Robust replication of barrier-style claims on price and volatility

Peter Carr [*] Matthew Lorig [†]

This version: August 3, 2015

Abstract

We show how to price and replicate a variety of barrier-style claims written on the log price X and quadratic variation $\langle X \rangle$ of a risky asset. Our framework assumes no arbitrage, frictionless markets and zero interest rates. We model the risky asset as a strictly positive continuous semimartingale with an independent volatility process. The volatility process may exhibit jumps and may be non-Markovian. As hedging instruments, we use only the underlying risky asset, a zero-coupon bond, and European calls and puts with the same maturity as the barrier-style claim. We consider both single-barrier and double barrier claims in three varieties: knock-in, knock-out and rebate.

Key words: robust pricing, robust hedging, knock-in, knock-out, rebate, barrier, quadratic variation

1 Introduction

Barrier options are the most liquid of the second generation options, (i.e., options whose payoffs are path-dependent). In his landmark work, [Mer73] first valued a down-and-out call in closed form when the underlying stock follows geometric Brownian motion. So long as the instantaneous volatility is a known function of the stock price and time, one can create any barrier claim by dynamic trading in the stock and a risk-free asset. If the volatility process is a continuous stochastic process driven by a second independent source of uncertainty, then one must also dynamically trade an option As with any hedge, the hedging strategy is invariant to the expected rate of return of the underlying stock.

[BC94] show how a static hedge in European options can be used to hedge down-and-out calls on futures in the Black model. Essentially, the payoff of a down-and-out call with barrier H can be replicated by buying a European call on the same underlying futures price with the same maturity T and strike K and also selling K/H puts with strike H^2/K. [CEG98] make clear that this static hedge works in any model with deterministic local volatility, provided that the volatility function is symmetric in the log of the futures price relative to the barrier. We thus see a continuation of the pattern initiated by Merton in which hedging strategies are invariant to aspects of the statistical process.

[*]Courant Institute, New York University, 251 Mercer Street, New York, NY 10012, USA. **e-mail**: pcarr@nyc.rr.com
[†]Department of Applied Mathematics, University of Washington, Lewis Hall #202, Box 353925, Seattle, WA 98195, USA. **e-mail**: mlorig@uw.edu

Electronic copy available at: http://ssrn.com/abstract=2639454

In [CEG98], increments in the instantaneous volatility process are conditionally perfectly correlated with increments in the underlying futures price. [And01] points out that the above hedge also works when increments in the instantaneous volatility process are conditionally independent of increments in the forward price. Similarly, [Bat97] observes that the above hedge works for "Hull and White-type stochastic volatility processes." While Bates does not define this terminology, it seems reasonable to assert that both authors are assuming that the instantaneous volatility process is a diffusion, i.e., that the volatility process is continuous over time and has the strong Markov property, as in [HW87]. Furthermore, as in [HW87], the volatility process should be autonomous in that its evolution coefficients refer only to volatility and time, but not the price of the underlying asset.

[CL09] make clear that these conditions are merely sufficient, but not necessary. The hedge described above for a down-and-out call works perfectly provided that there are no jumps over the barrier and the call and put have the same implied volatility at the first passage time to the barrier, if any. We refer to the latter condition as Put Call Symmetry (PCS), which was introduced to finance by [Bat88] as a way to measure skewness. Hence, the bivariate process for the futures price and its volatility need not be Markov in itself. Furthermore, jumps in price and volatility can occur and increments in volatility can be correlated with returns, although some restrictions are necessary. As a result, we refer to these hedge strategies as *semi-robust*.

Barrier options are not the only path-dependent claims for which a semi-robust hedge exists. All of the barrier option hedges also extend to lookbacks. For lookbacks, the hedge is semi-static in that standard options are traded each time a new maximum is reached. Furthermore, assuming only no arbitrage, frictionless markets, zero interest rates, a positive continuous futures price process, and an independent volatility process, [CL08] show how to replicate any claim on the quadratic variation of returns experienced between initiation and a fixed maturity date. Their hedging instruments consist of the underlying futures and European options written on these futures at all strikes and with the same maturity as the claim. In contrast to the hedges of barrier and lookback claims, their trading strategy in options is fully dynamic. Examples of claims whose payoffs can be spanned include volatility swaps and options on realized variance.

The purpose of this manuscript is to synthesize the literature on semi-robust hedging of barrier claims and claims linked to quadratic variation. In particular, we show how to price and hedge any claim on the log price X and the quadratic variation $\langle X \rangle$ of a risky asset subject to certain barrier events either occurring or not occurring. Examples of such claims include (i) a barrier start variance or volatility swap for which the nonnegative payoff is the variance or volatility of log price experienced between the first passage time and a fixed maturity date, (ii) a barrier start claim whose final payoff is the Sharpe ratio calculated between the first passage time and the fixed maturity date, and (iii) a barrier style claim on a levered exchange-traded fund (LETF). The value of an LETF depends both on the value of the underlying ETF and on the realized quadratic variation of the ETF's log returns. Hence, standard hedges for barrier options cannot be applied to options on LETFs. Depending on the type of barrier claim considered, the hedges we construct may be static, semi-static or dynamic.

Our analysis makes the same assumptions as in [CL08]. In particular, we consider a continuous time stochastic process for instantaneous volatility whose increments are uncorrelated with returns. Jumps in

Electronic copy available at: http://ssrn.com/abstract=2639454

the volatility process are allowed and the evolution coefficients of the volatility process can refer to past or present values of the instantaneous volatility, time, and other variables as well, provided that they are independent of the futures price (i.e., non-Markovian dynamics are allowed for the volatility process). Both foreign exchange and bond markets exhibit symmetric smiles, which, in a stochastic volatility setting, implies a volatility process that is uncorrelated with returns of the underlying [CL09, Theorem 3.4]. Thus, our results are particularly relevant for these markets.

The rest of this paper proceeds as follows. In Section 2, we introduce a general market model for a single risky asset $S = \mathrm{e}^X$ and state our main assumptions. In the subsequent sections, we show how to price and hedge a variety of claims on X and its quadratic variation $\langle X \rangle$. In Section 3 we examine European-style options on $(X_T, \langle X \rangle_T)$. Section 4 focuses on single barrier options of three varieties: knock-in (Section 4.1), knock-out (Section 4.2) and rebate (Section 4.3). Section 5 examines double barrier options of three varieties: knock-out (Section 5.1), knock-in (Section 5.2) and rebate (Section 5.3). In Section 6 we provide examples of barrier-style claims written on a levered portfolio and on the realized Sharpe ratio of an asset. We also consider buy at-first-touch investment strategies and show how claims that replicate these strategies can be priced and hedged. Concluding remarks and directions for future research are offered in Section 7.

2 Model and assumptions

We consider a frictionless market (i.e., no transaction costs) and fix an arbitrary but finite time horizon $T < \infty$. For simplicity, we assume zero interest rates, no arbitrage, and take as given an equivalent martingale measure (EMM) $\mathbb{P}$ chosen by the market on a complete filtered probability space $(\Omega, \mathcal{H}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$. The filtration $\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ represents the history of the market. All stochastic processes defined below live on this probability space and all expectations are with respect to $\mathbb{P}$ unless otherwise stated.

Let $B = (B_t)_{0 \leq t \leq t}$ represent the value of a zero-coupon bond maturing at time T. Since the risk-free rate of interest is zero, we have $B_t = 1$ for all $t \in [0, T]$. Let $S = (S_t)_{0 \leq t \leq T}$ represent the value of a risky asset. We assume S is strictly positive and has continuous sample paths. To rule out arbitrage, it is well-known that the asset S must be a martingale under the pricing measure $\mathbb{P}$. As such, there exists a non-negative, $\mathcal{F}$-adapted stochastic process $\sigma = (\sigma_t)_{0 \leq t \leq T}$ such that

$$\mathrm{d}S_t = \sigma_t S_t \mathrm{d}W_t, \qquad\qquad S_0 > 0,$$

where W is a Brownian motion with respect to the pricing measure $\mathbb{P}$ and the filtration $\mathcal{F}$. Henceforth, the process σ will be referred to as the *volatility process*. We assume that the volatility process σ is a semimartingale, that it evolves independently of W and that it satisfies

$$\int_0^T \sigma_t^2 \mathrm{d}t < c < \infty, \tag{2.1}$$

for some arbitrarily large but finite constant $c > 0$. Note that σ may experience jumps and is not required to be Markovian.

It will be convent to introduce $X = (X_t)_{0 \leq t \leq T}$, the log *returns process*

$$X_t = \log S_t.$$

Since S is strictly positive by assumption, the process X is well-defined and finite for all $t \in [0, T]$. A simple application of Itô's Lemma yields

$$\mathrm{d}X_t = -\tfrac{1}{2}\sigma_t^2 \mathrm{d}t + \sigma_t \mathrm{d}W_t, \qquad X_0 = \log S_0. \tag{2.2}$$

Note that a claim on (the path of) S can always be expressed as a claim on (the path of) $X = \log S$.

Let $C_t(K)$ denote the time t price of a European call written on S with maturity date T and strike price $K > 0$, and let $P_t(K)$ denote the price of a European put written on S with the same strike and maturity. By no-arbitrage arguments, we have

$$C_t(K) = \mathbb{E}[(S_T - K)^+ | \mathcal{F}_t] = \mathbb{E}[(\mathrm{e}^{X_T} - K)^+ | \mathcal{F}_t],$$
$$P_t(K) = \mathbb{E}[(K - S_T)^+ | \mathcal{F}_t] = \mathbb{E}[(K - \mathrm{e}^{X_T})^+ | \mathcal{F}_t].$$

For convenience, we will sometimes refer to a European call or put written on X rather than S with the understanding that these are equivalent. We assume that a European call or put with maturity T trades at every strike $K \in (0, \infty)$. As demonstrated by [BL78], this assumption is equivalent to knowing the density of X_T under $\mathbb{P}$. Additionally, the assumption guarantees that any T-maturity European-style claim on X_T can perfectly hedged with a static portfolio of the bond B, the underlying S and calls and puts [CM98]. Although in reality, calls and puts trade at only finitely many strikes, our results retain relevance; [LL15] show how to adjust static hedges optimally when calls and puts are traded at only discrete strikes in a finite interval.

3 European-style claims

Under the assumptions of Section 2, [CL08] show how to price and replicate a claim with a payoff of the form $\varphi(\langle X \rangle_T)$ by dynamically trading the bond B, the underlying S and T-maturity European options written on X. In this section, we extend their results by showing how to price and replicate a claim with a payoff of the form $\varphi(X_T, \langle X \rangle_T)$. We call such a claim *European-style*, since its payoff depends only on the values of X_T and $\langle X \rangle_T$. Throughout this paper, we will denote by $E = (E_t)_{0 \le t \le T}$ the price process of any European-style claim (E stands for "European-style"). Using risk-neutral pricing, we have

$$E_t = \mathbb{E}[\varphi(X_T, \langle X \rangle_T) | \mathcal{F}_t], \qquad t \in [0, T]. \tag{3.1}$$

Although the term "European-style" suggests that E has a path-independent payoff, it is clear that E has a payoff $\varphi(X_T, \langle X \rangle_T)$ that is path-*dependent*, as $\langle X \rangle_T$ depends on the entire path of X.

We begin our analysis of the European-style claim (3.1) with a proposition that relates the joint characteristic function of $(X_T - X_t, \langle X \rangle_T - \langle X \rangle_t) | \mathcal{F}_t$ to the characteristic function of $(X_T - X_t) | \mathcal{F}_t$.

Proposition 3.1. *Let $\omega, s \in \mathbb{C}$. Define $u : \mathbb{C}^2 \to \mathbb{C}$ as either of the following*

$$u \equiv u(\omega, s) = \mathtt{i} \left(-\tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} - \omega^2 - \mathtt{i}\omega + 2\mathtt{i}s} \right). \tag{3.2}$$

Then we have

$$\mathbb{E}[\mathrm{e}^{\mathtt{i}\omega(X_T - X_t) + \mathtt{i}s(\langle X \rangle_T - \langle X \rangle_t)} | \mathcal{F}_t] = \mathbb{E}[\mathrm{e}^{\mathtt{i}u(X_T - X_t)} | \mathcal{F}_t]. \tag{3.3}$$

Proof. Let $\mathcal{F}^\sigma_{t,T}$ denote the sigma-algebra generated by $(\sigma_s)_{t\le s\le T}$. Then $(\langle X\rangle_T - \langle X\rangle_t) \in \mathcal{F}^\sigma_{t,T}$ and

$$X_T - X_t|\mathcal{F}^\sigma_{t,T} \sim \mathcal{N}(m, v^2), \qquad m = -\tfrac{1}{2}(\langle X\rangle_T - \langle X\rangle_t), \qquad v^2 = \langle X\rangle_T - \langle X\rangle_t. \tag{3.4}$$

We recall the characteristic function of a normal random variable

$$Z \sim \mathcal{N}(m, v^2), \qquad \mathbb{E}[e^{i\omega Z}] = e^{im\omega - \frac{1}{2}v^2\omega^2}. \tag{3.5}$$

With the above in mind, we compute

$$\begin{aligned}
\mathbb{E}[e^{i\omega(X_T-X_t)+is(\langle X\rangle_T-\langle X\rangle_t)}|\mathcal{F}_t] &= \mathbb{E}[e^{is(\langle X\rangle_T-\langle X\rangle_t)}\mathbb{E}[e^{i\omega(X_T-X_t)}|\mathcal{F}_t \vee \mathcal{F}^\sigma_{t,T}]|\mathcal{F}_t] \\
&= \mathbb{E}[\mathbb{E}[e^{(is-(\omega^2+i\omega)/2)(\langle X\rangle_T-\langle X\rangle_t)}|\mathcal{F}_t \vee \mathcal{F}^\sigma_{t,T}]|\mathcal{F}_t] && \text{(by (3.4) and (3.5))} \\
&= \mathbb{E}[\mathbb{E}[e^{(-(u^2+iu)/2)(\langle X\rangle_T-\langle X\rangle_t)}|\mathcal{F}_t \vee \mathcal{F}^\sigma_{t,T}]|\mathcal{F}_t] && \text{(by (3.2))} \\
&= \mathbb{E}[\mathbb{E}[e^{iu(X_T-X_t)}|\mathcal{F}_t \vee \mathcal{F}^\sigma_{t,T}]|\mathcal{F}_t] && \text{(by (3.4) and (3.5))} \\
&= \mathbb{E}[e^{iu(X_T-X_t)}|\mathcal{F}_t],
\end{aligned}$$

which establishes (3.3). □

The importance of Proposition 3.1 is that it allows one to relate the value a claim on $(X_T, \langle X\rangle_T)$ to the value of a claim on X_T only. To see this, let us define the *generalized Fourier transform* $\mathbf{F}$ and *inverse transform* $\mathbf{F}^{-1}$

$$\text{Fourier Transform}: \quad \mathbf{F}[\varphi](\omega, s) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} dx dv \varphi(x, v)e^{-i\omega x - isv}, \qquad \omega, s \in \mathbb{C}, \tag{3.6}$$

$$\text{Inverse Transform}: \quad \mathbf{F}^{-1}[\widehat{\varphi}](x, v) = \int_{\mathbb{R}^2} d\omega_r ds_r \widehat{\varphi}(\omega, s)e^{i\omega x + isv}, \qquad (\omega_r, s_r) = (\mathrm{Re}(\omega), \mathrm{Re}(s)). \tag{3.7}$$

Assumption 3.2. For any payoff function $\varphi : \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}$, we assume the following holds

$$\mathbf{F}^{-1}[\widehat{\varphi}] = \varphi, \quad \text{a.e.}, \qquad \widehat{\varphi} = \mathbf{F}[\varphi]. \tag{3.8}$$

Expression (3.8) will hold if φ has a generalized Fourier transform $\widehat{\varphi}$, which satisfies $\widehat{\varphi} \in L^1$. We note, in particular, that call and put payoffs satisfy this condition.

Using equation (3.8) and Proposition 3.1, we can now relate the price of a claim on $(X_T, \langle X\rangle_T)$ to the price of a claim on X_T only.

Theorem 3.3 (Price of a European-style claim)**.** *Let E_t by given by (3.1) and let φ satisfy Assumption 3.2. Then*

$$E_t = \mathbb{E}[\chi_t(X_T)|\mathcal{F}_t], \tag{3.9}$$

where χ_t is an $\mathcal{F}_t$-measurable function given by

$$\chi_t(x) = \int_{\mathbb{R}^2} d\omega_r ds_r \widehat{\varphi}(\omega, s)e^{i(\omega - u(\omega,s))X_t + is\langle X\rangle_t}e^{iu(\omega,s)x}. \tag{3.10}$$

with $u(\omega, s)$ as defined in (3.2).

Proof. The proof is by computation. We have

$$
\begin{aligned}
E_t &= \mathbb{E}[\varphi(X_T, \langle X\rangle_T)|\mathcal{F}_t] && \text{(by (3.1))}\\
&= \mathbb{E}[\mathbf{F}^{-1}[\widehat{\varphi}](X_T, \langle X\rangle_T)|\mathcal{F}_t] && \text{(by (3.8))}\\
&= \int_{\mathbb{R}^2} \mathrm{d}\omega_r \mathrm{d}s_r \widehat{\varphi}(\omega, s)\mathbb{E}[\mathrm{e}^{\mathtt{i}\omega X_T + \mathtt{i}s\langle X\rangle_T}|\mathcal{F}_t] && \text{(by (3.7))}\\
&= \int_{\mathbb{R}^2} \mathrm{d}\omega_r \mathrm{d}s_r \widehat{\varphi}(\omega, s)\mathrm{e}^{\mathtt{i}\omega X_t + \mathtt{i}s\langle X\rangle_t}\mathbb{E}[\mathrm{e}^{\mathtt{i}\omega(X_T - X_t) + \mathtt{i}s(\langle X\rangle_T - \langle X\rangle_t)}|\mathcal{F}_t] && (X_t, \langle X\rangle_t \in \mathcal{F}_t)\\
&= \int_{\mathbb{R}^2} \mathrm{d}\omega_r \mathrm{d}s_r \widehat{\varphi}(\omega, s)\mathrm{e}^{\mathtt{i}(\omega - u(\omega,s))X_t + \mathtt{i}s\langle X\rangle_t}\mathbb{E}[\mathrm{e}^{\mathtt{i}u(\omega,s)X_T}|\mathcal{F}_t]. && \text{(by (3.3))} \qquad (3.11)\\
&= \mathbb{E}[\chi_t(X_T)|\mathcal{F}_t], && \text{(by (3.10))}
\end{aligned}
$$

which establishes (3.9). ☐

Next, we recall a classical result from [CM98]. Suppose a function f can be expressed as the difference of convex functions. Then f can be represented as a linear combination of call and put payoffs. Specifically, for any $\kappa \in \mathbb{R}^+$ we have

$$f(s) = f(\kappa) + f'(\kappa)\Big((s-\kappa)^+ - (\kappa - s)^+\Big) + \int_0^\kappa \mathrm{d}K f''(K)(K-s)^+ + \int_\kappa^\infty \mathrm{d}K f''(K)(s-K)^+. \quad (3.12)$$

Here, f' is the left-derivative of f and f'' is the second derivative, which exists as a generalized function. Replacing s in (3.12) with the random variable S_T, setting $\kappa = S_t$, taking a conditional expectation, and recalling that $B_t = 1$, one obtains

$$\mathbb{E}[f(S_T)|\mathcal{F}_t] = f(S_t)B_t + \int_0^{S_t} \mathrm{d}K f''(K)P_t(K) + \int_{S_t}^\infty \mathrm{d}K f''(K)C_t(K), \quad (3.13)$$

where we have used $\mathbb{E}[(S_T - K)^+|\mathcal{F}_t] = C_t(K)$ and $\mathbb{E}[(K - S_T)^+|\mathcal{F}_t] = P_t(K)$. Equation (3.13) leads to the following corollary to Theorem 3.3.

Corollary 3.4 (Price of a European-style claim relative to calls and puts)**.** *Let E_t by given by* (3.1) *and let φ satisfy Assumption 3.2. Define $\eta_t(s) = \chi_t(\log s)$ where χ_t is given by* (3.10)*. Then*

$$E_t = \eta_t(S_t)B_t + \int_0^{S_t} \mathrm{d}K \eta_t''(K)P_t(K) + \int_{S_t}^\infty \mathrm{d}K \eta_t''(K)C_t(K). \quad (3.14)$$

Proof. From Theorem 3.3 we have

$$E_t = \mathbb{E}[\chi_t(X_T)|\mathcal{F}_t] = \mathbb{E}[\chi_t(\log S_T)|\mathcal{F}_t] = \mathbb{E}[\eta_t(S_T)|\mathcal{F}_t].$$

Expression (3.14) now follows by applying (3.13) with $f = \eta_t$. ☐

Remark 3.5. From (3.10) we see that the x-dependence in $\chi_t(x)$ felt only through exponentials $\mathrm{e}^{\mathtt{i}ux}$. Let us define

$$\mathrm{e}^{\mathtt{i}uX_T} = f_r(S_T; u) + \mathtt{i} f_i(S_T; u), \qquad f_r(s; u) := \mathrm{Re}(\mathrm{e}^{\mathtt{i}u \log s}), \qquad f_i(s; u) := \mathrm{Im}(\mathrm{e}^{\mathtt{i}u \log s}). \quad (3.15)$$

Note that the conditional expectation $\mathbb{E}[\mathrm{e}^{\mathtt{i}u(\omega,s)X_T}|\mathcal{F}_t]$ in (3.11) can be obtained from T-maturity calls and puts by applying (3.13) to $f_r(S_T; u)$ and $f_i(S_T; u)$ separately.

Remark 3.6. The time t price of the European-style claim E_t given in Corollary 3.4 is entirely *nonparametric*, as it is constructed from a linear combination of the bond price and call and put prices. Thus, equation (3.14) can be considered a model-free pricing formula. Figure 1 provides a visual diagram of how European calls and puts reveal prices of path-dependent European-style derivatives.

We will now show how the payoff $\varphi(X_T, \langle X \rangle_T)$ of the European claim E can be replicated by trading the bond B, the underlying asset S, and European options on X_T. To begin, it will be helpful to review some basic stochastic calculus rules for $\mathbb{C}$-valued semimartingales. Suppose

$$T = U + \mathtt{i}V, \qquad\qquad X = Y + \mathtt{i}Z.$$

where (U, V, Y, Z) are $\mathbb{R}$-valued semimartingales. Then

$$\begin{aligned} T_{t-}\mathrm{d}X_t &= (U_{t-}\mathrm{d}Y_t - V_{t-}\mathrm{d}Z_t) + \mathtt{i}\,(U_{t-}\mathrm{d}Z_t + V_{t-}\mathrm{d}Y_t)\,, \\ \mathrm{d}[T, X]_t &= (\mathrm{d}[U, Y]_t - \mathrm{d}[V, Z]_t) + \mathtt{i}\,(\mathrm{d}[U, Z]_t + \mathrm{d}[V, Y]_t)\,. \end{aligned} \tag{3.16}$$

We will call any $\mathbb{C}$-valued process $\Pi = (\Pi_t)_{0 \le t \le T}$ *self-financing* if it satisfies

$$\Pi_t = \sum_{i=1}^{d} \Delta_t^i V_t^i, \qquad\qquad \mathrm{d}\Pi_t = \sum_{i=1}^{d} \Delta_{t-}^i \mathrm{d}V_t^i, \tag{3.17}$$

for some $d \in \mathbb{N}$ and $\mathbb{C}^d$-valued semimartingales $\Delta = (\Delta_t)_{0 \le t \le T}$ and $V = (V_t)_{0 \le t \le T}$. In the case where Δ and V are $\mathbb{R}^d$-valued, this definition corresponds to the usual notion of a self-financing portfolio. We now establish a self-financing replication strategy for $\mathbb{C}$-valued exponential claims.

Proposition 3.7. *Let $\omega, s, \in \mathbb{C}$ and define $\mathbb{C}$-valued processes $N = (N_t)_{0 \le t \le T}$ and $Q = (Q_t)_{0 \le t \le T}$ by*

$$N_t \equiv N_t(\omega, s) := \mathrm{e}^{\mathtt{i}(\omega - u)X_t + \mathtt{i}s\langle X \rangle_t}, \qquad\qquad Q_t \equiv Q_t(\omega, s) := \mathbb{E}[\mathrm{e}^{\mathtt{i}uX_T} | \mathcal{F}_t], \tag{3.18}$$

where $u \equiv u(\omega, s)$ is as given in (3.2). Define $\Pi = (\Pi_t)_{0 \le t \le T}$ by

$$\Pi_t \equiv \Pi_t(\omega, s) = (\widehat{\varphi} N_t)\, Q_t + \left(\frac{\widehat{\varphi}\mathtt{i}(\omega - u) N_t Q_{t-}}{S_t} \right) S_t + (-\widehat{\varphi}\mathtt{i}(\omega - u) N_t Q_{t-})\, B_t, \tag{3.19}$$

where $\widehat{\varphi} \equiv \widehat{\varphi}(\omega, s)$. Then Π is self-financing in the sense of (3.17) and satisfies

$$\Pi_T = \widehat{\varphi} \mathrm{e}^{\mathtt{i}\omega X_T + \mathtt{i}s\langle X \rangle_T}. \tag{3.20}$$

Proof. From (3.19) we see that the value of Π at time any time $t \in [0, T]$ is

$$\Pi_t = (\widehat{\varphi} N_t) Q_t + \left(\frac{\widehat{\varphi}\mathtt{i}(\omega - u) N_t Q_{t-}}{S_t} \right) S_t + (-\widehat{\varphi}\mathtt{i}(\omega - u) N_t Q_{t-})\, B_t = \widehat{\varphi} N_t Q_t,$$

where we have used $B_t = 1$. In particular, using (3.18), at the maturity date T, we have

$$\Pi_T = \widehat{\varphi} N_T Q_T = \widehat{\varphi} \mathrm{e}^{\mathtt{i}(\omega - u)X_T + \mathtt{i}s\langle X \rangle_T} \mathbb{E}[\mathrm{e}^{\mathtt{i}uX_T} | \mathcal{F}_T] = \widehat{\varphi} \mathrm{e}^{\mathtt{i}\omega X_T + \mathtt{i}s\langle X \rangle_T},$$

which establishes (3.20). To prove that Π is self-financing in the sense of (3.17) we observe that

$$\mathbb{E}[\widehat{\varphi} \mathrm{e}^{\mathtt{i}\omega X_T + \mathtt{i}s\langle X \rangle_T} | \mathcal{F}_t] = \widehat{\varphi} \mathrm{e}^{\mathtt{i}(\omega - u)X_t + \mathtt{i}s\langle X \rangle_t} \mathbb{E}[\mathrm{e}^{\mathtt{i}uX_T} | \mathcal{F}_t] = \widehat{\varphi} N_t Q_t = \Pi_t. \tag{3.21}$$

The left-hand side of (3.21) is a martingale by iterated conditioning. Thus, the process Π must also be a martingale. The process Q must be a martingale by the same reasoning. Next, we compute

$$\begin{aligned}
\mathrm{d}\Pi_t &= \mathrm{d}(\widehat{\varphi} N_t Q_t) = (\widehat{\varphi} N_t)\mathrm{d}Q_t + (\widehat{\varphi} Q_{t-})\mathrm{d}N_t + \widehat{\varphi}\mathrm{d}[N,Q]_t \\
&= (\widehat{\varphi} N_t)\mathrm{d}Q_t + \left(\frac{\widehat{\varphi}\mathtt{i}(\omega-u)N_t Q_{t-}}{S_t}\right)\mathrm{d}S_t + \mathrm{d}A_t,
\end{aligned}$$

where $A = (A_t)_{0\leq t\leq T}$ is a finite variation process. Since Π, Q and S are martingales, it follows that the finite variation process A must also be a martingale. Moreover, as sample paths of S are continuous, so too are the sample paths of Q. Continuity of A follows from continuity of Q. Since a finite variation, continuous martingale must be a constant, we conclude that A is a constant and thus $\mathrm{d}A_t = 0$. Therefore, the process Π has dynamics

$$\begin{aligned}
\mathrm{d}\Pi_t &= (\widehat{\varphi} N_t)\mathrm{d}Q_t + \left(\frac{\widehat{\varphi}\mathtt{i}(\omega-u)N_t Q_{t-}}{S_t}\right)\mathrm{d}S_t \\
&= (\widehat{\varphi} N_t)\mathrm{d}Q_t + \left(\frac{\widehat{\varphi}\mathtt{i}(\omega-u)N_t Q_{t-}}{S_t}\right)\mathrm{d}S_t + \left(-\widehat{\varphi}\mathtt{i}(\omega-u)N_t Q_{t-}\right)\mathrm{d}B_t,
\end{aligned} \tag{3.22}$$

where we have used $\mathrm{d}B_t = 0$. Combining (3.19) with (3.22), we see that Π is self-financing in the sense of (3.17). □

One would like to replicate not only exponential claims $\widehat{\varphi}\mathrm{e}^{\mathtt{i}\omega X_T + \mathtt{i}s\langle X\rangle_T}$, but also claims with more general payoffs. In the following Theorem, we provide a replication strategy for an option with a payoff of the form $\varphi(X_T, \langle X\rangle_T)$.

Theorem 3.8 (Replication of a European-style claim)**.** *Let E_t be as defined in (3.1) and suppose φ satisfies Assumption 3.2. Let $\Pi(\omega,s)$ be as given in (3.19). Then*

$$E_T = \int_{\mathbb{R}^2} \mathrm{d}\omega_r \mathrm{d}s_r\, \mathrm{Re}(\Pi_0(\omega,s)) + \int_0^T \int_{\mathbb{R}^2} \mathrm{d}\omega_r \mathrm{d}s_r\, \mathrm{Re}(\mathrm{d}\Pi_t(\omega,s)). \tag{3.23}$$

Proof. We have

$$\begin{aligned}
E_T &= \varphi(X_T, \langle X\rangle_T) && \text{(by (3.1))} \\
&= \mathbf{F}^{-1}[\widehat{\varphi}](X_T, \langle X\rangle_T) && \text{(by (3.8))} \\
&= \int_{\mathbb{R}^2} \mathrm{d}\omega_r \mathrm{d}s_r \widehat{\varphi}(\omega,s)\mathrm{e}^{\mathtt{i}\omega X_T + \mathtt{i}s\langle X\rangle_T} && \text{(by (3.7))} \\
&= \int_{\mathbb{R}^2} \mathrm{d}\omega_r \mathrm{d}s_r\, \mathrm{Re}\left(\widehat{\varphi}(\omega,s)\mathrm{e}^{\mathtt{i}\omega X_T + \mathtt{i}s\langle X\rangle_T}\right) && (\varphi \mapsto \mathbb{R}) \\
&= \int_{\mathbb{R}^2} \mathrm{d}\omega_r \mathrm{d}s_r\, \mathrm{Re}(\Pi_T(\omega,s)) && \text{(by (3.20))} \\
&= \int_{\mathbb{R}^2} \mathrm{d}\omega_r \mathrm{d}s_r\, \mathrm{Re}(\Pi_0(\omega,s)) + \int_0^T \int_{\mathbb{R}^2} \mathrm{d}\omega_r \mathrm{d}s_r\, \mathrm{Re}(\mathrm{d}\Pi_t(\omega,s)).
\end{aligned}$$

This establishes (3.23). □

Remark 3.9. Using (3.16) and (3.22), we have

$$\begin{aligned}\mathrm{Re}(\mathrm{d}\Pi_t) &= \mathrm{Re}(\widehat{\varphi} N_t)\,\mathrm{Re}(\mathrm{d}Q_t) + \mathrm{Im}(-\widehat{\varphi} N_t)\,\mathrm{Im}(\mathrm{d}Q_t) \\ &\quad + \mathrm{Re}\left(\frac{\widehat{\varphi}\mathtt{i}(\omega - u) N_t Q_{t-}}{S_t}\right)\mathrm{d}S_t + \mathrm{Re}\left(-\widehat{\varphi}\mathtt{i}(\omega - u) N_t Q_{t-}\right)\mathrm{d}B_t.\end{aligned}$$

Thus, the differential in (3.23) can be written more explicitly as

$$\begin{aligned}\int_{\mathbb{R}^2} \mathrm{d}\omega_r \mathrm{d}s_r\, \mathrm{Re}(\mathrm{d}\Pi_t(\omega,s)) &= \int_{\mathbb{R}^2} \mathrm{d}\omega_r \mathrm{d}s_r\, \mathrm{Re}(\widehat{\varphi}(\omega,s) N_t(\omega,s))\,\mathrm{Re}(\mathrm{d}Q_t(\omega,s)) \\ &\quad + \int_{\mathbb{R}^2} \mathrm{d}\omega_r \mathrm{d}s_r\, \mathrm{Im}(-\widehat{\varphi}(\omega,s) N_t(\omega,s))\,\mathrm{Im}(\mathrm{d}Q_t(\omega,s)) \\ &\quad + \int_{\mathbb{R}^2} \mathrm{d}\omega_r \mathrm{d}s_r\, \mathrm{Re}\left(\frac{\widehat{\varphi}(\omega,s)\mathtt{i}(\omega - u(\omega,s)) N_t(\omega,s) Q_{t-}(\omega,s)}{S_t}\right)\mathrm{d}S_t \\ &\quad + \int_{\mathbb{R}^2} \mathrm{d}\omega_r \mathrm{d}s_r\, \mathrm{Re}\left(-\widehat{\varphi}(\omega,s)\mathtt{i}(\omega - u(\omega,s)) N_t(\omega,s) Q_{t-}(\omega,s)\right)\mathrm{d}B_t. \qquad (3.24)\end{aligned}$$

From (3.24), we see that the replicating E_T involves trading the stock S, the bond B, and a basket of European options of the form $\mathrm{Re}(Q(\omega,s)) = \mathbb{E}[\mathrm{Re}(\mathrm{e}^{\mathtt{i}u(\omega,s)X_T})|\mathcal{F}]$ and $\mathrm{Im}(Q(\omega,s)) = \mathbb{E}[\mathrm{Im}(\mathrm{e}^{\mathtt{i}u(\omega,s)X_T})|\mathcal{F}]$.

Remark 3.10. As a sanity check, we note that, when the option payoff function of X_T only (i.e., $\varphi(X_T)$), the replicating portfolio (3.23) is *static*. To see this, observe that

$$\varphi(x,v) = f(x) \qquad \text{implies} \qquad \mathbf{F}[\varphi](\omega,s) = \widehat{\varphi}(\omega,s) = \widehat{f}(\omega)\delta(s).$$

where $\widehat{f}$ is the (one-dimensional) Fourier transform of f and δ is a Dirac delta function centered at zero. Inserting the above expression for $\widehat{\varphi}$ in (3.24) and integrating with respect to s_r (in this case, one can take $s = s_r$), we see that the only surviving terms are the $s = 0$ terms. When $s = 0$ we have by (3.2) that $u(\omega,s) = \omega$ and by (3.18) that $N_t(\omega,s) = 1$. Thus (3.24) becomes

$$\int_{\mathbb{R}^2} \mathrm{d}\omega_r \mathrm{d}s_r\, \mathrm{Re}(\mathrm{d}\Pi_t(\omega,s)) = \int_{\mathbb{R}} \mathrm{d}\omega_r \Big(\mathrm{Re}(\widehat{f}(\omega)\,\mathrm{Re}(\mathrm{d}Q_t(\omega,0)) - \mathrm{Im}(\widehat{f}(\omega)\,\mathrm{Im}(\mathrm{d}Q_t(\omega,0)) \Big).$$

Thus, the number of shares of $\mathrm{Re}(Q(\omega,0))$ and $\mathrm{Im}(Q(\omega,0))$ that one holds to replicate a European option written on X_T is constant in time. This, of course, is in agreement with the static hedging result given in (3.13).

To finish this section, let us write the replication strategy (3.23) in terms of standard European calls and puts on X_T.

Corollary 3.11 (Replication of a European-style claim relative to calls and puts)**.** *Suppose φ satisfies Assumption 3.2. Then the stochastic differential in* (3.23) *can be alternatively expressed as follows*

$$\int_{\mathbb{R}^2} \mathrm{d}\omega_r \mathrm{d}s_r\, \mathrm{Re}(\mathrm{d}\Pi_t(\omega,s)) = \int_0^{S_t} \mathrm{d}K \Delta_t^K \mathrm{d}P_t(K) + \int_{S_t}^{\infty} \mathrm{d}K \Delta_t^K \mathrm{d}C_t(K) + \Delta_t^S \mathrm{d}S_t + \Delta_t^B \mathrm{d}B_t. \qquad (3.25)$$

where Δ_t^K, Δ_t^S and Δ_t^B are given by

$$\Delta_t^K = \int_{\mathbb{R}^2} \mathrm{d}\omega_r \mathrm{d}s_r\, \mathrm{Re}(\widehat{\varphi}(\omega,s) N_t(\omega,s)) f_r''(K; u(\omega,s))$$

$$
\begin{aligned}
&\quad + \int_{\mathbb{R}^2} \mathrm{d}\omega_r \mathrm{d}s_r \operatorname{Im}(-\widehat{\varphi}(\omega,s) N_t(\omega,s)) f_i''(K;u(\omega,s)),\\
\Delta_t^S &= \int_{\mathbb{R}^2} \mathrm{d}\omega_r \mathrm{d}s_r \operatorname{Re}\left(\frac{\widehat{\varphi}(\omega,s)\mathtt{i}(\omega-u(\omega,s))N_t(\omega,s)Q_{t-}(\omega,s)}{S_t}\right),\\
\Delta_t^B &= \int_{\mathbb{R}^2} \mathrm{d}\omega_r \mathrm{d}s_r \operatorname{Re}(\widehat{\varphi}(\omega,s) N_t(\omega,s)) f_r(S_t;u(\omega,s))\\
&\quad + \int_{\mathbb{R}^2} \mathrm{d}\omega_r \mathrm{d}s_r \operatorname{Im}(-\widehat{\varphi}(\omega,s) N_t(\omega,s)) f_i(S_t;u(\omega,s))\\
&\quad + \int_{\mathbb{R}^2} \mathrm{d}\omega_r \mathrm{d}s_r \operatorname{Re}\left(-\widehat{\varphi}(\omega,s)\mathtt{i}(\omega-u(\omega,s))N_t(\omega,s)Q_{t-}(\omega,s)\right),
\end{aligned}
$$

with $u(\omega,s)$ as given in (3.2), the functions f_r and f_i as defined in (3.15) and $N_t(\omega,s)$ and $Q_t(\omega,s)$ as given in (3.18).

Proof. From (3.15) and (3.18) we have

$$
\begin{aligned}
\operatorname{Re}(Q_t(\omega,s)) &= \mathbb{E}[\operatorname{Re}(\mathrm{e}^{\mathtt{i}u(\omega,s)\log S_T})|\mathcal{F}_t] = \mathbb{E}[f_r(S_T;u(\omega,s))|\mathcal{F}_t],\\
\operatorname{Im}(Q_t(\omega,s)) &= \mathbb{E}[\operatorname{Im}(\mathrm{e}^{\mathtt{i}u(\omega,s)\log S_T})|\mathcal{F}_t] = \mathbb{E}[f_i(S_T;u(\omega,s))|\mathcal{F}_t].
\end{aligned}
$$

Hence, using (3.13), we see that

$$
\operatorname{Re}(\mathrm{d}Q_t(\omega,s)) = f_r(S_t)\mathrm{d}B_t + \int_0^{S_t} \mathrm{d}K f_r''(K)\mathrm{d}P_t(K) + \int_{S_t}^{\infty} \mathrm{d}K f_r''(K)\mathrm{d}C_t(K), \tag{3.26}
$$

$$
\operatorname{Im}(\mathrm{d}Q_t(\omega,s)) = f_i(S_t)\mathrm{d}B_t + \int_0^{S_t} \mathrm{d}K f_i''(K)\mathrm{d}P_t(K) + \int_{S_t}^{\infty} \mathrm{d}K f_i''(K)\mathrm{d}C_t(K). \tag{3.27}
$$

Expression (3.25) is then obtained by inserting expressions (3.26) and (3.27) into (3.24). □

We have now provided a nonparametric pricing formula and replication strategy (Corollaries 3.4 and 3.11, respectively) for European-style claims of the form (3.1). In subsequent section, we will use these results to price and replicate double and single barrier knock-in, knock-out, and rebate options.

4 Single barrier claims

In this section, we will show how to price and replicate three types of single-barrier claims: knock-in, knock-out and rebate. To this end, for any $x \in \mathbb{R}$, we define τ_x, the *first hitting time to level x* by

$$
\tau_x := \inf\{t \geq 0 : X_t = x\}, \qquad \inf\{\emptyset\} = \infty. \tag{4.1}
$$

Note that τ_x is an $\mathcal{F}$-stopping time. Moreover, we have $\mathbb{I}_{\{\tau_x<\infty\}}X_{\tau_x} = x$.

4.1 Single barrier knock-in claims

In this section, we consider a claim with a payoff of the form $\mathbb{I}_{\{\tau_L \leq T\}}\varphi(X_T, \langle X\rangle_T - \langle X\rangle_{\tau_L})$. We will refer to such a claim as *knock-in*, since the claim has a non-zero payoff only if X hits a level L prior to the maturity date T. Throughout this paper, we will denote by $I = (I_t)_{0 \leq t \leq T}$ the price process of any knock-in claim (I stands for "in"). Using risk-neutral pricing, we have

$$I_t = \mathbb{E}[\mathbb{I}_{\{\tau_L \leq T\}} \varphi(X_T, \langle X \rangle_T - \langle X \rangle_{\tau_L}) | \mathcal{F}_t]. \qquad t \in [0, T]. \tag{4.2}$$

Our goals are to (i) find the value I_t relative to T-maturity options on X and (ii) provide a hedging strategy that replicates the option payoff. We begin our analysis with the following proposition.

Proposition 4.1. *Fix $L < X_0$. Let $\omega, s \in \mathbb{C}$ and let $u \equiv u(\omega, s)$ be as given in (3.2). Define*

$$\psi_L(x) \equiv \psi_L(x; \omega, s) = \mathbb{I}_{\{x<L\}} \mathrm{e}^{\mathrm{i}\omega L} \left(\mathrm{e}^{(x-L) - \mathrm{i}u(x-L)} + \mathrm{e}^{\mathrm{i}u(x-L)} \right). \tag{4.3}$$

Then, with τ_L as given in (4.1) we have

$$\mathbb{I}_{\{\tau_L \leq T\}} \mathbb{E}[\psi_L(X_T) | \mathcal{F}_{\tau_L}] = \mathbb{I}_{\{\tau_L \leq T\}} \mathbb{E}[\mathrm{e}^{\mathrm{i}\omega X_T + \mathrm{i}s(\langle X \rangle_T - \langle X \rangle_{\tau_L})} | \mathcal{F}_{\tau_L}]. \tag{4.4}$$

Proof. We have

$$\begin{aligned} \mathbb{I}_{\{\tau_L \leq T\}} \mathbb{E}[\psi_L(X_T) | \mathcal{F}_{\tau_L}] &= \mathbb{I}_{\{\tau_L \leq T\}} \mathrm{e}^{\mathrm{i}\omega L} \mathbb{E}[\mathbb{I}_{\{x<L\}} \left(\mathrm{e}^{(X_T-L) - \mathrm{i}u(X_T-L)} + \mathrm{e}^{\mathrm{i}u(X_T-L)} \right) | \mathcal{F}_{\tau_L}] \\ &= \mathbb{I}_{\{\tau_L \leq T\}} \mathrm{e}^{\mathrm{i}\omega L} \mathbb{E}[Z_T \mathbb{I}_{\{x<L\}} \mathrm{e}^{\frac{1}{2}(X_T-L) + \frac{1}{8}(\langle X \rangle_T - \langle X \rangle_{\tau_L})} \mathrm{e}^{-\mathrm{i}u(X_T-L)} | \mathcal{F}_{\tau_L}] \\ &\quad + \mathbb{I}_{\{\tau_L \leq T\}} \mathrm{e}^{\mathrm{i}\omega L} \mathbb{E}[Z_T \mathbb{I}_{\{x<L\}} \mathrm{e}^{-\frac{1}{2}(X_T-L) + \frac{1}{8}(\langle X \rangle_T - \langle X \rangle_{\tau_L})} \mathrm{e}^{\mathrm{i}u(X_T-L)} | \mathcal{F}_{\tau_L}], \end{aligned} \tag{4.5}$$

where we have introduced a process $Z = (Z_t)_{0 \leq t \leq T}$, defined as

$$Z_t := \exp\left(-\tfrac{1}{2} \int_0^t (\tfrac{1}{2}\sigma_t \mathbb{I}_{\{\tau_L \leq t\}})^2 \mathrm{d}t + \int_0^t (\tfrac{1}{2}\sigma_t \mathbb{I}_{\{\tau_L \leq t\}}) \mathrm{d}W_t \right). \tag{4.6}$$

Note that, in deriving (4.5), we have used the fact that

$$Z_T | \mathcal{F}_{\tau_L} = \mathrm{e}^{\frac{1}{2}(X_T-L) + \frac{1}{8}(\langle X \rangle_T - \langle X \rangle_{\tau_L})}.$$

Now, observe that $Z_T > 0$ and $\mathbb{E}[Z_T] = 1$. Thus, we can define a new probability measure $\widetilde{\mathbb{P}}$ via

$$\widetilde{\mathbb{P}}(A) = \mathbb{E}[\mathbb{I}_{\{A\}} Z_T] = \widetilde{\mathbb{E}}[\mathbb{I}_{\{A\}}],$$

where $Z_T \equiv \mathrm{d}\widetilde{\mathbb{P}}/\mathrm{d}\mathbb{P}$ is the Radon-Nikodym derivative of $\widetilde{\mathbb{P}}$ with respect to $\mathbb{P}$. Since $Z_T > 0$ we also have $1/Z_T = \mathrm{d}\mathbb{P}/\mathrm{d}\widetilde{\mathbb{P}}$. By Girsanov's theorem, the dynamics of X under $\widetilde{\mathbb{P}}$ are given by

$$\mathrm{d}X_t = -\tfrac{1}{2}\sigma_t^2 \mathbb{I}_{\{\tau_L > t\}} \mathrm{d}t + \sigma_t \mathrm{d}\widetilde{W}_t, \tag{4.7}$$

where $\widetilde{W}$ is a $\widetilde{\mathbb{P}}$-Brownian motion. We now re-write (4.5) as

$$\begin{aligned} \mathbb{I}_{\{\tau_L \leq T\}} \mathbb{E}[\psi_L(X_T) | \mathcal{F}_{\tau_L}] &= \mathbb{I}_{\{\tau_L \leq T\}} \mathrm{e}^{\mathrm{i}\omega L} \widetilde{\mathbb{E}}[\mathbb{I}_{\{X_T<L\}} \mathrm{e}^{\frac{1}{2}(X_T-L) + \frac{1}{8}(\langle X \rangle_T - \langle X \rangle_{\tau_L})} \mathrm{e}^{-\mathrm{i}u(X_T-L)} | \mathcal{F}_{\tau_L}] \\ &\quad + \mathbb{I}_{\{\tau_L \leq T\}} \mathrm{e}^{\mathrm{i}\omega L} \widetilde{\mathbb{E}}[\mathbb{I}_{\{X_T<L\}} \mathrm{e}^{-\frac{1}{2}(X_T-L) + \frac{1}{8}(\langle X \rangle_T - \langle X \rangle_{\tau_L})} \mathrm{e}^{\mathrm{i}u(X_T-L)} | \mathcal{F}_{\tau_L}]. \end{aligned} \tag{4.8}$$

From (4.7), we see that $\mathrm{d}X_t = \sigma_t \mathrm{d}\widetilde{W}_t$ when $\tau_L \leq t$. Hence, under $\widetilde{\mathbb{P}}$, and on the set $\{\tau_L \leq T\}$, we have $(X_T - L) \stackrel{\mathcal{D}}{=} -(X_T - L)$. Thus, in the first expectation on the right-hand side of (4.8), we can replace $(X_T - L)$ with $-(X_T - L)$. Upon making this replacement, we obtain

$$\begin{aligned}\mathbb{I}_{\{\tau_L \leq T\}}\mathbb{E}[\psi_L(X_T)|\mathcal{F}_{\tau_L}] &= \mathbb{I}_{\{\tau_L \leq T\}}\mathrm{e}^{\mathrm{i}\omega L}\widetilde{\mathbb{E}}[\mathbb{I}_{\{X_T>L\}}\mathrm{e}^{-\frac{1}{2}(X_T-L)+\frac{1}{8}(\langle X\rangle_T - \langle X\rangle_{\tau_L})}\mathrm{e}^{\mathrm{i}u(X_T-L)}|\mathcal{F}_{\tau_L}] \\ &\quad + \mathbb{I}_{\{\tau_L \leq T\}}\mathrm{e}^{\mathrm{i}\omega L}\widetilde{\mathbb{E}}[\mathbb{I}_{\{X_T<L\}}\mathrm{e}^{-\frac{1}{2}(X_T-L)+\frac{1}{8}(\langle X\rangle_T - \langle X\rangle_{\tau_L})}\mathrm{e}^{\mathrm{i}u(X_T-L)}|\mathcal{F}_{\tau_L}] \\ &= \mathbb{I}_{\{\tau_L \leq T\}}\mathrm{e}^{\mathrm{i}\omega L}\widetilde{\mathbb{E}}[\mathrm{e}^{-\frac{1}{2}(X_T-L)+\frac{1}{8}(\langle X\rangle_T - \langle X\rangle_{\tau_L})}\mathrm{e}^{\mathrm{i}u(X_T-L)}|\mathcal{F}_{\tau_L}]. \end{aligned} \tag{4.9}$$

Now, we observe that

$$\frac{1}{Z_T}|\mathcal{F}_{\tau_L} = \mathrm{e}^{-\frac{1}{2}(X_T-L)+\frac{1}{8}(\langle X\rangle_T - \langle X\rangle_{\tau_L})}.$$

Thus, we can re-write (4.9) as

$$\begin{aligned}\mathbb{I}_{\{\tau_L \leq T\}}\mathbb{E}[\psi_L(X_T)|\mathcal{F}_{\tau_L}] &= \mathbb{I}_{\{\tau_L \leq T\}}\mathrm{e}^{\mathrm{i}\omega L}\widetilde{\mathbb{E}}\left[\frac{1}{Z_T}\mathrm{e}^{\mathrm{i}u(X_T-L)}\Big|\mathcal{F}_{\tau_L}\right] = \mathbb{I}_{\{\tau_L \leq T\}}\mathrm{e}^{\mathrm{i}\omega L}\mathbb{E}[\mathrm{e}^{\mathrm{i}u(X_T-L)}|\mathcal{F}_{\tau_L}] \\ &= \mathbb{I}_{\{\tau_L \leq T\}}\mathrm{e}^{\mathrm{i}\omega L}\mathbb{E}[\mathrm{e}^{\mathrm{i}\omega(X_T-L)+\mathrm{i}s(\langle X\rangle_T - \langle X\rangle_{\tau_L})}|\mathcal{F}_{\tau_L}] \\ &= \mathbb{I}_{\{\tau_L \leq T\}}\mathbb{E}[\mathrm{e}^{\mathrm{i}\omega X_T+\mathrm{i}s(\langle X\rangle_T - \langle X\rangle_{\tau_L})}|\mathcal{F}_{\tau_L}]. \end{aligned}$$

where, in the last step, we have used (3.3). Thus, we have established (4.4). □

Remark 4.2. If one were to make the replacement $\mathbb{I}_{\{\tau_L \leq t\}} \to 1$ in (4.6), then the measure $\widetilde{\mathbb{P}}$ would correspond to what is commonly referred to as the *half measure*.

Remark 4.3. Suppose $U > X_0$ and $u \equiv u(\omega, s)$ is given by (3.2). Define

$$\psi_U(x) \equiv \psi_U(x;\omega,s) = \mathbb{I}_{\{x>U\}}\mathrm{e}^{\mathrm{i}\omega U}\left(\mathrm{e}^{(x-U)-\mathrm{i}u(x-U)} + \mathrm{e}^{\mathrm{i}u(x-U)}\right). \tag{4.10}$$

Then we have

$$\mathbb{I}_{\{\tau_U \leq T\}}\mathbb{E}[\psi_U(X_T)|\mathcal{F}_{\tau_U}] = \mathbb{I}_{\{\tau_U \leq T\}}\mathbb{E}[\mathrm{e}^{\mathrm{i}\omega X_T+\mathrm{i}s(\langle X\rangle_T - \langle X\rangle_{\tau_U})}|\mathcal{F}_{\tau_U}]. \tag{4.11}$$

where τ_U as given in (4.1). The proof of this result follows line-by-line the proof of Proposition 4.1 with only minor modifications.

Corollary 4.4. *Fix $L < X_0$ and $\omega, s \in \mathbb{C}$. Let $u \equiv u(\omega, s)$ and $\psi_L(x) \equiv \psi_L(x;\omega,s)$ be as given in (3.2) and (4.3), respectively. Then, with τ_L as defined in (4.1) we have*

$$\mathbb{I}_{\{\tau_L>t\}}\mathbb{E}[\mathbb{I}_{\{\tau_L \leq T\}}\mathrm{e}^{\mathrm{i}\omega X_T+\mathrm{i}s(\langle X\rangle_T - \langle X\rangle_{\tau_L})}|\mathcal{F}_t] = \mathbb{I}_{\{\tau_L>t\}}\mathbb{E}[\psi_L(X_T)|\mathcal{F}_t]. \tag{4.12}$$

Proof. Using $\{\tau_L > t\} = \{T \geq \tau_L > t\} \cup \{\tau_L > T\}$, we have

$$\begin{aligned}&\mathbb{I}_{\{\tau_L>t\}}\mathbb{E}[\mathbb{I}_{\{\tau_L \leq T\}}\mathrm{e}^{\mathrm{i}\omega X_T+\mathrm{i}s(\langle X\rangle_T - \langle X\rangle_{\tau_L})}|\mathcal{F}_t] \\ &= \mathbb{I}_{\{T\geq\tau_L>t\}}\mathbb{E}[\mathbb{I}_{\{\tau_L \leq T\}}\mathrm{e}^{\mathrm{i}\omega X_T+\mathrm{i}s(\langle X\rangle_T - \langle X\rangle_{\tau_L})}|\mathcal{F}_t] \\ &\quad + \mathbb{I}_{\{\tau_L>T\}}\mathbb{E}[\mathbb{I}_{\{\tau_L \leq T\}}\mathrm{e}^{\mathrm{i}\omega X_T+\mathrm{i}s(\langle X\rangle_T - \langle X\rangle_{\tau_L})}|\mathcal{F}_t]\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{I}_{\{T\geq\tau_L>t\}}\mathbb{E}[\mathbb{I}_{\{\tau_L\leq T\}}\mathrm{e}^{\mathrm{i}\omega X_T+\mathrm{i}s(\langle X\rangle_T-\langle X\rangle_{\tau_L})}|\mathcal{F}_t] && (\text{as } \mathbb{I}_{\{\tau_L>T\}}\mathbb{I}_{\{\tau_L\leq T\}}=0)\\
&= \mathbb{I}_{\{\tau_L>t\}}\mathbb{E}[\mathbb{I}_{\{\tau_L\leq T\}}\mathrm{e}^{\mathrm{i}\omega X_T+\mathrm{i}s(\langle X\rangle_T-\langle X\rangle_{\tau_L})}|\mathcal{F}_t] && (\mathbb{I}_{\{T\geq\tau_L>t\}}=\mathbb{I}_{\{\tau_L>t\}}\mathbb{I}_{\{\tau_L\leq T\}})\\
&= \mathbb{I}_{\{\tau_L>t\}}\mathbb{E}[\mathbb{I}_{\{\tau_L\leq T\}}\mathbb{E}[\mathrm{e}^{\mathrm{i}\omega X_T+\mathrm{i}s(\langle X\rangle_T-\langle X\rangle_{\tau_L})}|\mathcal{F}_{\tau_L}]|\mathcal{F}_t]\\
&= \mathbb{I}_{\{\tau_L>t\}}\mathbb{E}[\mathbb{I}_{\{\tau_L\leq T\}}\mathbb{E}[\psi_L(X_T)|\mathcal{F}_{\tau_L}]|\mathcal{F}_t] && (\text{by } (4.4))\\
&= \mathbb{I}_{\{\tau_L>t\}}\mathbb{E}[\mathbb{I}_{\{\tau_L\leq T\}}\psi_L(X_T)|\mathcal{F}_t]\\
&= \mathbb{I}_{\{\tau_L>t\}}\mathbb{E}[\left(\mathbb{I}_{\{\tau_L\leq T\}}+\mathbb{I}_{\{\tau_L>T\}}\right)\psi_L(X_T)|\mathcal{F}_t] && (\text{because } \mathbb{I}_{\{\tau_L>T\}}\psi_L(X_T)=0)\\
&= \mathbb{I}_{\{\tau_L>t\}}\mathbb{E}[\psi_L(X_T)|\mathcal{F}_t]. && (\text{as } \mathbb{I}_{\{\tau_L\leq T\}}+\mathbb{I}_{\{\tau_L>T\}}=1)
\end{aligned}
$$

Thus, we have proved (4.12). □

Using Corollary 4.4, we are now able to price the knock-in claim (4.2).

Theorem 4.5 (Price of a single barrier knock-in claim)**.** *Fix $L < X_0$. Let I_t be as given in (4.2) and let φ satisfy Assumption 3.2. Then, with τ_L as defined in (4.1), we have*

$$
I_t = \mathbb{I}_{\{\tau_L>t\}}\mathbb{E}[\Psi_L(X_T)|\mathcal{F}_t] + \mathbb{I}_{\{\tau_L\leq t\}}\mathbb{E}[\varphi_{\tau_L}(X_T,\langle X\rangle_T)|\mathcal{F}_t], \tag{4.13}
$$

where φ_{τ_L}, which is an $\mathcal{F}_t$-measurable function on the set $\{\tau_L \leq t\}$, and Ψ_L are given by

$$
\varphi_{\tau_L}(x,v) = \varphi(x, v-\langle X\rangle_{\tau_L}), \tag{4.14}
$$

$$
\Psi_L(x) = \int_{\mathbb{R}^2} \mathrm{d}\omega_r \mathrm{d}s_r \widehat{\varphi}(\omega,s)\psi_L(x;\omega,s), \tag{4.15}
$$

with $\psi_L(x;\omega,s)$ as defined in (4.3).

Proof. We have

$$
\begin{aligned}
I_t &= \mathbb{E}[\mathbb{I}_{\{\tau_L\leq T\}}\varphi(X_T,\langle X\rangle_T-\langle X\rangle_{\tau_L})|\mathcal{F}_t] && (\text{by } (4.2))\\
&= \mathbb{I}_{\{\tau_L\leq t\}}\mathbb{E}[\mathbb{I}_{\{\tau_L\leq T\}}\varphi(X_T,\langle X\rangle_T-\langle X\rangle_{\tau_L})|\mathcal{F}_t]\\
&\qquad + \mathbb{I}_{\{(\tau_L>t\}}\mathbb{E}[\mathbb{I}_{\{\tau_L\leq T\}}\varphi(X_T,\langle X\rangle_T-\langle X\rangle_{\tau_L})|\mathcal{F}_t]\\
&= \mathbb{I}_{\{\tau_L\leq t\}}\mathbb{E}[\varphi_{\tau_L}(X_T,\langle X\rangle_T)|\mathcal{F}_t]\\
&\quad + \mathbb{I}_{\{\tau_L>t\}}\mathbb{E}[\mathbb{I}_{\{\tau_L\leq T\}}\mathbf{F}^{-1}[\widehat{\varphi}](X_T,\langle X\rangle_T-\langle X\rangle_{\tau_L})|\mathcal{F}_t] && (\text{by } (3.8) \text{ and } (4.14))\\
&= \mathbb{I}_{\{\tau_L\leq t\}}\mathbb{E}[\varphi_{\tau_L}(X_T,\langle X\rangle_T)|\mathcal{F}_t]\\
&\quad + \mathbb{I}_{\{\tau_L>t\}}\int_{\mathbb{R}^2}\mathrm{d}\omega_r\mathrm{d}s_r\widehat{\varphi}(\omega,s)\mathbb{E}[\mathbb{I}_{\{\tau_L\leq T\}}\mathrm{e}^{\mathrm{i}\omega X_T+\mathrm{i}s(\langle X\rangle_T-\langle X\rangle_{\tau_L})}|\mathcal{F}_t] && (\text{by } (3.7))\\
&= \mathbb{I}_{\{\tau_L\leq t\}}\mathbb{E}[\varphi_{\tau_L}(X_T,\langle X\rangle_T)|\mathcal{F}_t]\\
&\quad + \mathbb{I}_{\{\tau_L>t\}}\int_{\mathbb{R}^2}\mathrm{d}\omega_r\mathrm{d}s_r\widehat{\varphi}(\omega,s)\mathbb{E}[\psi_L(X_T;\omega,s)|\mathcal{F}_t] && (\text{by } (4.12))\\
&= \mathbb{I}_{\{\tau_L\leq t\}}\mathbb{E}[\varphi_{\tau_L}(X_T,\langle X\rangle_T)|\mathcal{F}_t] + \mathbb{I}_{\{\tau_L>t\}}\mathbb{E}[\Psi_L(X_T)|\mathcal{F}_t]. && (\text{by } (4.15))
\end{aligned}
$$

This establishes (4.13). □

Example 4.6 (Knock-in power claims)**.** Consider a knock-in option with payoff $\varphi(X_T, \langle X\rangle_T - \langle X\rangle_{\tau_L})$ where $\varphi(x,v) = x^m v^n$ with $m,n \in \mathbb{Z}^+$. Using (3.6) we have $\widehat{\varphi}(\omega,s) = (\mathtt{i}\partial_\omega)^m(\mathtt{i}\partial_s)^q\delta(\omega)\delta(s)$ where δ is the Dirac delta function. Inserting the expression for $\widehat{\varphi}$ into (4.15) and integrating by parts, the integrating with respect to ω_r and s_r, we obtain

$$\Psi_L(x) = \Big[(-\mathtt{i}\partial_\omega)^m(-\mathtt{i}\partial_s)^n \mathbb{I}_{\{x<L\}}\mathrm{e}^{\mathtt{i}\omega L}\left(\mathrm{e}^{(x-L)-\mathtt{i}u(\omega,s)(x-L)} + \mathrm{e}^{\mathtt{i}u(\omega,s)(x-L)}\right)\Big|_{(\omega,s)=(0,0)}.$$

With $m = 0$ and $n = 1, 2$ we have explicitly

$$\begin{aligned}
(m,n) &= (0,1): & \Psi_L(x) &= \mathbb{I}_{\{x<L\}}2\mathrm{e}^{-L}\left(\mathrm{e}^L - \mathrm{e}^x\right)(L-x),\\
(m,n) &= (0,2): & \Psi_L(x) &= \mathbb{I}_{\{x<L\}}4\mathrm{e}^{-L}\left(\mathrm{e}^L(L-x-2) + \mathrm{e}^x(L-x+2)\right)(L-x).
\end{aligned}$$

We plot these functions in Figure 2.

Example 4.7 (Knock-in exponential claims)**.** Consider a knock-in option with payoff $\varphi(X_T, \langle X\rangle_T - \langle X\rangle_{\tau_L})$ where $\varphi(x,v) = \mathrm{e}^{px+qv}$ with $p,q \in \mathbb{R}$. Using (3.6) we have $\widehat{\varphi}(\omega,s) = \delta(\omega + \mathtt{i}p)\delta(s+\mathtt{i}q)$ where δ is the Dirac delta function. Inserting the expression for $\widehat{\varphi}$ into (4.15) and integrating with respect to ω_r and s_r, we obtain

$$\Psi_L(x) = \Big[\mathbb{I}_{\{x<L\}}\mathrm{e}^{\mathtt{i}\omega L}\left(\mathrm{e}^{(x-L)-\mathtt{i}u(\omega,s)(x-L)} + \mathrm{e}^{\mathtt{i}u(\omega,s)(x-L)}\right)\Big|_{(\omega,s)=(-\mathtt{i}p,-\mathtt{i}q)}.$$

If we define $\beta := \frac{1}{4} + p^2 - p + 2q$, then we obtain

$$\begin{aligned}
\beta \geq 0: && \Psi_L(x) &= \mathbb{I}_{\{x<L\}}\mathrm{e}^{pL}\left(\mathrm{e}^{\frac{1}{2}(x-L)}2\cosh\left[\sqrt{\beta}(x-L)\right]\right),\\
\beta < 0: && \Psi_L(x) &= \mathbb{I}_{\{x<L\}}\mathrm{e}^{pL}\left(\mathrm{e}^{\frac{1}{2}(x-L)}2\cos\left[\sqrt{|\beta|}(x-L)\right]\right).
\end{aligned}$$

We plot $\Psi_L(x)$ for various values of p, q in Figure 2.

Before giving the replications strategy for a single-barrier knock-in claim it will be helpful to present the following lemma.

Lemma 4.8. *Let τ be an $\mathcal{F}$ stopping time. Let $\Pi = (\Pi_t)_{0\leq t\leq T}$ be given by*

$$\Pi_t = (1-D_t)V_t^1 + D_tV_t^2, \qquad D_t := \mathbb{I}_{\{\tau\leq t\}}. \tag{4.16}$$

Then Π is self-financing if $V_\tau^1 = V_\tau^2$. That is

$$V_\tau^1 = V_\tau^2 \qquad \Rightarrow \qquad \mathrm{d}\Pi_t = (1-D_{t-})\mathrm{d}V_t^1 + D_{t-}\mathrm{d}V_t^2. \tag{4.17}$$

Proof. We compute

$$\begin{aligned}
\mathrm{d}\Pi_t &= \mathrm{d}\Big((1-D_t)V_t^1 + D_tV_t^2\Big) && \text{(by (4.16))}\\
&= (1-D_{t-})\mathrm{d}V_t^1 + D_{t-}\mathrm{d}V_t^2 + \left(V_{t-}^2 - V_{t-}^1\right)\mathrm{d}D_t + \mathrm{d}[V^2 - V^1, D]_t. && (4.18)
\end{aligned}$$

Now, note that

$$\mathrm{d}D_t = \begin{cases} 0 & t \neq \tau,\\ 1 & t = \tau. \end{cases} \tag{4.19}$$

Therefore, we need only to analyze the last two terms in (4.18) at the stopping time τ. Note that

$$\begin{aligned}\left(V^2_{\tau-} - V^1_{\tau-}\right) \mathrm{d}D_\tau &= V^2_{\tau-} - V^1_{\tau-},\\ \mathrm{d}[V^2 - V^1, D]_\tau &= \left((V^2_\tau - V^1_\tau) - (V^2_{\tau-} - V^1_{\tau-})\right) \mathrm{d}D_\tau = (V^2_\tau - V^1_\tau) - (V^2_{\tau-} - V^1_{\tau-}).\end{aligned}$$

Thus, we obtain

$$\left(V^2_{\tau-} - V^1_{\tau-}\right) \mathrm{d}D_\tau + \mathrm{d}[V^2 - V^1, D]_\tau = V^2_\tau - V^1_\tau. \tag{4.20}$$

Expression (4.17) follows from (4.18), (4.19) and (4.20). □

Theorem 4.9 (Replication of a single barrier knock-in claim)**.** *Fix $L < X_0$. Let τ_L and I_t be as defined in (4.1) and (4.2), respectively and let φ satisfy Assumption 3.2. Define the hitting time indicator process $D = (D_t)_{0 \le t \le T}$ and European-style options $E^1 = (E^1_t)_{0 \le t \le T}$ and $E^2 = (E^2_t)_{\tau_L \le t \le T}$ by*

$$D_t = \mathbb{I}_{\{\tau_L \le t\}}, \qquad E^1_t = \mathbb{E}[\Psi_L(X_T)|\mathcal{F}_t], \qquad E^2_t = \mathbb{E}[\varphi_{\tau_L}(X_T, \langle X\rangle_T)|\mathcal{F}_t]. \tag{4.21}$$

where Ψ_L and φ_{τ_L} as given in (4.15) and (4.14), respectively. Then

$$I_t = (1 - D_t)E^1_t + D_t E^2_t, \tag{4.22}$$

and the portfolio (4.22) is self-financing portfolio in the sense of (3.17). That is,

$$\mathrm{d}I_t = (1 - D_{t-})\mathrm{d}E^1_t + D_{t-}\mathrm{d}E^2_t. \tag{4.23}$$

Proof. From (4.13) and (4.21) we have

$$I_t = (1 - D_t)E^1_t + D_t E^2_t,$$

which proves (4.22). To prove (4.23), we must show, from (4.17), that $\mathbb{I}_{\{\tau_L \le T\}} E^1_{\tau_L} = \mathbb{I}_{\{\tau_L \le T\}} E^2_{\tau_L}$. Observe that

$$\begin{aligned}
\mathbb{I}_{\{\tau_L \le T\}} E^1_{\tau_L} &= \mathbb{I}_{\{\tau_L \le T\}} \mathbb{E}[\Psi_L(X_T)|\mathcal{F}_{\tau_L}] && \text{(by (4.21))}\\
&= \mathbb{I}_{\{\tau_L \le T\}} \int_{\mathbb{R}^2} \mathrm{d}\omega_r \mathrm{d}s_r \widehat{\varphi}(\omega, s) \mathbb{E}[\psi_L(X_T; \omega, s)|\mathcal{F}_{\tau_L}] && \text{(by (4.15))}\\
&= \mathbb{I}_{\{\tau_L \le T\}} \int_{\mathbb{R}^2} \mathrm{d}\omega_r \mathrm{d}s_r \widehat{\varphi}(\omega, s) \mathbb{E}[\mathrm{e}^{\mathtt{i}\omega X_T + \mathtt{i}s(\langle X\rangle_T - \langle X\rangle_{\tau_L})}|\mathcal{F}_{\tau_L}] && \text{(by (4.4))}\\
&= \mathbb{I}_{\{\tau_L \le T\}} \mathbb{E}[\varphi(X_T, \langle X\rangle_t - \langle X\rangle_{\tau_L})|\mathcal{F}_{\tau_L}] && \text{(by (3.7) and (3.8))}\\
&= \mathbb{I}_{\{\tau_L \le T\}} \mathbb{E}[\varphi_{\tau_L}(X_T, \langle X\rangle_t)|\mathcal{F}_{\tau_L}] && \text{(by (4.14))}\\
&= \mathbb{I}_{\{\tau_L \le T\}} E^2_{\tau_L}. && \text{(by (4.21)).}
\end{aligned}$$

We therefore conclude that the portfolio is self-financing. □

Remark 4.10. Note that E^1 has a payoff $\Psi_L(X_T)$ that in independent of $\langle X\rangle_T$. Thus, in accordance with Remark 3.10, the hedging portfolio (4.22) is static over the interval $[0, \tau_L \wedge T]$. Since the payoff of E^2 depends both on X_T and $\langle X\rangle_T$, on the interval $(\tau_L \wedge T, T]$ the hedging portfolio (4.22) involves dynamic trading of the underlying S, the bond B and European calls and puts on X, as described in Section 3.

4.2 Single barrier knock-out claims

In this section, we will consider a claim whose payoff is of the form $\mathbb{I}_{\{\tau_L > T\}}\varphi(X_T, \langle X\rangle_T)$. We will refer to such a claim as *knock-out*, since the claim pays nothing if X hits a level L prior to the maturity date T. Throughout this paper, we will denote by $O = (O_t)_{0\le t\le T}$ the price process of any knock-out claim (O stands for "out"). Using risk-neutral pricing, we have

$$O_t = \mathbb{E}[\mathbb{I}_{\{\tau_L > T\}}\varphi(X_T, \langle X\rangle_T)|\mathcal{F}_t]. \qquad\qquad t \in [0, T]. \tag{4.24}$$

Our goals are to (i) find the value O_t relative to T-maturity options on X and (ii) provide a hedging portfolio that replicates the option payoff.

Our strategy will be to represent the value of the knock-out claim O_t at a time $t < \tau_L$ as the difference of two European-style claims of the form (3.1). On the set $\{\tau_L > T\}$, one claim will pay $\varphi(X_T, \langle X\rangle_T)$ at time T and the other claim will pay nothing. On the set $\{\tau_L \le T\}$, the two claims will have the same value at time τ_L and thus, at the hitting time τ_L, the position in the two European-style options can be cleared with no value. To construct a pair European-style claims that satisfy the above requirements, we need the following proposition.

Proposition 4.11. *Let τ be an $\mathcal{F}$-stopping time Then, for any $G : \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}$ we have*

$$\mathbb{I}_{\{\tau\le T\}}\mathbb{E}[G(X_T, \langle X\rangle_T)|\mathcal{F}_\tau] = \mathbb{I}_{\{\tau\le T\}}\mathbb{E}[\mathrm{e}^{X_T - X_\tau}G(2X_\tau - X_T, \langle X\rangle_T)|\mathcal{F}_\tau]. \tag{4.25}$$

Proof. For any $G : \mathbb{R} \to \mathbb{R}$, [CL09, Theorem 5.2] prove that

$$\mathbb{I}_{\{\tau\le T\}}\mathbb{E}[G(X_T)|\mathcal{F}_\tau] = \mathbb{I}_{\{\tau\le T\}}\mathbb{E}[\mathrm{e}^{X_T - X_\tau}G(2X_\tau - X_T)|\mathcal{F}_\tau]. \tag{4.26}$$

Using this result, we have

$$\begin{aligned}\mathbb{I}_{\{\tau\le T\}}\mathbb{E}[G(X_T, \langle X\rangle_T)|\mathcal{F}_\tau] &= \mathbb{I}_{\{\tau\le T\}}\mathbb{E}[\mathbb{E}[G(X_T, \langle X\rangle_T)|\mathcal{F}^\sigma_{\tau,T}\vee\mathcal{F}_\tau]|\mathcal{F}_\tau] \\ &= \mathbb{I}_{\{\tau\le T\}}\mathbb{E}[\mathbb{E}[\mathrm{e}^{X_T - X_\tau}G(2X_\tau - X_T, \langle X\rangle_T)|\mathcal{F}^\sigma_{\tau,T}\vee\mathcal{F}_\tau]|\mathcal{F}_\tau] \\ &= \mathbb{I}_{\{\tau\le T\}}\mathbb{E}[\mathrm{e}^{X_T - X_\tau}G(2X_\tau - X_T, \langle X\rangle_T)|\mathcal{F}_\tau],\end{aligned}$$

where, in moving from the second to the third equality, we have used (4.26), the fact that $\langle X\rangle_T$ is $(\mathcal{F}^\sigma_{\tau,T}\vee\mathcal{F}_\tau)$-measurable and the fact that $(X_t)_{\tau\le t<T}$ conditioned on $(\mathcal{F}^\sigma_{\tau,T}\vee\mathcal{F}_\tau)$ is a special case of dynamics (2.2) with time-dependent but deterministic drift and diffusion coefficients. □

Using Proposition 4.11, it is now straightforward to price the knock-out option (4.24).

Theorem 4.12 (Price of a single barrier knock-out claim)**.** *Fix $L < X_0$. Let τ_L and O_t be as defined in* (4.1) *and* (4.24)*, respectively, and let φ satisfy Assumption 3.2. Then*

$$O_t = \mathbb{I}_{\{\tau_L > t\}}\mathbb{E}[\varphi_1(X_T, \langle X\rangle_T) - \varphi_2(X_T, \langle X\rangle_T)|\mathcal{F}_t], \tag{4.27}$$

where the functions φ_1 and φ_2 are given by

$$\varphi_1(x, v) = \mathbb{I}_{\{x>L\}}\varphi(x, v), \qquad\qquad \varphi_2(x, v) = \mathbb{I}_{\{x<L\}}\mathrm{e}^{x-L}\varphi(2L - x, v). \tag{4.28}$$

Proof. First, we note that by applying (4.25) with $G = \varphi_1$ that

$$\mathbb{I}_{\{\tau_L \le T\}} \mathbb{E}[\varphi_1(X_T, \langle X \rangle_T) - \varphi_2(X_T, \langle X \rangle_T) | \mathcal{F}_{\tau_L}] = 0. \tag{4.29}$$

Next, since $\tau_L > T$ implies $X_T > L$, we observe from (4.28) that

$$\mathbb{I}_{\{\tau_L > T\}} \varphi(X_T, \langle X \rangle_T) = \mathbb{I}_{\{\tau_L > T\}} \left(\varphi_1(X_T, \langle X \rangle_T) + \varphi_2(X_T, \langle X \rangle_T)\right). \tag{4.30}$$

Now, note that

$$O_t = O_t \Big(\mathbb{I}_{\{\tau_L \le t\}} + \mathbb{I}_{\{T \ge \tau_L > t\}} + \mathbb{I}_{\{\tau_L > T\}} \Big). \tag{4.31}$$

On the set $\{\tau_L \le t\}$ we have

$$\begin{aligned} \mathbb{I}_{\{\tau_L \le t\}} O_t &= \mathbb{I}_{\{\tau_L \le t\}} \mathbb{E}[\mathbb{I}_{\{\tau_L > T\}} \varphi(X_T, \langle X \rangle_T) | \mathcal{F}_t] = 0 \\ &= \mathbb{I}_{\{\tau_L \le t\}} \Big(\mathbb{I}_{\{\tau_L > t\}} \mathbb{E}[\varphi_1(X_T, \langle X \rangle_T) - \varphi_2(X_T, \langle X \rangle_T) | \mathcal{F}_t] \Big), \end{aligned} \tag{4.32}$$

on the set $\{T \ge \tau_L > t\}$, we have

$$\begin{aligned} \mathbb{I}_{\{T \ge \tau_L > t\}} O_t &= \mathbb{I}_{\{T \ge \tau_L > t\}} \mathbb{E}[\mathbb{I}_{\{\tau_L > T\}} \varphi(X_T, \langle X \rangle_T) | \mathcal{F}_t] = 0 \\ &= \mathbb{I}_{\{T \ge \tau_L > t\}} \mathbb{E}[\mathbb{E}[\varphi_1(X_T, \langle X \rangle_T) - \varphi_2(X_T, \langle X \rangle_T) | \mathcal{F}_{\tau_L}] | \mathcal{F}_t] && \text{(by (4.29))} \\ &= \mathbb{I}_{\{T \ge \tau_L > t\}} \mathbb{E}[\varphi_1(X_T, \langle X \rangle_T) - \varphi_2(X_T, \langle X \rangle_T) | \mathcal{F}_t] \\ &= \mathbb{I}_{\{T \ge \tau_L > t\}} \Big(\mathbb{I}_{\{\tau_L > t\}} \mathbb{E}[\varphi_1(X_T, \langle X \rangle_T) - \varphi_2(X_T, \langle X \rangle_T) | \mathcal{F}_t] \Big), \end{aligned} \tag{4.33}$$

and on the set $\{\tau_L > T\}$, we have

$$\begin{aligned} \mathbb{I}_{\{\tau_L > T\}} O_t &= \mathbb{I}_{\{\tau_L > T\}} \mathbb{E}[\mathbb{I}_{\{\tau_L > T\}} \varphi(X_T, \langle X \rangle_T) | \mathcal{F}_t] \\ &= \mathbb{I}_{\{\tau_L > T\}} \mathbb{E}[\varphi(X_T, \langle X \rangle_T) | \mathcal{F}_t] \\ &= \mathbb{I}_{\{\tau_L > T\}} \mathbb{E}[\varphi_1(X_T, \langle X \rangle_T) - \varphi_2(X_T, \langle X \rangle_T) | \mathcal{F}_t] && \text{(by (4.30))} \\ &= \mathbb{I}_{\{\tau_L > T\}} \Big(\mathbb{I}_{\{\tau_L > t\}} \mathbb{E}[\varphi_1(X_T, \langle X \rangle_T) - \varphi_2(X_T, \langle X \rangle_T) | \mathcal{F}_t] \Big). \end{aligned} \tag{4.34}$$

Combining (4.31), (4.32), (4.33) and (4.34) yields (4.27). □

Theorem 4.13 (Replication of a single barrier knock-out claim)**.** *Fix $L < X_0$. Let τ_L and O_t be as defined in (4.1) and (4.24), respectively, and let φ satisfy Assumption 3.2. Let D be as given in (4.21) and define European-style claims $E^1 = (E^1_t)_{0 \le t \le T}$ and $E^2 = (E^2_t)_{0 \le t \le T}$ by*

$$E^1_t = \mathbb{E}[\varphi_1(X_T, \langle X \rangle_T) | \mathcal{F}_t], \qquad E^2_t = \mathbb{E}[\varphi_2(X_T, \langle X \rangle_T) | \mathcal{F}_t], \tag{4.35}$$

where φ_1 and φ_2 are given in (4.28). Then

$$O_t = (1 - D_t)(E^1_t - E^2_t) \tag{4.36}$$

and the portfolio (4.36) is self-financing portfolio in the sense of (3.17). That is,

$$\mathrm{d}O_t = (1 - D_{t-})(\mathrm{d}E^1_t - \mathrm{d}E^2_t). \tag{4.37}$$

Proof. Expression (4.36) follows directly from (4.21) and (4.27) and (4.35). Moreover, from (4.29) and (4.35) we have $E^1_{\tau_L} = E^2_{\tau_L}$. The self-financing property (4.37) follows from (4.17). $\square$

Remark 4.14. From (4.37) we see that the hedging portfolio for the knock-out option O involves a static long position in European-style option E^1 and a static short position in European-style option E^2. Since the payoffs of E^1 and E^2 depend on X_T and $\langle X \rangle_T$, these options can dynamically replicated by trading the bond B, the stock S, and European claims on X, as described in Section 3.

Remark 4.15 (Knock-in claims (II))**.** In Section 4.1 we showed how to price and hedge a knock-in claim I of the form (4.2). Now, using the results of Section 4.2, we can price and hedge a knock-in claim of a different form. Specifically, consider the following payoff: $\mathbb{I}_{\{\tau_L \leq T\}} \varphi(X_T, \langle X \rangle_T)$. We have

$$\mathbb{I}_{\{\tau_L \leq T\}} \varphi(X_T, \langle X \rangle_T) = \varphi(X_T, \langle X \rangle_T) - \mathbb{I}_{\{\tau_L > T\}} \varphi(X_T, \langle X \rangle_T). \tag{4.38}$$

The right-hand side of (4.38) is the difference of a European-style payoff and a knock-out payoff. Pricing and hedging results for the former are treated in Section 3 while pricing and hedging results for the latter are given in the current section.

4.3 Single barrier rebate claims

In this section, we will consider a claim with a payoff of the form $\mathbb{I}_{\{\tau_L \leq T\}} \varphi(\langle X \rangle_{\tau_L})$. We will refer to such a claim as *rebate claim*, since it pays a rebate $\varphi(\langle X \rangle_{\tau_L})$ to the option holder if X hits a level L prior to the maturity date T. Throughout this paper, we will denote by $R = (R_t)_{0 \leq t \leq T}$ the price process of the any rebate claim (R stands for "rebate"). Using risk-neutral pricing, we have

$$R_t = \mathbb{E}[\mathbb{I}_{\{\tau_L \leq T\}} \varphi(\langle X \rangle_{\tau_L}) | \mathcal{F}_t], \qquad t \in [0, T]. \tag{4.39}$$

Our goals are to (i) find the value R_t relative to T-maturity options on X and (ii) provide a hedging strategy that replicates the option payoff. We begin with a short lemma.

Lemma 4.16. *Define a function $v : \mathbb{C} \to \mathbb{C}$ and a stochastic process $M \equiv (M_t)_{0 \leq t \leq T}$ by*

$$v \equiv v(s) = \mathtt{i}\Big(-\tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} - 2\mathtt{i}s} \Big), \qquad M_t \equiv M_t(s) = \mathrm{e}^{\mathtt{i}vX_t + \mathtt{i}s\langle X \rangle_t}, \qquad t \in [0, T]. \tag{4.40}$$

Then M is a martingale.

Proof. Using (2.1), one can show by direct computation that $\mathbb{E}|M_t| < \infty$ for all $t < \infty$. Thus, we need only to show that M satisfies the martingale property. Using (4.40), we compute

$$\begin{aligned}
\mathrm{d}M_t &= M_t(\mathtt{i}v\mathrm{d}X_t + \mathtt{i}s\mathrm{d}\langle X \rangle_t) - \tfrac{1}{2}v^2 M_t \mathrm{d}\langle X \rangle_t \\
&= M_t \left(\mathtt{i}v \left(\frac{\mathrm{d}S_t}{S_t} - \frac{1}{2}\mathrm{d}\langle X \rangle_t \right) + \mathtt{i}s\mathrm{d}\langle X \rangle_t \right) - \tfrac{1}{2}v^2 M_t \mathrm{d}\langle X \rangle_t \\
&= \frac{\mathtt{i}vM_t}{S_t}\mathrm{d}S_t + \left(-\tfrac{1}{2}\mathtt{i}v + \mathtt{i}s - \tfrac{1}{2}v^2\right) M_t \mathrm{d}\langle X \rangle_t \\
&= \frac{\mathtt{i}vM_t}{S_t}\mathrm{d}S_t,
\end{aligned}$$

where we have used $\left(-\tfrac{1}{2}\mathtt{i}v + \mathtt{i}s - \tfrac{1}{2}v^2\right) = 0$. Since S is a martingale, it follows that M is a martingale. $\square$

Using the above lemma, we can relate the value of an rebate claim with an exponential payoff to the value of a European-style option and a knock-out option.

Proposition 4.17. *Let τ_L and v be as defined in (4.1) and (4.40), respectively. Then*

$$\begin{aligned}&\mathbb{I}_{\{\tau_L>t\}}\mathbb{E}[\mathbb{I}_{\{\tau_L\le T\}}\mathrm{e}^{\mathrm{i}s\langle X\rangle_{\tau_L}}|\mathcal{F}_t]\\&=\mathbb{I}_{\{\tau_L>t\}}\mathrm{e}^{-\mathrm{i}vL}\left(\mathbb{E}[\mathrm{e}^{\mathrm{i}vX_T+\mathrm{i}s\langle X\rangle_T}|\mathcal{F}_t]-\mathbb{E}[\mathbb{I}_{\{\tau_L>T\}}\mathrm{e}^{\mathrm{i}vX_T+\mathrm{i}s\langle X\rangle_T}|\mathcal{F}_t]\right).\end{aligned}\tag{4.41}$$

Proof. Note that $\{\tau_L > t\} = \{\tau_L > T\} \cup \{T \ge \tau_L > t\}$. On the set $\{\tau_L > T\}$ we have

$$\begin{aligned}&\mathbb{I}_{\{\tau_L>T\}}\mathrm{e}^{-\mathrm{i}vL}\left(\mathbb{E}[\mathrm{e}^{\mathrm{i}vX_T+\mathrm{i}s\langle X\rangle_T}|\mathcal{F}_t]-\mathbb{E}[\mathbb{I}_{\{\tau_L>T\}}\mathrm{e}^{\mathrm{i}vX_T+\mathrm{i}s\langle X\rangle_T}|\mathcal{F}_t]\right)\\&=\mathbb{I}_{\{\tau_L>T\}}\mathrm{e}^{-\mathrm{i}vL}\left(\mathbb{E}[\mathrm{e}^{\mathrm{i}vX_T+\mathrm{i}s\langle X\rangle_T}|\mathcal{F}_t]-\mathbb{E}[\mathrm{e}^{\mathrm{i}vX_T+\mathrm{i}s\langle X\rangle_T}|\mathcal{F}_t]\right)=0\\&=\mathbb{I}_{\{\tau_L>T\}}\mathbb{E}[\mathbb{I}_{\{\tau_L\le T\}}\mathrm{e}^{\mathrm{i}s\langle X\rangle_{\tau_L}}|\mathcal{F}_t],\end{aligned}\tag{4.42}$$

where, in the last step, we have used $\{\tau_L > T\} \cap \{\tau_L \le T\} = \emptyset$. Next, on the set $\{T \ge \tau_L > t\}$ we have

$$\begin{aligned}&\mathbb{I}_{\{T\ge\tau_L>t\}}\mathrm{e}^{-\mathrm{i}vL}\left(\mathbb{E}[\mathrm{e}^{\mathrm{i}vX_T+\mathrm{i}s\langle X\rangle_T}|\mathcal{F}_t]-\mathbb{E}[\mathbb{I}_{\{\tau_L>T\}}\mathrm{e}^{\mathrm{i}vX_T+\mathrm{i}s\langle X\rangle_T}|\mathcal{F}_t]\right)\\&=\mathbb{I}_{\{T\ge\tau_L>t\}}\mathrm{e}^{-\mathrm{i}vL}\mathbb{E}[\mathrm{e}^{\mathrm{i}vX_T+\mathrm{i}s\langle X\rangle_T}|\mathcal{F}_t] && (\text{as } \mathbb{I}_{\{T\ge\tau_L>t\}}\mathbb{I}_{\{\tau_L>T\}}=0)\\&=\mathbb{I}_{\{T\ge\tau_L>t\}}\mathrm{e}^{-\mathrm{i}vL}\mathbb{E}[\mathbb{E}[M_T|\mathcal{F}_{\tau_L}]|\mathcal{F}_t] && (\text{by (4.40)})\\&=\mathbb{I}_{\{T\ge\tau_L>t\}}\mathrm{e}^{-\mathrm{i}vL}\mathbb{E}[M_{\tau_L}|\mathcal{F}_t] && (M \text{ is a martingale})\\&=\mathbb{I}_{\{T\ge\tau_L>t\}}\mathrm{e}^{-\mathrm{i}vL}\mathbb{E}[\mathrm{e}^{\mathrm{i}vL+\mathrm{i}s\langle X\rangle_{\tau_L}}|\mathcal{F}_t] && (\text{by (4.40)})\\&=\mathbb{I}_{\{T\ge\tau_L>t\}}\mathbb{E}[\mathbb{I}_{\{\tau_L\le T\}}\mathrm{e}^{\mathrm{i}s\langle X\rangle_{\tau_L}}|\mathcal{F}_t],\end{aligned}\tag{4.43}$$

where, in the last step, we have used the fact that $T \ge \tau_L > t$ implies $\mathbb{I}_{\{\tau_L\le T\}} = 1$. Equation (4.41) follows from (4.42) and (4.43). □

Theorem 4.18 (Price and replication of a single barrier rebate claim)**.** *Fix $L < X_0$. Let τ_L and R_t be as defined in (4.1) and (4.39), respectively. Assume φ satisfies Assumption 3.2 and define a European option E and knock-out option O by*

$$E_t=\mathbb{E}[\varphi_1(X_T,\langle X\rangle_T)|\mathcal{F}_t], \qquad O_t=\mathbb{E}[\mathbb{I}_{\{\tau_L>T\}}\varphi_1(X_T,\langle X\rangle_T)|\mathcal{F}_t], \qquad t\in[0,T],\tag{4.44}$$

where the payoff function φ_1 is given by

$$\varphi_1(x,w)=\int_{\mathbb{R}}\mathrm{d}s_r\widehat{\varphi}(s)\mathrm{e}^{-\mathrm{i}v(s)L}\mathrm{e}^{\mathrm{i}v(s)x+\mathrm{i}sw},\tag{4.45}$$

with $v(s)$ as given in (4.40). Here, $\widehat{\varphi}$ is the one-dimensional Fourier transform of φ. Then, with D as defined in (4.21), the price of the rebate claim is given by

$$R_t=(1-D_t)(E_t-O_t)+D_t\Phi_t, \qquad \Phi_t=\mathbb{I}_{\{\tau_L\le t\}}\varphi(\langle X\rangle_{\tau_L}).\tag{4.46}$$

Additionally, the portfolio (4.46) is self-financing

$$\mathrm{d}R_t=(1-D_{t-})(\mathrm{d}E_t-\mathrm{d}O_t)+D_{t-}\mathrm{d}\Phi_t.\tag{4.47}$$

Proof. We examine the value of R_t on two sets: $\{\tau_L > t\}$ and $\{\tau_L \leq t\}$. On the set $\{\tau_L > t\}$ we have

$$
\begin{aligned}
\mathbb{I}_{\{\tau_L>t\}} R_t &= \mathbb{I}_{\{\tau_L>t\}} \mathbb{E}[\mathbb{I}_{\{\tau_L\leq T\}} \varphi(\langle X\rangle_{\tau_L})|\mathcal{F}_t] && \text{by ((4.39))} \\
&= \mathbb{I}_{\{\tau_L>t\}} \mathbb{E}[\mathbb{I}_{\{\tau_L\leq T\}} \mathbf{F}^{-1}[\widehat{\varphi}](\langle X\rangle_{\tau_L})|\mathcal{F}_t] && \text{(by (3.8))} \\
&= \mathbb{I}_{\{\tau_L>t\}} \int_{\mathbb{R}} \mathrm{d}s_r \widehat{\varphi}(s) \mathbb{E}[\mathbb{I}_{\{\tau_L\leq T\}} \mathrm{e}^{\mathtt{i}s\langle X\rangle_{\tau_L}}|\mathcal{F}_t] && \text{(by (3.7))} \\
&= \mathbb{I}_{\{\tau_L>t\}} \int_{\mathbb{R}} \mathrm{d}s_r \widehat{\varphi}(s) \mathrm{e}^{-\mathtt{i}v(s)L} \mathbb{E}[\mathrm{e}^{\mathtt{i}v(s)X_T+\mathtt{i}s\langle X\rangle_T}|\mathcal{F}_t] \\
&\quad - \mathbb{I}_{\{\tau_L>t\}} \int_{\mathbb{R}} \mathrm{d}s_r \widehat{\varphi}(s) \mathrm{e}^{-\mathtt{i}v(s)L} \mathbb{E}[\mathbb{I}_{\{\tau_L>T\}} \mathrm{e}^{\mathtt{i}v(s)X_T+\mathtt{i}s\langle X\rangle_T}|\mathcal{F}_t] && \text{(by (4.41))} \\
&= \mathbb{I}_{\{\tau_L>t\}} \big(\mathbb{E}[\varphi_1(X_T,\langle X\rangle_T)|\mathcal{F}_t] - \mathbb{E}[\mathbb{I}_{\{\tau_L>T\}}\varphi_1(X_T,\langle X\rangle_T)|\mathcal{F}_t]\big) && \text{(by (4.45))} \\
&= (1-D_t)(E_t - O_t) && \text{(by (4.21) and (4.44))} \\
&= \mathbb{I}_{\{\tau_L>t\}} \Big((1-D_t)(E_t-O_t) + D_t\Phi_t\Big). && \text{(by (4.21))} \qquad (4.48)
\end{aligned}
$$

Next, on the set $\{\tau_L \leq t\}$ we have, trivially

$$
\begin{aligned}
\mathbb{I}_{\{\tau_L\leq t\}} R_t &= \mathbb{I}_{\{\tau_L\leq t\}} \mathbb{E}[\mathbb{I}_{\{\tau_L\leq T\}} \varphi(\langle X\rangle_{\tau_L})|\mathcal{F}_t] = D_t \varphi(\langle X\rangle_{\tau_L}) = D_t \Phi_t \\
&= \mathbb{I}_{\{\tau_L\leq t\}} \Big((1-D_t)(E_t-O_t) + D_t\Phi_t\Big).
\end{aligned} \qquad (4.49)
$$

where, in the second equality, we used the definition (4.21) of D, and the fact that $\tau_L \leq t$ implies $\mathbb{I}_{\{\tau_L\leq T\}} = 1$ and $\varphi(\langle X\rangle_{\tau_L}) \in \mathcal{F}_t$. Equation (4.46) follows from (4.48) and (4.49).

To establish the self-financing property (4.47), we must show, from (4.17) that $\mathbb{I}_{\{\tau_L\leq T\}}(E_{\tau_L} - O_{\tau_L}) = \mathbb{I}_{\{\tau_L\leq T\}}\Phi_{\tau_L}$. Note that

$$
\begin{aligned}
\mathbb{I}_{\{\tau_L\leq T\}} O_{\tau_L} &= 0, && \text{(by (4.44))} \\
\mathbb{I}_{\{\tau_L\leq T\}} E_{\tau_L} &= \mathbb{I}_{\{\tau_L\leq T\}} \mathbb{E}[\varphi_1(X_T,\langle X\rangle_T)|\mathcal{F}_{\tau_L}] && \text{(by (4.44))} \\
&= \mathbb{I}_{\{\tau_L\leq T\}} \int_{\mathbb{R}} \mathrm{d}s \widehat{\varphi}(s) \mathrm{e}^{-\mathtt{i}v(s)L} \mathbb{E}[\mathrm{e}^{\mathtt{i}v(s)X_T+\mathtt{i}s\langle X\rangle_T}|\mathcal{F}_{\tau_L}] && \text{(by (4.45))} \\
&= \mathbb{I}_{\{\tau_L\leq T\}} \int_{\mathbb{R}} \mathrm{d}s \widehat{\varphi}(s) \mathrm{e}^{-\mathtt{i}v(s)L} \mathbb{E}[M_T(s)|\mathcal{F}_{\tau_L}] && \text{(by (4.40))} \\
&= \mathbb{I}_{\{\tau_L\leq T\}} \int_{\mathbb{R}} \mathrm{d}s \widehat{\varphi}(s) \mathrm{e}^{-\mathtt{i}v(s)L} M_{\tau_L}(s) && \text{(M is a martingale)} \\
&= \mathbb{I}_{\{\tau_L\leq T\}} \int_{\mathbb{R}} \mathrm{d}s \widehat{\varphi}(s) \mathrm{e}^{-\mathtt{i}s\langle X\rangle_{\tau_L}} && \text{(by (4.40))} \\
&= \mathbb{I}_{\{\tau_L\leq T\}} \varphi(\langle X\rangle_{\tau_L}). && \text{(by (3.7) and (3.8))} \\
&= \mathbb{I}_{\{\tau_L\leq T\}} \Phi_{\tau_L}. && \text{(by (4.46))}
\end{aligned}
$$

Thus, $\mathbb{I}_{\{\tau_L\leq T\}}(E_{\tau_L} - O_{\tau_L}) = \mathbb{I}_{\{\tau_L\leq T\}}\Phi_{\tau_L}$, and we conclude that the hedging portfolio is self-financing. □

Remark 4.19. Although the hedging strategy (4.47) involves a static long position in a European claim and a static short position in a knock-out option, the payoffs of these options depend on $\langle X\rangle_T$. As such, hedging with a bond B, the underlying stock S, and European calls and puts involves dynamic trading.

5 Double barrier claims

In this section, we will show how to price and hedge three kinds of double barrier claims: knock-out, knock-in and rebate. To this end, for any $L, U \in \mathbb{R}$, we define

$$\tau_{L,U} := \tau_L \wedge \tau_U. \tag{5.1}$$

5.1 Double barrier knock-out claims

We will first consider a double barrier knock-out claim, whose payoff is of the form $\mathbb{I}_{\{\tau_{L,U} > T\}} \varphi(X_T, \langle X \rangle_T)$, where we assume $L < X_0 < U$. Let $O = (O_t)_{0 \le t \le T}$ denote the price process of a claim with this payoff. Using risk-neutral valuation, we have

$$O_t = \mathbb{E}[\mathbb{I}_{\{\tau_{L,U} > T\}} \varphi(X_T, \langle X \rangle_T) | \mathcal{F}_t]. \tag{5.2}$$

Our analysis of the knock-out claim (5.2) begins with the following proposition.

Proposition 5.1. *Fix $L < X_0 < U$ and let $G : (L, U) \times \mathbb{R}_+ \to \mathbb{R}$ be bounded. Define*

$$H(x, v) := \sum_{n=-\infty}^{\infty} \left(\mathrm{e}^{n(L-U)} G^*(2n(U - L) + x, v) - \mathrm{e}^{n(L-U)+x-L} G^*(2n(U - L) + 2L - x, v) \right),$$
$$G^*(x, v) := G(x, v) \mathbb{I}_{\{L < x < U\}}.$$

Then, with $\tau_{L,U}$ as defined in (5.1), we have

$$H(x, v) \mathbb{I}_{\{L < x < U\}} = G(x, v), \qquad \mathbb{I}_{\{\tau_{L,U} \le T\}} \mathbb{E}[H(X_T, \langle X \rangle_T) | \mathcal{F}_{\tau_{L,U}}] = 0. \tag{5.3}$$

Proof. One can easily verify that $H(x, v) \mathbb{I}_{\{L < x < U\}} = G(x, v)$ by inspection. Next, to establish that the conditional expectation in (5.3) is equal to zero, we have from [CL09, Theorem 5.18] that

$$\mathbb{I}_{\{\tau_{L,U} \le T\}} \mathbb{E}[H(X_T, v) | \mathcal{F}_{\tau_{L,U}}] = 0,$$

which holds for any fixed $v \in \mathbb{R}_+$. Using this result, we compute

$$\mathbb{I}_{\{\tau_{L,U} \le T\}} \mathbb{E}[H(X_T, \langle X \rangle_T) | \mathcal{F}_{\tau_{L,U}}] = \mathbb{I}_{\{\tau_{L,U} \le T\}} \mathbb{E}[\mathbb{E}[H(X_T, \langle X \rangle_T) | \mathcal{F}^{\sigma}_{\tau_{L,U},T} \vee \mathcal{F}_{\tau_{L,U}}] | \mathcal{F}_{\tau_{L,U}}] = 0.$$

where we have used the fact that $\langle X \rangle_T$ is $(\mathcal{F}^{\sigma}_{\tau_{L,U},T} \vee \mathcal{F}_{\tau_{L,U}})$-measurable and the fact that $(X_t)_{\tau_{L,U} \le t < T}$ conditioned on $(\mathcal{F}^{\sigma}_{\tau_{L,U},T} \vee \mathcal{F}_{\tau_{L,U}})$ is a special case of dynamics (2.2) with time-dependent (but deterministic) drift and diffusion coefficients. □

Theorem 5.2 (Price of a double barrier knock-out claim)**.** *Let O_t be given by (5.2) and suppose the payoff function φ is bounded. Then*

$$O_t = \mathbb{I}_{\{\tau_{L,U} > t\}} \mathbb{E}[\varphi_1(X_T, \langle X \rangle_T) | \mathcal{F}_t], \tag{5.4}$$

where

$$\varphi_1(x, v) = \sum_{n=-\infty}^{\infty} \left(\mathrm{e}^{n(L-U)} \varphi^*(2n(U - L) + x, v) - \mathrm{e}^{n(L-U)+x-L} \varphi^*(2n(U - L) + 2L - x, v) \right), \tag{5.5}$$
$$\varphi^*(x, v) := \varphi(x, v) \mathbb{I}_{\{L < x < U\}}$$

Proof. First, by applying Proposition 5.1 with $G = \varphi$, we have $H = \varphi_1$. Thus, by (5.3), we have

$$\mathbb{I}_{\{\tau_{L,U} \le T\}} \mathbb{E}[\varphi_1(X_T, \langle X \rangle_T) | \mathcal{F}_{\tau_{L,U}}] = 0. \tag{5.6}$$

Since $\tau_{L,U} > T$ implies $U < X_T < L$, we also have from (5.3) that

$$\mathbb{I}_{\{\tau_{L,U} > T\}} \varphi(X_T, \langle X \rangle_T) = \mathbb{I}_{\{\tau_{L,U} > T\}} \varphi_1(X_T, \langle X \rangle_T). \tag{5.7}$$

Now, note that

$$O_t = O_t \Big(\mathbb{I}_{\{\tau_{L,U} \le t\}} + \mathbb{I}_{\{T \ge \tau_{L,U} > t\}} + \mathbb{I}_{\{\tau_{L,U} > T\}} \Big). \tag{5.8}$$

On the set $\{\tau_{L,U} \le t\}$ we have

$$\begin{aligned}
\mathbb{I}_{\{\tau_{L,U} \le t\}} O_t &= \mathbb{I}_{\{\tau_{L,U} \le t\}} \mathbb{E}[\mathbb{I}_{\{\tau_{L,U} > T\}} \varphi(X_T, \langle X \rangle_T) | \mathcal{F}_t] && \text{(by (5.2)}\\
&= 0 && \text{(as } \mathbb{I}_{\{\tau_{L,U} \le t\}} \mathbb{I}_{\{\tau_{L,U} > T\}} = 0)\\
&= \mathbb{I}_{\{\tau_{L,U} \le t\}} \Big(\mathbb{I}_{\{\tau_{L,U} > t\}} \mathbb{E}[\varphi_1(X_T, \langle X \rangle_T) | \mathcal{F}_t] \Big), && \text{(as } \mathbb{I}_{\{\tau_{L,U} \le t\}} \mathbb{I}_{\{\tau_{L,U} > t\}} = 0)
\end{aligned} \tag{5.9}$$

on the set $\{T \ge \tau_{L,U} > t\}$, we have

$$\begin{aligned}
\mathbb{I}_{\{T \ge \tau_{L,U} > t\}} O_t &= \mathbb{I}_{\{T \ge \tau_{L,U} > t\}} \mathbb{E}[\mathbb{I}_{\{\tau_{L,U} > T\}} \varphi(X_T, \langle X \rangle_T) | \mathcal{F}_t] && \text{(by (5.2)}\\
&= 0 && \text{(as } \mathbb{I}_{\{T \ge \tau_{L,U} > t\}} \mathbb{I}_{\{\tau_{L,U} > T\}} = 0)\\
&= \mathbb{I}_{\{T \ge \tau_{L,U} > t\}} \mathbb{E}[\mathbb{I}_{\{\tau_{L,U} \le T\}} \mathbb{E}[\varphi_1(X_T, \langle X \rangle_T) | \mathcal{F}_{\tau_{L,U}}] | \mathcal{F}_t] && \text{(by (5.6))}\\
&= \mathbb{I}_{\{T \ge \tau_{L,U} > t\}} \mathbb{E}[\varphi_1(X_T, \langle X \rangle_T) | \mathcal{F}_t]\\
&= \mathbb{I}_{\{T \ge \tau_{L,U} > t\}} \Big(\mathbb{I}_{\{\tau_{L,U} > t\}} \mathbb{E}[\varphi_1(X_T, \langle X \rangle_T) | \mathcal{F}_t] \Big),
\end{aligned} \tag{5.10}$$

and on the set $\{\tau_{L,U} > T\}$, we have

$$\begin{aligned}
\mathbb{I}_{\{\tau_{L,U} > T\}} O_t &= \mathbb{I}_{\{\tau_{L,U} > T\}} \mathbb{E}[\mathbb{I}_{\{\tau_{L,U} > T\}} \varphi(X_T, \langle X \rangle_T) | \mathcal{F}_t] && \text{(by (5.2)}\\
&= \mathbb{I}_{\{\tau_{L,U} > T\}} \mathbb{E}[\varphi_1(X_T, \langle X \rangle_T) | \mathcal{F}_t] && \text{(by (5.7))}\\
&= \mathbb{I}_{\{\tau_{L,U} > T\}} \Big(\mathbb{I}_{\{\tau_{L,U} > t\}} \mathbb{E}[\varphi_1(X_T, \langle X \rangle_T) | \mathcal{F}_t] \Big).
\end{aligned} \tag{5.11}$$

Combining (5.8), (5.9), (5.10) and (5.11) yields (5.4). □

Theorem 5.3 (Replication of a double barrier knock-out claim)**.** *Fix $L < X_0 < U$. Let $\tau_{L,U}$ and O_t be as defined in (5.1) and (5.2), respectively, and let φ be bounded. Define the hitting time indicator process $D = (D_t)_{0 \le t \le T}$ and a European-style claims $E^1 = (E^1_t)_{0 \le t \le T}$ by*

$$D_t = \mathbb{I}_{\{\tau_{L,U} \le t\}}, \qquad E^1_t = \mathbb{E}[\varphi_1(X_T, \langle X \rangle_T) | \mathcal{F}_t], \tag{5.12}$$

where φ_1 is given in (5.5). Then

$$O_t = (1 - D_t) E^1_t \tag{5.13}$$

and the portfolio (5.13) is self-financing portfolio in the sense of (3.17). That is,

$$\mathrm{d}O_t = (1 - D_{t-}) \mathrm{d}E^1_t. \tag{5.14}$$

Proof. Expression (5.13) follows from (5.4) and (5.12). The self-financing property (5.14) will follow from (4.17) if we show that $\mathbb{I}_{\{\tau_{L,U}\leq T\}}E^1_{\tau_{L,U}} = 0$. We have

$$\mathbb{I}_{\{\tau_{L,U}\leq T\}}E^1_{\tau_{L,U}} = \mathbb{I}_{\{\tau_{L,U}\leq T\}}\mathbb{E}[\varphi_1(X_T, \langle X\rangle_T)|\mathcal{F}_{\tau_{L,U}}] = 0,$$

where we have used (5.6). □

5.2 Double barrier knock-in claims

We now examine a double barrier knock-in claim, whose payoff is of the form $\mathbb{I}_{\{\tau_{L,U}\leq T\}}\varphi(X_T, \langle X\rangle_T - \langle X\rangle_{\tau_{L,U}})$, where we assume $L < X_0 < U$. Let $I = (I_t)_{0\leq t\leq T}$ denote the price process of a claim with this payoff. Using risk-neutral pricing, we have

$$I_t = \mathbb{E}[\mathbb{I}_{\{\tau_{L,U}\leq T\}}\varphi(X_T, \langle X\rangle_T - \langle X\rangle_{\tau_{L,U}})|\mathcal{F}_t], \tag{5.15}$$

Our analysis of the knock-in claim (5.15) begins with the following proposition.

Proposition 5.4. *Fix $L < X_0 < U$. Suppose $G_L : \mathbb{R}\times\mathbb{R}^+ \to \mathbb{C}$ and $G_U : \mathbb{R}\times\mathbb{R}^+ \to \mathbb{C}$ are bounded. Define $H : \mathbb{R}\times\mathbb{R}^+ \to \mathbb{C}$ by*

$$\begin{aligned} H(x,v) = G_L(x,v) &- \sum_{n=1}^{\infty} \mathrm{e}^{n(L-U)}\Big(G_U(2n(U-L)+x,v) - G_L(2n(U-L)+x,v)\Big) \\ &+ \sum_{n=1}^{\infty} \mathrm{e}^{n(L-U)+x-L}\Big(G_U(2n(U-L)+2L-x,v) - G_L(2n(U-L)+2L-x,v)\Big). \end{aligned} \tag{5.16}$$

Then, with τ_L and τ_U given by (4.1), we have

$$\mathbb{I}_{\{\tau_L\leq T\}}\mathbb{E}[H(X_T,\langle X\rangle_T)|\mathcal{F}_{\tau_L}] = \mathbb{I}_{\{\tau_L\leq T\}}\mathbb{E}[G_L(X_T,\langle X\rangle_T)|\mathcal{F}_{\tau_L}], \tag{5.17}$$

$$\mathbb{I}_{\{\tau_U\leq T\}}\mathbb{E}[H(X_T,\langle X\rangle_T)|\mathcal{F}_{\tau_U}] = \mathbb{I}_{\{\tau_U\leq T\}}\mathbb{E}[G_U(X_T,\langle X\rangle_T)|\mathcal{F}_{\tau_U}]. \tag{5.18}$$

Proof. The proof is given in Appendix A. □

Proposition 5.5. *Fix $L < X_0 < R$ and $\omega, s \in \mathbb{C}$. Let $u \equiv u(\omega,s)$, $\psi_L(x) \equiv \psi_L(x;\omega,s)$ and $\psi_U(x) \equiv \psi_U(x;\omega,s)$ be as given in (3.2), (4.3) and (4.10), respectively. Define $\phi(x) \equiv \phi(x;\omega,s)$ and $\psi_{L,U}(x) \equiv \psi_{L,U}(x;\omega,s)$ by*

$$\begin{aligned} \phi(x) = \psi_U(x) &- \sum_{n=1}^{\infty} \mathrm{e}^{n(L-U)}\Big(\psi_L(2n(U-L)+x) - \psi_U(2n(U-L)+x)\Big) \\ &+ \sum_{n=1}^{\infty} \mathrm{e}^{n(L-U)+x-L}\Big(\psi_L(2n(U-L)+2L-x) - \psi_U(2n(U-L)+2L-x)\Big). \\ \psi_{L,U}(x) &= \psi_L(x) + \psi_U(x) - \phi(x). \end{aligned} \tag{5.19}$$

Then, with $\tau_{L,U}$ as given in (5.1) we have

$$\mathbb{I}_{\{\tau_{L,U}\leq T\}}\mathbb{E}[\psi_{L,U}(X_T)|\mathcal{F}_{\tau_{L,U}}] = \mathbb{I}_{\{\tau_{L,U}\leq T\}}\mathbb{E}[\mathrm{e}^{\mathrm{i}\omega X_T + \mathrm{i}s(\langle X\rangle_T - \langle X\rangle_{\tau_{L,U}})}|\mathcal{F}_{\tau_{L,U}}]. \tag{5.20}$$

Proof. Setting $G_L = \psi_U$ and $G_U = \psi_L$ in (5.16) we obtain $H = \phi$. Thus, from (5.17) and (5.18) we have

$$\mathbb{I}_{\{\tau_L \le T\}} \mathbb{E}[\phi(X_T)|\mathcal{F}_{\tau_L}] = \mathbb{I}_{\{\tau_L \le T\}} \mathbb{E}[\psi_U(X_T)|\mathcal{F}_{\tau_L}], \tag{5.21}$$

$$\mathbb{I}_{\{\tau_U \le T\}} \mathbb{E}[\phi(X_T)|\mathcal{F}_{\tau_U}] = \mathbb{I}_{\{\tau_U \le T\}} \mathbb{E}[\psi_L(X_T)|\mathcal{F}_{\tau_U}]. \tag{5.22}$$

Hence, from (4.4), (5.19) and (5.21) we have

$$\mathbb{I}_{\{\tau_L \le T\}} \mathbb{E}[\psi_{L,U}(X_T)|\mathcal{F}_{\tau_L}] = \mathbb{I}_{\{\tau_L \le T\}} \mathbb{E}[\mathrm{e}^{\mathrm{i}\omega X_T + \mathrm{i}s(\langle X\rangle_T - \langle X\rangle_{\tau_L})}|\mathcal{F}_{\tau_L}].$$

and from (4.11), (5.19) and (5.22) we have

$$\mathbb{I}_{\{\tau_U \le T\}} \mathbb{E}[\psi_{L,U}(X_T)|\mathcal{F}_{\tau_U}] = \mathbb{I}_{\{\tau_U \le T\}} \mathbb{E}[\mathrm{e}^{\mathrm{i}\omega X_T + \mathrm{i}s(\langle X\rangle_T - \langle X\rangle_{\tau_U})}|\mathcal{F}_{\tau_U}].$$

Since $\tau_{L,U} = \tau_L \wedge \tau_U$, this proves the proposition. □

Theorem 5.6 (Price of a double barrier knock-in claim)**.** *Fix $L < X_0 < U$. Let I_t be as given in (5.15) and let φ satisfy Assumption 3.2. Then, with $\tau_{L,U}$ as defined in (5.1), we have*

$$\begin{aligned} I_t &= \mathbb{I}_{\{\tau_{L,U} > t\}} \left(\mathbb{E}[\Psi_{L,U}(X_T)|\mathcal{F}_t] - \mathbb{E}[\mathbb{I}_{\{\tau_{L,U} > T\}} \Psi_{L,U}(X_T)|\mathcal{F}_t] \right) \\ &\quad + \mathbb{I}_{\{\tau_{L,U} \le t\}} \mathbb{E}[\varphi_{\tau_{L,U}}(X_T, \langle X\rangle_T)|\mathcal{F}_t], \end{aligned} \tag{5.23}$$

where $\varphi_{\tau_{L,U}}$, which is an $\mathcal{F}_t$-measurable function on the set $\{\tau_{L,U} \le t\}$, and $\Psi_{L,U}$ are given by

$$\varphi_{\tau_{L,U}}(x, v) = \varphi(x, v - \langle X\rangle_{\tau_{L,U}}), \tag{5.24}$$

$$\Psi_{L,U}(x) = \int_{\mathbb{R}^2} \mathrm{d}\omega_r \mathrm{d}s_r \widehat{\varphi}(\omega, s) \psi_{L,U}(x; \omega, s), \tag{5.25}$$

with $\psi_{L,U}(x;\omega,s)$ as defined in (5.19).

Proof. We begin by noting that

$$I_t = I_t \left(\mathbb{I}_{\{\tau_{L,U} > T\}} + \mathbb{I}_{\{T \ge \tau_{L,U} > t\}} + \mathbb{I}_{\{\tau_{L,U} \le t\}} \right). \tag{5.26}$$

On the set $\{\tau_{L,U} > T\}$ we have

$$\begin{aligned} &\mathbb{I}_{\{\tau_{L,U} > T\}} I_t \\ &= \mathbb{I}_{\{\tau_{L,U} > T\}} \mathbb{E}[\mathbb{I}_{\{\tau_{L,U} \le T\}} \varphi(X_T, \langle X\rangle_T - \langle X\rangle_{\tau_{L,U}})|\mathcal{F}_t] && \text{(by (5.15))} \\ &= 0 && (\text{as } \mathbb{I}_{\{\tau_{L,U} > T\}} \mathbb{I}_{\{\tau_{L,U} \le T\}} = 0) \\ &= \mathbb{I}_{\{\tau_{L,U} > T\}} \mathbb{I}_{\{\tau_{L,U} > t\}} \left(\mathbb{E}[\Psi_{L,U}(X_T)|\mathcal{F}_t] - \mathbb{E}[\mathbb{I}_{\{\tau_{L,U} > T\}} \Psi_{L,U}(X_T)|\mathcal{F}_t] \right) \\ &\quad + \mathbb{I}_{\{\tau_{L,U} > T\}} \left(\mathbb{I}_{\{\tau_{L,U} \le t\}} \mathbb{E}[\varphi_{\tau_{L,U}}(X_T, \langle X\rangle_T)|\mathcal{F}_t] \right). && (\text{as } \mathbb{I}_{\{\tau_{L,U} > T\}} \mathbb{I}_{\{\tau_{L,U} \le t\}} = 0) \end{aligned} \tag{5.27}$$

Next, on the set $\{T \ge \tau_{L,U} > t\}$ we have

$$\begin{aligned} &\mathbb{I}_{\{T \ge \tau_{L,U} > t\}} I_t \\ &= \mathbb{I}_{\{T \ge \tau_{L,U} > t\}} \mathbb{E}[\mathbb{I}_{\{\tau_{L,U} \le T\}} \varphi(X_T, \langle X\rangle_T - \langle X\rangle_{\tau_{L,U}})|\mathcal{F}_t] && \text{(by (5.15))} \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}[\mathbb{I}_{\{\tau_{L,U}\le T\}}\mathbf{F}^{-1}[\widehat{\varphi}](X_T, \langle X\rangle_T - \langle X\rangle_{\tau_{L,U}})|\mathcal{F}_t] && \text{(by (3.8))}\\
&= \mathbb{I}_{\{T\ge\tau_{L,U}>t\}}\int_{\mathbb{R}^2} \mathrm{d}\omega_r \mathrm{d}s_r \widehat{\varphi}(\omega,s)\mathbb{E}[\mathbb{I}_{\{\tau_{L,U}\le T\}}\mathrm{e}^{\mathrm{i}\omega X_T + \mathrm{i}s(\langle X\rangle_T - \langle X\rangle_{\tau_{L,U}})}|\mathcal{F}_t] && \text{(by (3.7))}\\
&= \mathbb{I}_{\{T\ge\tau_{L,U}>t\}}\int_{\mathbb{R}^2} \mathrm{d}\omega_r \mathrm{d}s_r \widehat{\varphi}(\omega,s)\mathbb{E}[\mathbb{I}_{\{\tau_{L,U}\le T\}}\mathbb{E}[\mathrm{e}^{\mathrm{i}\omega X_T + \mathrm{i}s(\langle X\rangle_T - \langle X\rangle_{\tau_{L,U}})}|\mathcal{F}_{\tau_{L,U}}]|\mathcal{F}_t]\\
&= \mathbb{I}_{\{T\ge\tau_{L,U}>t\}}\int_{\mathbb{R}^2} \mathrm{d}\omega_r \mathrm{d}s_r \widehat{\varphi}(\omega,s)\mathbb{E}[\mathbb{I}_{\{\tau_{L,U}\le T\}}\mathbb{E}[\psi_{L,U}(X_T)|\mathcal{F}_{\tau_{L,U}}]|\mathcal{F}_t] && \text{(by (5.20))}\\
&= \mathbb{I}_{\{T\ge\tau_{L,U}>t\}}\mathbb{E}[\Psi_{L,U}(X_T)|\mathcal{F}_t] && \text{(by (5.25))}\\
&= \mathbb{I}_{\{T\ge\tau_{L,U}>t\}}\mathbb{I}_{\{\tau_{L,U}>t\}}\left(\mathbb{E}[\Psi_{L,U}(X_T)|\mathcal{F}_t] - \mathbb{E}[\mathbb{I}_{\{\tau_{L,U}>T\}}\Psi_{L,U}(X_T)|\mathcal{F}_t]\right)\\
&\quad + \mathbb{I}_{\{T\ge\tau_{L,U}>t\}}\left(\mathbb{I}_{\{\tau_{L,U}\le t\}}\mathbb{E}[\varphi_{\tau_{L,U}}(X_T,\langle X\rangle_T)|\mathcal{F}_t]\right), && (5.28)
\end{aligned}
$$

where, in the last equality, we have used $\mathbb{I}_{\{T\ge\tau_{L,U}>t\}}\mathbb{I}_{\{\tau_{L,U}>T\}} = 0$ and $\mathbb{I}_{\{T\ge\tau_{L,U}>t\}}\mathbb{I}_{\{\tau_{L,U}\le t\}} = 0$. Finally, on the set $\{\tau_{L,U}\le t\}$ we have

$$
\begin{aligned}
&\mathbb{I}_{\{\tau_{L,U}\le t\}} I_t\\
&= \mathbb{I}_{\{\tau_{L,U}\le t\}}\mathbb{E}[\mathbb{I}_{\{\tau_{L,U}\le T\}}\varphi(X_T,\langle X\rangle_T - \langle X\rangle_{\tau_{L,U}})|\mathcal{F}_t] && \text{(by (5.15))}\\
&= \mathbb{I}_{\{\tau_{L,U}\le t\}}\mathbb{E}[\varphi_{\tau_{L,U}}(X_T,\langle X\rangle_T)|\mathcal{F}_t] && \text{(by (5.24))}\\
&= \mathbb{I}_{\{\tau_{L,U}\le t\}}\mathbb{I}_{\{\tau_{L,U}>t\}}\left(\mathbb{E}[\Psi_{L,U}(X_T)|\mathcal{F}_t] - \mathbb{E}[\mathbb{I}_{\{\tau_{L,U}>T\}}\Psi_{L,U}(X_T)|\mathcal{F}_t]\right)\\
&\quad + \mathbb{I}_{\{\tau_{L,U}\le t\}}\left(\mathbb{I}_{\{\tau_{L,U}\le t\}}\mathbb{E}[\varphi_{\tau_{L,U}}(X_T,\langle X\rangle_T)|\mathcal{F}_t]\right). && \text{(as } \mathbb{I}_{\{\tau_{L,U}\le t\}}\mathbb{I}_{\{\tau_{L,U}>t\}} = 0) \ (5.29)
\end{aligned}
$$

Combining (5.26), (5.27), (5.28) and (5.29) yields (5.23). □

Theorem 5.7 (Replication of a double barrier knock-in claim)**.** *Fix $L < X_0 < U$. Let $\tau_{L,U}$ and I_t be as defined in (5.1) and (5.15), respectively and let φ satisfy Assumption 3.2. Let the hitting time indicator process $D = (D_t)_{0\le t\le T}$ be given by (5.12) and define European-style options $E^1 = (E^1_t)_{0\le t\le T}$ and $E^2 = (E^2_t)_{\tau_{L,U}\le t\le T}$ and a knock-out option $O = (O_t)_{0\le t\le T}$ by*

$$
E^1_t = \mathbb{E}[\Psi_{L,R}(X_T)|\mathcal{F}_t], \qquad E^2_t = \mathbb{E}[\varphi_{\tau_{L,U}}(X_T,\langle X\rangle_T)|\mathcal{F}_t], \qquad O_t = \mathbb{E}[\mathbb{I}_{\{\tau_{L,U}>T\}}\Psi_{L,U}(X_T)|\mathcal{F}_t]. \tag{5.30}
$$

where $\Psi_{L,R}$ and $\varphi_{\tau_{L,U}}$ as given in (5.24) and (5.25), respectively. Then

$$
I_t = (1 - D_t)\left(E^1_t - O_t\right) + D_t E^2_t \tag{5.31}
$$

and the portfolio (5.31) is self-financing portfolio in the sense of (3.17). That is,

$$
\mathrm{d}I_t = (1 - D_{t-})\left(\mathrm{d}E^1_t - \mathrm{d}O_t\right) + D_{t-}\mathrm{d}E^2_t. \tag{5.32}
$$

Proof. Expression (5.31) follows from (5.12), (5.23) and (5.30). To establish the self-financing property (5.32), we must show by (4.17) that

$$
\mathbb{I}_{\{\tau_{L,U}\le T\}}\left(E^1_{\tau_{L,U}} - O_{\tau_{L,U}}\right) = \mathbb{I}_{\{\tau_{L,U}\le T\}} E^2_{\tau_{L,U}}. \tag{5.33}
$$

We compute

$$
\mathbb{I}_{\{\tau_{L,U}\le T\}}\left(E^1_{\tau_{L,U}} - O_{\tau_{L,U}}\right)
$$

$$
\begin{aligned}
&= \mathbb{I}_{\{\tau_{L,U}\le T\}}\left(\mathbb{E}[\Psi_{L,U}(X_T)|\mathcal{F}_{\tau_{L,U}}] - \mathbb{E}[\mathbb{I}_{\{\tau_{L,U}>T\}}\Psi_{L,U}(X_T)|\mathcal{F}_{\tau_{L,U}}]\right) && \text{(by (5.30))}\\
&= \mathbb{I}_{\{\tau_{L,U}\le T\}}\mathbb{E}[\Psi_{L,U}(X_T)|\mathcal{F}_{\tau_{L,U}}] && \text{(as } \mathbb{I}_{\{\tau_{L,U}\le T\}}\mathbb{I}_{\{\tau_{L,U}>T\}}=0)\\
&= \mathbb{I}_{\{\tau_{L,U}\le T\}}\int_{\mathbb{R}^2}\mathrm{d}\omega_r\mathrm{d}s_r\widehat{\varphi}(\omega,s)\mathbb{E}[\psi_{L,U}(X_T;\omega,s)|\mathcal{F}_{\tau_{L,U}}] && \text{(by (5.25))}\\
&= \mathbb{I}_{\{\tau_{L,U}\le T\}}\int_{\mathbb{R}^2}\mathrm{d}\omega_r\mathrm{d}s_r\widehat{\varphi}(\omega,s)\mathbb{E}[\mathrm{e}^{\mathtt{i}\omega X_T+\mathtt{i}s(\langle X\rangle_T-\langle X\rangle_{\tau_{L,U}})}|\mathcal{F}_{\tau_{L,U}}] && \text{(by (5.20))}\\
&= \mathbb{I}_{\{\tau_{L,U}\le T\}}\int_{\mathbb{R}^2}\mathrm{d}\omega_r\mathrm{d}s_r\mathbb{E}[\mathbf{F}^{-1}[\widehat{\varphi}](X_T,\langle X\rangle_T-\langle X\rangle_{\tau_{L,U}})|\mathcal{F}_{\tau_{L,U}}] && \text{(by (3.7))}\\
&= \mathbb{I}_{\{\tau_{L,U}\le T\}}\mathbb{E}[\varphi(X_T,\langle X\rangle_T-\langle X\rangle_{\tau_{L,U}})|\mathcal{F}_{\tau_{L,U}}] && \text{(by (3.8))}\\
&= \mathbb{I}_{\{\tau_{L,U}\le T\}}\mathbb{E}[\varphi_{\tau_{L,U}}(X_T,\langle X\rangle_T)|\mathcal{F}_{\tau_{L,U}}] && \text{(by (5.24))}\\
&= \mathbb{I}_{\{\tau_{L,U}\le T\}}E^2_{\tau_{L,U}}. && \text{(by (5.30))}
\end{aligned}
$$

Thus, we have established (5.33) and, hence, (5.32) as well. □

Remark 5.8 (Knock-in claims (III))**.** Define $\psi'_L(x) \equiv \psi'_L(x;\omega,s)$ and $\psi'_U(x) \equiv \psi'_U(x;\omega,s)$ by

$$
\psi'_L(x) := \mathrm{e}^{-\mathtt{i}\omega L}\psi_L(x), \qquad\qquad \psi'_U(x) := \mathrm{e}^{-\mathtt{i}\omega U}\psi_U(x),
$$

and construct $\phi'(x) \equiv \phi'(x;\omega,s)$ and $\psi'_{L,U}(x) \equiv \psi'_{L,U}(x;\omega,s)$ from ψ'_L and ψ'_U as ϕ and $\psi_{L,U}$ are constructed from ψ_L and ψ_U. Then (5.20) becomes

$$
\mathbb{I}_{\{\tau_{L,U}\le T\}}\mathbb{E}[\psi'_{L,U}(X_T)|\mathcal{F}_{\tau_{L,U}}] = \mathbb{I}_{\{\tau_{L,U}\le T\}}\mathbb{E}[\mathrm{e}^{\mathtt{i}\omega(X_T-X_{\tau_{L,U}})+\mathtt{i}s(\langle X\rangle_T-\langle X\rangle_{\tau_{L,U}})}|\mathcal{F}_{\tau_{L,U}}]. \tag{5.34}
$$

Using (5.34), we can price a claim with a payoff of the form $\mathbb{I}_{\{\tau_{L,U}\le T\}}\varphi(X_T - X_{\tau_{L,U}}, \langle X\rangle_T - \langle X\rangle_{\tau_{L,U}})$. Specifically, one simply replaces $\varphi_{\tau_{L,U}}$ and $\Psi_{L,U}$ in Theorem 5.6 with $\varphi'_{\tau_{L,U}}$ and $\Psi'_{L,U}$ where

$$
\begin{aligned}
\varphi'_{\tau_{L,U}}(x,v) &= \varphi(x - X_{\tau_{L,U}}, v - \langle X\rangle_{\tau_{L,U}}),\\
\Psi'_{L,U}(x) &= \int_{\mathbb{R}^2}\mathrm{d}\omega_r\mathrm{d}s_r\widehat{\varphi}(\omega,s)\psi'_{L,U}(x;\omega,s).
\end{aligned}
$$

5.3 Double barrier rebate claims

We now examine a double barrier rebate claim, whose payoff is of the form $\mathbb{I}_{\{\tau_{L,U}\le T\}}\varphi(\langle X\rangle_{\tau_{L,U}})$, where we assume $L < X_0 < U$. Let $R = (R_t)_{0\le t\le T}$ denote the price process of a claim with this payoff. Using risk-neutral valuation, we have

$$
R_t = \mathbb{E}[\mathbb{I}_{\{\tau_{L,U}\le T\}}\varphi(\langle X\rangle_{\tau_{L,U}})|\mathcal{F}_t], \tag{5.35}
$$

Our analysis of the knock-in claim (5.35) begins with the following proposition.

Proposition 5.9. *Fix $s \in \mathbb{C}$ and $L < X_0 < U$. Let $v \equiv v(s)$ be as given in (4.40). Define $\chi_L(x,w) \equiv \chi_L(x,w;s)$ and $\chi_U(x,w) \equiv \chi_U(x,w;s)$ by*

$$
\chi_L(x,w) = \mathrm{e}^{-\mathtt{i}vL}\mathrm{e}^{\mathtt{i}vx+\mathtt{i}sw}, \qquad\qquad \chi_U(x,w) = \mathrm{e}^{-\mathtt{i}vU}\mathrm{e}^{\mathtt{i}vx+\mathtt{i}sw}.
$$

Also define $\theta(x,w) \equiv \theta(x,w;s)$ *and* $\zeta(x,w) \equiv \zeta(x,w;s)$ *by*

$$\theta(x,w) = \chi_U(x,w) - \sum_{n=1}^{\infty} \mathrm{e}^{n(L-U)} \Big(\chi_L(2n(U-L)+x,w) - \chi_U(2n(U-L)+x,w) \Big)$$
$$+ \sum_{n=1}^{\infty} \mathrm{e}^{n(L-U)+x-L} \Big(\chi_L(2n(U-L)+2L-x,w) - \chi_U(2n(U-L)+2L-x,w) \Big).$$
$$\zeta(x,w) = \chi_L(x,w) + \chi_U(x,w) - \theta(x,w). \tag{5.36}$$

Then, with $\tau_{L,U}$ *as given in* (5.1) *we have*

$$\mathbb{I}_{\{\tau_{L,U} \le T\}} \mathbb{E}[\zeta(X_T, \langle X \rangle_T) | \mathcal{F}_{\tau_{L,U}}] = \mathbb{I}_{\{\tau_{L,U} \le T\}} \mathrm{e}^{\mathrm{i}s\langle X \rangle_{\tau_{L,U}}}. \tag{5.37}$$

Proof. First, from (4.40), we recall that $M_t = \mathrm{e}^{\mathrm{i}vX_t + \mathrm{i}s\langle X \rangle_t}$ is a martingale. It therefore follows that

$$\mathbb{I}_{\{\tau_L \le T\}} \mathbb{E}[\chi_L(X_T, \langle X \rangle_T) | \mathcal{F}_{\tau_L}] = \mathbb{I}_{\{\tau_L \le T\}} \mathrm{e}^{\mathrm{i}s\langle X \rangle_{\tau_L}}, \tag{5.38}$$
$$\mathbb{I}_{\{\tau_U \le T\}} \mathbb{E}[\chi_U(X_T, \langle X \rangle_T) | \mathcal{F}_{\tau_U}] = \mathbb{I}_{\{\tau_L \le T\}} \mathrm{e}^{\mathrm{i}s\langle X \rangle_{\tau_U}}. \tag{5.39}$$

Next, we note that, setting $G_L = \chi_U$ and $G_U = \chi_L$ in (5.16) we obtain $H = \theta$. Thus, from (5.17) and (5.18) we have

$$\mathbb{I}_{\{\tau_L \le T\}} \mathbb{E}[\theta(X_T, \langle X \rangle_T) | \mathcal{F}_{\tau_L}] = \mathbb{I}_{\{\tau_L \le T\}} \mathbb{E}[\chi_U(X_T, \langle X \rangle_T) | \mathcal{F}_{\tau_L}], \tag{5.40}$$
$$\mathbb{I}_{\{\tau_U \le T\}} \mathbb{E}[\theta(X_T, \langle X \rangle_T) | \mathcal{F}_{\tau_U}] = \mathbb{I}_{\{\tau_U \le T\}} \mathbb{E}[\chi_L(X_T, \langle X \rangle_T) | \mathcal{F}_{\tau_U}]. \tag{5.41}$$

It follows from (5.38) and (5.40) and (5.36) that

$$\mathbb{I}_{\{\tau_L \le T\}} \mathbb{E}[\zeta(X_T, \langle X \rangle_T) | \mathcal{F}_{\tau_L}] = \mathbb{I}_{\{\tau_L \le T\}} \mathrm{e}^{\mathrm{i}s\langle X \rangle_{\tau_L}},$$

and it follows from (5.39) and (5.41) and (5.36) that

$$\mathbb{I}_{\{\tau_U \le T\}} \mathbb{E}[\zeta(X_T, \langle X \rangle_T) | \mathcal{F}_{\tau_U}] = \mathbb{I}_{\{\tau_U \le T\}} \mathrm{e}^{\mathrm{i}s\langle X \rangle_{\tau_U}},$$

Since $\tau_{L,U} = \tau_L \wedge \tau_U$, this proves (5.37). □

Theorem 5.10 (Price and replication of a double barrier rebate claim)**.** *Fix* $L < X_0 < U$*. Let* $\tau_{L,U}$ *and* R_t *be as defined in* (5.1) *and* (5.35)*, respectively. Assume* φ *satisfies Assumption 3.2 and define a European option* E *and a double-barrier knock-out option* O *by*

$$E_t = \mathbb{E}[\varphi_1(X_T, \langle X \rangle_T) | \mathcal{F}_t], \qquad O_t = \mathbb{E}[\mathbb{I}_{\{\tau_{L,U} > T\}} \varphi_1(X_T, \langle X \rangle_T) | \mathcal{F}_t], \qquad t \in [0,T], \tag{5.42}$$

where the payoff function φ_1 *is given by*

$$\varphi_1(x,w) = \int_{\mathbb{R}} \mathrm{d}s_r \widehat{\varphi}(s) \zeta(x,w;s), \tag{5.43}$$

with $v(s)$ *and* $\zeta(x,w,s)$ *as given in* (4.40) *and* (5.36)*, respectively. Here,* $\widehat{\varphi}$ *is the one-dimensional Fourier transform of* φ*. Then, with* D *as defined in* (5.12)*, the price of the rebate claim is given by*

$$R_t = (1-D_t)(E_t - O_t) + D_t \Phi_t, \qquad \Phi_t = \mathbb{I}_{\{\tau_{L,U} \le t\}} \varphi(\langle X \rangle_{\tau_{L,U}}). \tag{5.44}$$

Additionally, the portfolio (5.44) *is self-financing*

$$\mathrm{d}R_t = (1 - D_{t-})(\mathrm{d}E_t - \mathrm{d}O_t) + D_{t-} \mathrm{d}\Phi_t. \tag{5.45}$$

Proof. We examine the value of R_t on three sets: $\{\tau_{L,U} > T\}$, $\{T \geq \tau_{L,U} > t\}$ and $\{\tau_{L,U} \leq t\}$. On the set $\{\tau_{L,U} > T\}$ we have

$$\begin{aligned}
&\mathbb{I}_{\{\tau_{L,U}>T\}} R_t \\
&= \mathbb{I}_{\{\tau_{L,U}>T\}} \mathbb{E}[\mathbb{I}_{\{\tau_{L,U}\leq T\}} \varphi(\langle X\rangle_{\tau_{L,U}})|\mathcal{F}_t] && \text{by ((5.35))} \\
&= 0 && (\text{as } \mathbb{I}_{\{\tau_{L,U}>T\}}\mathbb{I}_{\{\tau_{L,U}\leq T\}} = 0) \\
&= \mathbb{I}_{\{\tau_{L,U}>T\}} \left(\mathbb{E}[\varphi_1(X_T, \langle X\rangle_T)|\mathcal{F}_t] - \mathbb{E}[\mathbb{I}_{\{\tau_{L,U}>T\}}\varphi_1(X_T, \langle X\rangle_T)|\mathcal{F}_t]\right) \\
&= \mathbb{I}_{\{\tau_{L,U}>T\}} \left((1-D_t)(E_t - O_t) + D_t\Phi_t\right). && (\text{by(5.12), (5.42) and (5.44)}) \quad (5.46)
\end{aligned}$$

On the set $\{T \geq \tau_{L,U} > t\}$ we have

$$\begin{aligned}
&\mathbb{I}_{\{T\geq\tau_{L,U}>t\}} R_t \\
&= \mathbb{I}_{\{T\geq\tau_{L,U}>t\}} \mathbb{E}[\mathbb{I}_{\{\tau_{L,U}\leq T\}} \varphi(\langle X\rangle_{\tau_{L,U}})|\mathcal{F}_t] && \text{by ((5.35))} \\
&= \mathbb{I}_{\{T\geq\tau_{L,U}>t\}} \mathbb{E}[\mathbb{I}_{\{\tau_{L,U}\leq T\}} \mathbf{F}^{-1}[\widehat{\varphi}](\langle X\rangle_{\tau_{L,U}})|\mathcal{F}_t] && (\text{by (3.8)}) \\
&= \mathbb{I}_{\{T\geq\tau_{L,U}>t\}} \int_{\mathbb{R}} \mathrm{d}s_r \widehat{\varphi}(s) \mathbb{E}[\mathbb{I}_{\{\tau_{L,U}\leq T\}} \mathrm{e}^{\mathrm{i}s\langle X\rangle_{\tau_{L,U}}}|\mathcal{F}_t] && (\text{by (3.7)}) \\
&= \mathbb{I}_{\{T\geq\tau_{L,U}>t\}} \int_{\mathbb{R}} \mathrm{d}s_r \widehat{\varphi}(s) \mathbb{E}[\mathbb{I}_{\{\tau_{L,U}\leq T\}} \mathbb{E}[\zeta(X_T, \langle X\rangle_T; s)|\mathcal{F}_{\tau_{L,U}}]|\mathcal{F}_t] && (\text{by (5.37)}) \\
&= \mathbb{I}_{\{T\geq\tau_{L,U}>t\}} \int_{\mathbb{R}} \mathrm{d}s_r \widehat{\varphi}(s) \mathbb{E}[\zeta(X_T, \langle X\rangle_T; s)|\mathcal{F}_t] \\
&= \mathbb{I}_{\{T\geq\tau_{L,U}>t\}} \mathbb{E}[\varphi_1(X_T, \langle X\rangle_T)|\mathcal{F}_t] && (\text{by (5.43)}) \\
&= \mathbb{I}_{\{T\geq\tau_{L,U}>t\}} \left(\mathbb{E}[\varphi_1(X_T, \langle X\rangle_T)|\mathcal{F}_t] + \mathbb{E}[\mathbb{I}_{\{\tau_{L,U}>T\}}\varphi_1(X_T, \langle X\rangle_T)|\mathcal{F}_t]\right) && (\text{as } \mathbb{I}_{\{T\geq\tau_{L,U}>t\}}\mathbb{I}_{\{\tau_{L,U}>T\}} = 0) \\
&= \mathbb{I}_{\{T\geq\tau_{L,U}>t\}} \left((1-D_t)(E_t - O_t) + D_t\Phi_t\right). && (\text{by(5.12), (5.42) and (5.44)}) \quad (5.47)
\end{aligned}$$

And on the set $\{\tau_{L,U} \leq t\}$ we have, trivially

$$\begin{aligned}
\mathbb{I}_{\{\tau_{L,U}\leq t\}} R_t &= \mathbb{I}_{\{\tau_{L,U}\leq t\}} \mathbb{E}[\mathbb{I}_{\{\tau_{L,U}\leq T\}} \varphi(\langle X\rangle_{\tau_{L,U}})|\mathcal{F}_t] = D_t \varphi(\langle X\rangle_{\tau_{L,U}}) = D_t\Phi_t \\
&= \mathbb{I}_{\{\tau_{L,U}\leq t\}} \left((1-D_t)(E_t - O_t) + D_t\Phi_t\right),
\end{aligned} \tag{5.48}$$

where we have used $\mathbb{I}_{\{\tau_{L,U}\leq t\}}(1-D_t) = 0$. Combining (5.46), (5.47) and (5.48) yields (5.44). To prove (5.45), we must show from (4.17) that

$$\mathbb{I}_{\{\tau_{L,U}\leq T\}} \left(E_{\tau_{L,U}} - O_{\tau_{L,U}}\right) = \mathbb{I}_{\{\tau_{L,U}\leq T\}} \Phi_{\tau_{L,U}}. \tag{5.49}$$

We have

$$\begin{aligned}
&\mathbb{I}_{\{\tau_{L,U}\leq T\}} \left(E_{\tau_{L,U}} - O_{\tau_{L,U}}\right) \\
&= \mathbb{I}_{\{\tau_{L,U}\leq T\}} \left(\mathbb{E}[\varphi_1(X_T, \langle X\rangle_T)|\mathcal{F}_{\tau_{L,U}}] - \mathbb{E}[\mathbb{I}_{\{\tau_{L,U}>T\}}\varphi_1(X_T, \langle X\rangle_T)|\mathcal{F}_{\tau_{L,U}}]\right) && (\text{by (5.42)}) \\
&= \mathbb{I}_{\{\tau_{L,U}\leq T\}} \mathbb{E}[\varphi_1(X_T, \langle X\rangle_T)|\mathcal{F}_{\tau_{L,U}}] && (\text{as } \mathbb{I}_{\{\tau_{L,U}\leq T\}}\mathbb{I}_{\{\tau_{L,U}>T\}} = 0) \\
&= \mathbb{I}_{\{\tau_{L,U}\leq T\}} \int_{\mathbb{R}} \mathrm{d}s_r \widehat{\varphi}(s) \mathbb{E}[\zeta(X_T, \langle X\rangle_T; s)|\mathcal{F}_{\tau_{L,U}}] && (\text{by (5.43)}) \\
&= \mathbb{I}_{\{\tau_{L,U}\leq T\}} \int_{\mathbb{R}} \mathrm{d}s_r \widehat{\varphi}(s) \mathrm{e}^{\mathrm{i}s\langle X\rangle_{\tau_{L,U}}} && (\text{by (5.37)})
\end{aligned}$$

$$= \mathbb{I}_{\{\tau_{L,U} \le T\}} \mathbf{F}^{-1}[\widehat{\varphi}](\langle X \rangle_{\tau_{L,U}}) \qquad \text{(by (3.7))}$$
$$= \mathbb{I}_{\{\tau_{L,U} \le T\}} \varphi(\langle X \rangle_{\tau_{L,U}}) \qquad \text{(by (3.8))}$$
$$= \mathbb{I}_{\{\tau_{L,U} \le T\}} \Phi_{\tau_{L,U}}. \qquad \text{(by (5.44))}$$

Thus, we have proved (5.49) and, hence, (5.45) as well. □

6 Examples

6.1 Realized Sharpe ratio options

The *Sharpe ratio* was introduced in [Sha66] as a simple way to measure the performance of an investment while adjusting for its risk. Let us define

$$\mathrm{d}Y_t = \frac{1}{S_t}\mathrm{d}S_t = \mathrm{d}X_t + \frac{1}{2}\mathrm{d}\langle X \rangle_t.$$

The *ex-ante Sharpe ratio* is typically defined as the expected return of Y divided by the return's standard deviation

$$\text{ex-ante Sharpe ratio}: \qquad \frac{\mathbb{E}^*[Y_T - Y_t|\mathcal{F}_t]}{\sqrt{\mathbb{V}^*[Y_T - Y_t|\mathcal{F}_t]}},$$

where $\mathbb{E}^*$ and $\mathbb{V}^*$ represent the expectation and variance operators under the physical measure $\mathbb{P}^*$. One difficulty associated with the ex-ante Sharpe ratio is that expected returns $\mathbb{E}^*[Y_T - Y_t|\mathcal{F}_t]$ are difficult to estimate. Rather than rely on expected returns estimates, investors may prefer to invest in a claim that pays a function of the *realized Sharpe ratio*, which, over the interval $[t, T]$, is defined as

$$\text{realized Sharpe ratio}: \qquad \Lambda_{t,T} := \frac{Y_T - Y_t}{\sqrt{\langle Y \rangle_T - \langle Y \rangle_t}} = \frac{X_T - X_t + \frac{1}{2}\left(\langle X \rangle_T - \langle X \rangle_t\right)}{\sqrt{\langle X \rangle_T - \langle X \rangle_t}}.$$

For example, a European-style call on $\Lambda_{0,T}$ guarantees an investor a payoff whose realized Sharpe ratio over the interval $[0, T]$ is at least as large as the strike price K. European-style options on the realized Sharpe ratio

$$\varphi(X_T, \langle X \rangle_T) = \vartheta(\Lambda_{0,T}),$$

can be priced and hedged using the results of Section 3.

A variety of barrier-style claims could also be written on Λ. For example, consider the following knock-in, knock-out and rebate claims

$$\begin{aligned}
\text{Knock in}: &\quad \mathbb{I}_{\{\tau_L \le T\}}\vartheta(\Lambda_{\tau_L,T}) = \mathbb{I}_{\{\tau_L \le T\}}\varphi(X_T, \langle X \rangle_T - \langle X \rangle_{\tau_L}), &\quad \varphi(x,v) &= \vartheta\left((x - L + \tfrac{1}{2}v)/\sqrt{v}\right),\\
\text{Knock out}: &\quad \mathbb{I}_{\{\tau_{L,U} > T\}}\vartheta(\Lambda_{0,T}) = \mathbb{I}_{\{\tau_{L,U} > T\}}\varphi(X_T, \langle X \rangle_T), &\quad \varphi(x,v) &= \vartheta\left((x - X_0 + \tfrac{1}{2}v)/\sqrt{v}\right),\\
\text{Rebate}: &\quad \mathbb{I}_{\{\tau_U \le T\}}\vartheta(\Lambda_{0,\tau_U}) = \mathbb{I}_{\{\tau_U \le T\}}\varphi(\langle X \rangle_{\tau_U}), &\quad \varphi(v) &= \vartheta\left((U - X_0 + \tfrac{1}{2}v)/\sqrt{v}\right).
\end{aligned}$$

These options can be priced and hedged using the results from Sections 4.1, 5.1 and 4.3, respectively.

6.2 Buy at-the-touch European options

Suppose at time $t < T$ an investor believes that if X hits a level $L < X_t$, then the value X will subsequently rise. To capitalize on his belief, the investor forms a strategy to purchase an at-the-money call if X hits a level L prior to time T. That is, on the event $\{\tau_L \leq T\}$, the investor will purchase $C_{\tau_L}(L)$ at time τ_L. One possible downside of this strategy is that it leaves the investor exposed to volatility risk over the interval $[t, \tau_L]$. Indeed, since volatility is typically negatively correlated with price movements (the leverage effect), a decrease in the value of X from X_t to L will likely be accompanied by an increase in the instantaneous value of the volatility process σ over the interval $[t, \tau_L]$. As a result, the investor will likely pay a higher price (in units of implied volatility) to purchase the call at time τ_L than he would have if he had purchased the call at time $t < \tau_L$.

In order to mitigate the risk that volatility rises over the interval $[t, \tau_L]$, the investor might alternatively choose to purchase a claim at time t that replicates his original strategy of buying an at-the-money call at time τ_L. The value V_t of the alternative claim at time $t < \tau_L$ is given by

$$\mathbb{I}_{\{t<\tau_L\}} V_t = \mathbb{I}_{\{t<\tau_L\}} \mathbb{E}[\mathbb{I}_{\{\tau_L \leq T\}} C_{\tau_L}(L) | \mathcal{F}_t]. \tag{6.1}$$

We wish to price and hedge the claim V.

Let $C^{\text{BS}}(x, k, v)$ denote the Black-Scholes price of a call option on an underlying with initial log price x, log strike k and total variance [1] v, i.e.,

$$C^{\text{BS}}(x, k, v) = \mathrm{e}^x \mathcal{N}(d_+) - \mathrm{e}^k \mathcal{N}(d_-), \qquad d_\pm = \frac{1}{\sqrt{v}} \left(x - k \pm \tfrac{1}{2} v \right).$$

Here, $\mathcal{N}$ is the CDF of a standard normal random variable. Observe that

$$\begin{aligned}
\mathbb{I}_{\{\tau_L \leq T\}} C_{\tau_L}(L) &= \mathbb{I}_{\{\tau_L \leq T\}} \mathbb{E}[C^{\text{BS}}(L, L, \langle X \rangle_T - \langle X \rangle_{\tau_L}) | \mathcal{F}_{\tau_L}] && \text{(Hull-White formula)} \\
&= \mathbb{I}_{\{\tau_L \leq T\}} \mathbb{E}[\mathrm{e}^L \mathcal{N}(\langle X \rangle_T - \langle X \rangle_{\tau_L}) | \mathcal{F}_{\tau_L}], \\
&= \mathbb{E}[\mathbb{I}_{\{\tau_L \leq T\}} \mathrm{e}^L \mathcal{N}(\langle X \rangle_T - \langle X \rangle_{\tau_L}) | \mathcal{F}_{\tau_L}]. && (\{\tau_L \leq T\} \in \mathcal{F}_{\tau_L})
\end{aligned} \tag{6.2}$$

Thus, using (6.1) and (6.2), we obtain

$$\begin{aligned}
\mathbb{I}_{\{t<\tau_L\}} V_t &= \mathbb{I}_{\{t<\tau_L\}} \mathbb{E}[\mathbb{E}[\mathbb{I}_{\{\tau_L \leq T\}} \mathrm{e}^L \mathcal{N}(\langle X \rangle_T - \langle X \rangle_{\tau_L}) | \mathcal{F}_{\tau_L}] | \mathcal{F}_t] \\
&= \mathbb{I}_{\{t<\tau_L\}} \mathbb{E}[\mathbb{I}_{\{\tau_L \leq T\}} \mathrm{e}^L \mathcal{N}(\langle X \rangle_T - \langle X \rangle_{\tau_L}) | \mathcal{F}_t].
\end{aligned} \tag{6.3}$$

The expression $\mathbb{I}_{\{\tau_L \leq T\}} \mathrm{e}^L \mathcal{N}(\langle X \rangle_T - \langle X \rangle_{\tau_L})$ inside the expectation on the right-hand side of (6.3) is the payoff of a single barrier knock-in claim. Single barrier knock-in claims can be priced and hedged using the results of Section 4.1.

More generally, let $E_t = \mathbb{E}[h(X_T) | \mathcal{F}_t]$ be the time t price of a European claim with payoff $h(X_T)$ at time $T > t$. Suppose an investor wishes to purchase a claim at time $t < \tau_L \leq T$ whose value V satisfies $\mathbb{I}_{\{\tau_L \leq T\}} V_{\tau_L} = \mathbb{I}_{\{\tau_L \leq T\}} E_{\tau_L}$. We will price and hedge this claim.

[1] Here, *total variance* means Black-Scholes variance multiplied by time to maturity.

Let $E^{\mathrm{BS}}(x, v; h)$ be the Black-Scholes price of a European option with initial log price x, total variance v^2 and payoff function h. Assuming h has a generalized Fourier transform $\widehat{h}$ and $h = \mathbf{F}^{-1}\widehat{h}$ we have

$$E^{\mathrm{BS}}(x, v; h) = \int_{\mathbb{R}} \mathrm{d}\omega_r \, \mathrm{e}^{\mathrm{i}\omega x + v(-\omega^2 - \mathrm{i}\omega)/2} \widehat{h}(\omega), \qquad \widehat{h}(\omega) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{d}x \, \mathrm{e}^{-\mathrm{i}\omega x} h(x).$$

Using the same arguments as above, we obtain

$$\mathbb{I}_{\{\tau_L \le T\}} E_{\tau_L} = \mathbb{I}_{\{\tau_L \le T\}} E^{\mathrm{BS}}(L, \langle X \rangle_T - \langle X \rangle_{\tau_L}; h),$$

and therefore

$$\mathbb{I}_{\{t < \tau_L\}} V_t = \mathbb{I}_{\{t < \tau_L\}} \mathbb{E}[\mathbb{I}_{\{\tau_L \le T\}} E^{\mathrm{BS}}(L, \langle X \rangle_T - \langle X \rangle_{\tau_L}; h) | \mathcal{F}_t].$$

Note that, once again, the expression inside the conditional expectation is the payoff of a knock-in claim; it can be priced and hedged using the results of Section 4.1.

6.3 Claims on levered exchange traded funds

The first US-listed exchange traded fund (ETF), the SPDR S&P500 ETF (SPY), was launched on January 29th, 1993. Since this time, the market for ETFs has grown rapidly; at the end of 2012, the global ETF industry had over \$1.8 trillion in assets under management. [2] A developing sub-class of ETFs are the *levered exchange traded funds* (LETFs). LETFs are actively managed funds that are designed to multiply instantaneous market returns by a factor of β. Specifically, if $S = \mathrm{e}^X$ is the price process of an ETF and $V = \mathrm{e}^Z$ is the value of an associated LETF then the relation between S and V is

$$\frac{\mathrm{d}V_t}{V_t} = \beta \frac{\mathrm{d}S_t}{S_t}.$$

Typical values of β are $\beta = \{-3, -2, -1, 2, 3\}$. One can show [AZ10] that

$$Z_T - Z_t = \beta(X_T - X_t) - \tfrac{1}{2}\beta(1-\beta)\left(\langle X \rangle_T - \langle X \rangle_t\right). \tag{6.4}$$

From (6.4), we see clearly that an option written on Z_T can be expressed as an option written on $(X_T, \langle X \rangle_T)$. For example, consider a single barrier knock-in claim that pays some function ϑ of the return on LETF V after the log of the ETF X hits either L or U, where $L < X_0 < U$. The payoff is given by

$$\mathbb{I}_{\{\tau_L \le T\}} \vartheta(V_T / V_{\tau_{L,U}}) = \mathbb{I}_{\{\tau_L \le T\}} \varphi(X_T, \langle X \rangle_T - \langle X \rangle_{\tau_{L,U}}), \qquad \varphi(x, v) = \vartheta(\mathrm{e}^{\beta(x-L) - \frac{1}{2}\beta(1-\beta)v}).$$

Such a claim can be priced and hedged using the results from Section 5.2.

7 Summary and future research

Assuming only that the price of a risky asset $S = \mathrm{e}^X$ is strictly positive and continuous and driven by an independent volatility process σ, we have shown how to price and hedge a variety of barrier-style claims

[2] Source: "2013 ETF & Investment Outlook" by David Mazza, SPDR ETF Strategy & Consulting, State Street Global Advisors. Available at http://www.spdr-etfs.com.

written on the log returns X and the quadratic variation of log returns $\langle X \rangle$. In particular, we have studied single and double barrier knock-in, knock-out, and rebate claims. The pricing formula we obtain are semi-robust in that they make no assumption about the market price of volatility risk. Moreover, our hedging strategies hold with probability one.

Future research will focus three areas (i) weakening the independence assumption on log returns and volatility, (ii) pricing and hedging when calls and puts are available only at discrete strikes or only within a finite interval, (iii) considering richer payoff structures, which may depend on the running maximum or minimum of the asset in addition to log returns and quadratic variation of log returns.

Acknowledgments

The authors are grateful to Roger Lee, Sergey Nadtochiy and Stephan Sturm for their feedback. They are not responsible for any errors that may appear in this manuscript. Part of this research was performed while the authors were visiting the Institute for Pure and Applied Mathematics (IPAM), which is supported by the National Science Foundation.

A Proof of Proposition 5.4

The proof is by construction.

Step 0: Create a European-style derivative with payoff H_0 such that (5.17) is satisfied with $H = H_0$. To do this, we set

$$H_0(x, v) := G_L(x, v). \tag{A.1}$$

We clearly have

$$\mathbb{I}_{\{\tau_L \le T\}} \mathbb{E}[H_0(X_T, \langle X \rangle_T) | \mathcal{F}_{\tau_L}] = \mathbb{I}_{\{\tau_L \le T\}} \mathbb{E}[G_L(X_T, \langle X \rangle_T) | \mathcal{F}_{\tau_L}], \tag{A.2}$$

satisfying (5.17). However, H_0 does not satisfy (5.18).

Step 1(a): Create a European-style derivative with payoff h_1^a so that (5.18) is satisfied with $H = H_0 + h_1^a$. We seek

$$\mathbb{I}_{\{\tau_U \le T\}} \mathbb{E}[H_0(X_T, \langle X \rangle_T) + h_1^a(X_T, \langle X \rangle_T) | \mathcal{F}_{\tau_U}] = \mathbb{I}_{\{\tau_U \le T\}} \mathbb{E}[G_U(X_T, \langle X \rangle_T) | \mathcal{F}_{\tau_U}].$$

Solving for $\mathbb{I}_{\{\tau_U \le T\}} \mathbb{E}[h_1^a(X_T, \langle X \rangle_T) | \mathcal{F}_{\tau_U}]$, we obtain

$$\begin{aligned}
&\mathbb{I}_{\{\tau_U \le T\}} \mathbb{E}[h_1^a(X_T, \langle X \rangle_T) | \mathcal{F}_{\tau_U}] \\
&= \mathbb{I}_{\{\tau_U \le T\}} \mathbb{E}[G_U(X_T, \langle X \rangle_T) - H_0(X_T, \langle X \rangle_T) | \mathcal{F}_{\tau_U}] \\
&= \mathbb{I}_{\{\tau_U \le T\}} \mathbb{E}[G_U(X_T, \langle X \rangle_T) - G_L(X_T, \langle X \rangle_T) | \mathcal{F}_{\tau_U}] && \text{(by (A.1))} \\
&= \mathbb{I}_{\{\tau_U \le T\}} \mathbb{E}[\mathrm{e}^{X_T - U} \left(G_U(2U - X_T, \langle X \rangle_T) - G_L(2U - X_T, \langle X \rangle_T) \right) | \mathcal{F}_{\tau_U}] && \text{(by (4.25))}
\end{aligned}$$

Thus, we set

$$h_1^a(x, v) = \mathrm{e}^{x-U} \left(G_U(2U - x, v) - G_L(2U - x, v) \right). \tag{A.3}$$

By construction, the European-style derivative with payoff function $H = H_0 + h_1^a$ satisfies (5.18), but it does not satisfy (5.17) due to the additional term h_1^a in the payoff.

Step 1(b): Create a European-style derivative with payoff function h_1^b so that (5.17) is satisfied with $H = H_0 + h_1^a + h_1^b$. We seek

$$\mathbb{I}_{\{\tau_L \le T\}}\mathbb{E}[H_0(X_T, \langle X\rangle_T) + h_1^a(X_T, \langle X\rangle_T) + h_1^b(X_T, \langle X\rangle_T)|\mathcal{F}_{\tau_L}] = \mathbb{I}_{\{\tau_L \le T\}}\mathbb{E}[G_L(X_T, \langle X\rangle_T)|\mathcal{F}_{\tau_L}].$$

Solving for $\mathbb{I}_{\{\tau_L \le T\}}\mathbb{E}[h_1^b(X_T, \langle X\rangle_T)|\mathcal{F}_{\tau_L}]$, we obtain

$$\begin{aligned}
&\mathbb{I}_{\{\tau_L \le T\}}\mathbb{E}[h_1^b(X_T, \langle X\rangle_T)|\mathcal{F}_{\tau_L}] \\
&= \mathbb{I}_{\{\tau_L \le T\}}\mathbb{E}[G_L(X_T, \langle X\rangle_T) - H_0(X_T, \langle X\rangle_T) - h_1^a(X_T, \langle X\rangle_T)|\mathcal{F}_{\tau_L}] \\
&= \mathbb{I}_{\{\tau_L \le T\}}\mathbb{E}[-h_1^a(X_T, \langle X\rangle_T)|\mathcal{F}_{\tau_L}] && \text{(by (A.2))} \\
&= \mathbb{I}_{\{\tau_L \le T\}}\mathbb{E}[-\mathrm{e}^{X_T - L}h_1^a(2U - X_T, \langle X\rangle_T)|\mathcal{F}_{\tau_L}] && \text{(by (4.25))}
\end{aligned}$$

Thus, we set

$$\begin{aligned}
h_1^b(x, v) &= -\mathrm{e}^{x-U}h_1^a(2U - x, v) \\
&= -\mathrm{e}^{L-U}\left(G_U(2(U-L) + x, v) - G_L(2(U-L) + x, v)\right). && \text{(by (A.3))}
\end{aligned}$$

To complete Step 1, set

$$H_1 = H_0 + h_1^a + h_1^b.$$

By construction, the European-style derivative with payoff function $H_1 = H_0 + h_1^a + h_1^b$ satisfies (5.17), but it does not satisfy (5.18) due to the term h_1^b in the payoff.

Step n: We now repeat Steps 1(a) and 1(b) ad infinitum. At the nth step we set

$$\begin{aligned}
h_n^a(x, v) &= -\mathrm{e}^{x-U}h_{n-1}^b(2U - x, v) \\
&= \mathrm{e}^{n(L-U)+x-L}\Big(G_U(2n(U-L) + 2L - x, v) - G_L(2n(U-L) + 2L - x, v)\Big), \\
h_n^b(x, v) &= -\mathrm{e}^{x-L}h_n^a(2L - x, v) \\
&= -\mathrm{e}^{n(L-U)}\Big(G_U(2n(U-L) + x, v) - G_L(2n(U-L) + x, v)\Big), \\
H_n(x, v) &= H_{n-1}(x, v) + h_n^a(x, v) + h_n^b(x, v).
\end{aligned}$$

By construction, we have

$$\begin{aligned}
&\mathbb{I}_{\{\tau_L \le T\}}\mathbb{E}[H_n(X_T, \langle X\rangle_T)|\mathcal{F}_{\tau_L}] = 0, \\
&\mathbb{I}_{\{\tau_U \le T\}}\mathbb{E}[H_n(X_T, \langle X\rangle_T)|\mathcal{F}_{\tau_U}] = \mathbb{I}_{\{\tau_U \le T\}}\mathbb{E}[h_n^b(X_T, \langle X\rangle_T)|\mathcal{F}_{\tau_U}].
\end{aligned}$$

Now, recall that G_L and G_U are bounded by assumption. Also notice that $h_n^a(x, v) \sim \mathrm{e}^{n(L-U)}$ and $h_n^b(x, v) \sim \mathrm{e}^{n(L-U)}$ with $L < U$. Thus, we have that $H_n \to H_\infty \equiv H$ *uniformly* on every compact set. We therefore have

$$0 = \lim_{n\to\infty}\mathbb{I}_{\{\tau_L \le T\}}\mathbb{E}[H_n(X_T, \langle X\rangle_T)|\mathcal{F}_{\tau_L}] = \mathbb{I}_{\{\tau_L \le T\}}\mathbb{E}[\lim_{n\to\infty}H_n(X_T, \langle X\rangle_T)|\mathcal{F}_{\tau_L}]$$

$$
\begin{aligned}
&= \mathbb{I}_{\{\tau_L \le T\}} \mathbb{E}[H(X_T, \langle X \rangle_T) | \mathcal{F}_{\tau_L}], \\
0 &= \lim_{n\to\infty} \mathbb{I}_{\{\tau_U \le T\}} \mathbb{E}[h_n^b(X_T, \langle X \rangle_T) | \mathcal{F}_{\tau_U}] = \lim_{n\to\infty} \mathbb{I}_{\{\tau_U \le T\}} \mathbb{E}[H_n(X_T, \langle X \rangle_T) | \mathcal{F}_{\tau_U}], \\
&= \mathbb{I}_{\{\tau_U \le T\}} \mathbb{E}[\lim_{n\to\infty} H_n(X_T, \langle X \rangle_T) | \mathcal{F}_{\tau_U}] = \mathbb{I}_{\{\tau_U \le T\}} \mathbb{E}[H(X_T, \langle X \rangle_T) | \mathcal{F}_{\tau_U}],
\end{aligned}
$$

which proves (5.17) and (5.18).

References

[And01] J. Andreasen. Behind the mirror. *Risk*, 2001.

[AZ10] M. Avellaneda and S. Zhang. Path-dependence of leveraged ETF returns. *SIAM Journal on Financial Mathematics*, 1:586–603, 2010.

[Bat88] David Bates. The crash premium: Option pricing under asymmetric processes, with applications to options on deutschemark futures. *Working Paper, University of Pennsylvania*, 1988.

[Bat97] David S Bates. The skewness premium: Option pricing under asymmetric processes. *Advances in Futures and Options Research*, 9:51–82, 1997.

[BC94] J. Bowie and P. Carr. Static simplicity. *Risk*, 8 1994.

[BL78] Douglas T. Breeden and Robert H. Litzenberger. Prices of state-contingent claims implicit in option prices. *The Journal of Business*, 51(4):621–651, 1978.

[CEG98] Peter Carr, Katrina Ellis, and Vishal Gupta. Static hedging of exotic options. *The Journal of Finance*, 53(3):1165–1190, 1998.

[CL08] Peter Carr and Roger Lee. Robust replication of volatility derivatives. In *PRMIA award for Best Paper in Derivatives, MFA 2008 Annual Meeting*, 2008.

[CL09] Peter Carr and Roger Lee. Put-call symmetry: Extensions and applications. *Mathematical Finance*, 19(4):523–560, 2009.

[CM98] P. Carr and D. Madan. Towards a theory of volatility trading. *Volatility: new estimation techniques for pricing derivatives*, page 417, 1998.

[HW87] John Hull and Alan White. The pricing of options on assets with stochastic volatilities. *The Journal of Finance*, 42(2):281–300, 1987.

[LL15] Tim Leung and Matthew Lorig. Optimal static quadratic hedging. *arXiv preprint*, 2015.

[Mer73] R.C. Merton. Theory of rational option pricing. *The Bell Journal of Economics and Management Science*, 4(1):141–183, 1973.

[Sha66] William F Sharpe. Mutual fund performance. *Journal of business*, pages 119–138, 1966.

$$
\begin{array}{ccc}
\mathbb{E}[\mathrm{e}^{\mathrm{i}u(\omega,s)(X_T-X_t)}|\mathcal{F}_t] & \longrightarrow & \mathbb{E}[\mathrm{e}^{\mathrm{i}\omega(X_T-X_t)+\mathrm{i}s(\langle X\rangle_T-\langle X\rangle_t)}|\mathcal{F}_t] \\
\uparrow & & \downarrow \\
K \mapsto (C_t(K), P_t(K)) & & E_t := \mathbb{E}[\varphi(X_T, \langle X\rangle_T)|\mathcal{F}_t]
\end{array}
$$

Figure 1: A visual representation of how European calls and puts reveal prices for path-dependent derivatives.

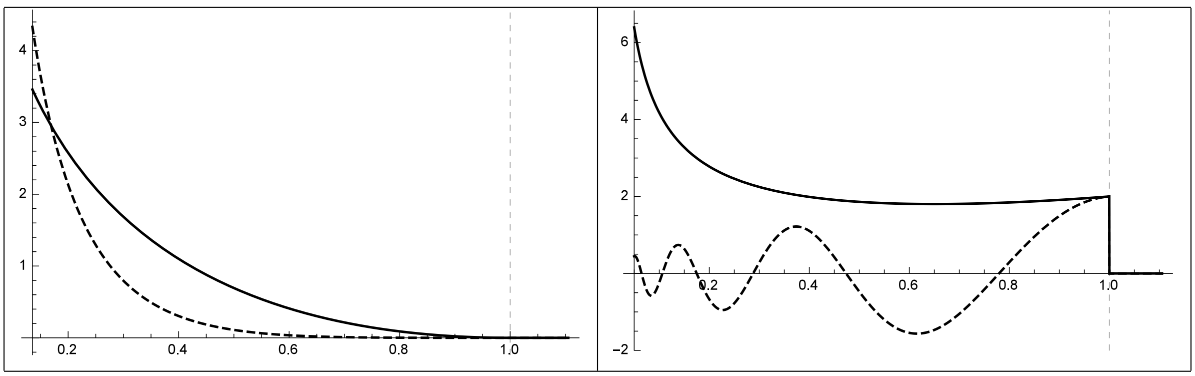

Figure 2: Consider a single barrier knock-in claim with payoff $\mathbb{I}_{\{\tau_L \leq T\}}\varphi(X_T, \langle X\rangle_T - \langle X\rangle_{\tau_L})$. From (4.13), at any time $t < \tau_L$, this claims has the same value as a European-style claim with payoff $\Psi_L(X_T)$, where Ψ_L is given by (4.15). On the left, we plot $\Psi_L(\log s)$ when $\varphi(x, v) = v^n$ for $n = 1$ (solid) and $n = 2$ (dashed). On the right, we plot $\Psi_L(\log s)$ when $\varphi(x, v) = \mathrm{e}^{qv}$ for $q = 1/2$ (solid) and $q = -20$ (dashed). We also plot a vertical dashed line at e^L.

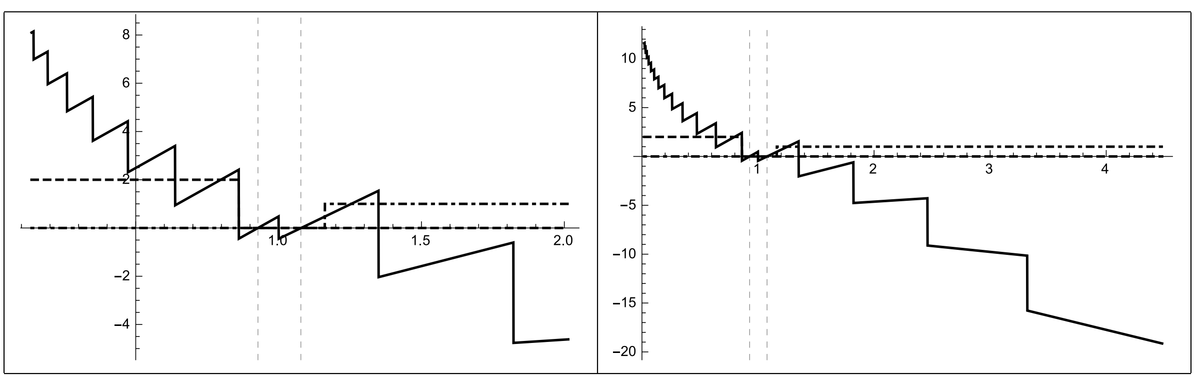

Figure 3: Let $G_L(x) = 2\mathbb{I}_{\{x<K_L\}}$ be the payoff 2 digital puts and $G_U(x) = \mathbb{I}_{\{x>K_U\}}$ be the payoff of a digital call. Fix $L < U$ and construct H from G_L and G_U as in (5.16). Then from (5.17) and (5.18) we have $\mathbb{E}[G_L(X_T)|X_t = L] = \mathbb{E}[H(X_T)|X_t = L]$ and $\mathbb{E}[G_U(X_T)|X_t = U] = \mathbb{E}[H(X_T)|X_t = U]$. Above, we plot $G_L(\log s)$ (black dashed), $G_U(\log s)$ (black dot-dashed) and $H(\log s)$ (solid black), as a function of s. We also plot vertical lines (grey dashed) at $s = \mathrm{e}^L$ and $s = \mathrm{e}^U$. Parameters for both plot are: $K_L = -0.15$, $L = -0.075$, $U = 0.075$ and $K_U = 0.15$.