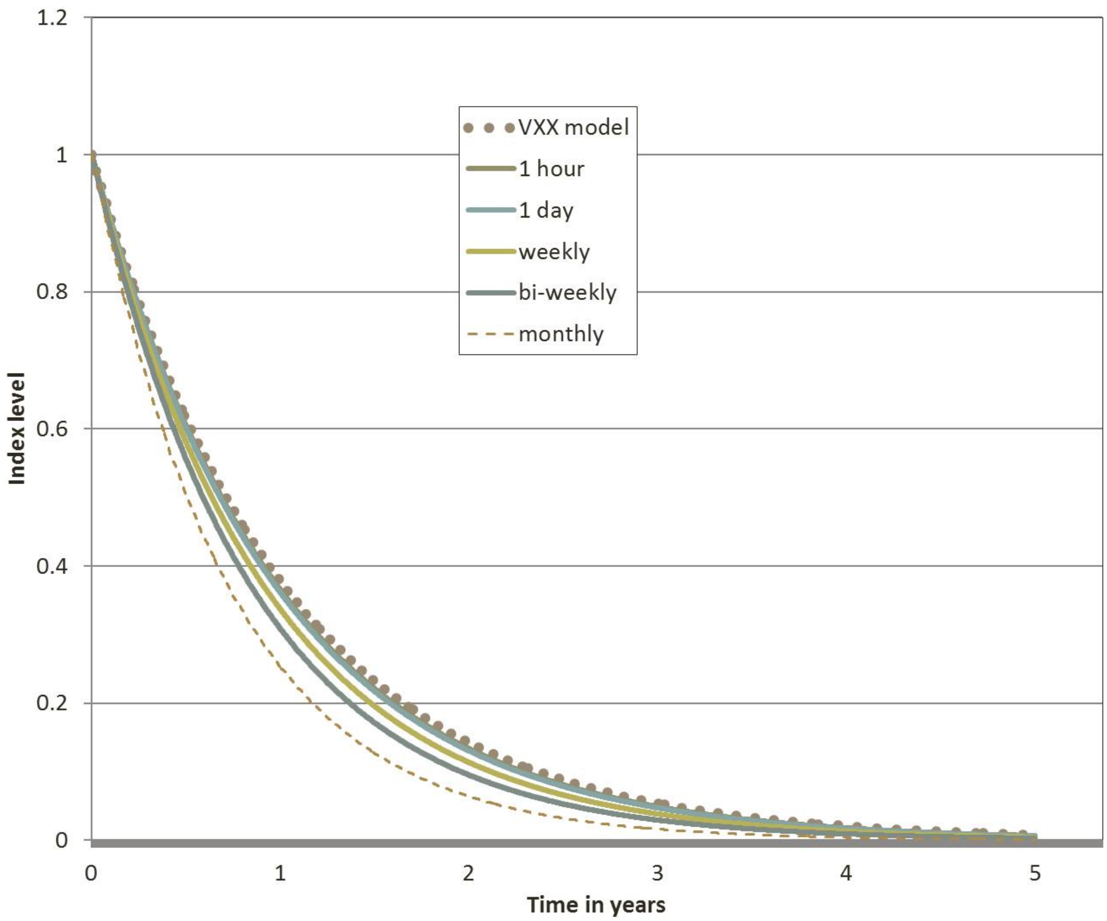

Figure 9: **Simulated index using $V_t = 0.14 > \theta^*$.** This Figure shows the time series of the simulated SPVXSTR, when V_t is set constant at 0.14 which is higher than $\theta^* = 0.1$ forcing a downward sloping VIX futures term structure. To calculate the futures prices we use the volatility of volatility $\sigma_V = 0.4$ and the risk-neutral parameter estimates $\kappa^* = 5.4642$ and $\theta^* = 0.1$ are used. The label of each time series corresponds to the rebalancing frequency used.