Multi-asset optimal execution and statistical arbitrage strategies under Ornstein-Uhlenbeck dynamics

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Abstract

In recent years, academics, regulators, and market practitioners have increasingly addressed liquidity issues. Amongst the numerous problems addressed, the optimal execution of large orders is probably the one that has attracted the most research works, mainly in the case of single-asset portfolios. In practice, however, optimal execution problems often involve large portfolios comprising numerous assets, and models should consequently account for risks at the portfolio level. In this paper, we address multi-asset optimal execution in a model where prices have multivariate Ornstein-Uhlenbeck dynamics and where the agent maximizes the expected (exponential) utility of her P&L. We use the tools of stochastic optimal control and simplify the initial multidimensional Hamilton-Jacobi-Bellman equation into a system of ordinary differential equations (ODEs) involving a Matrix Riccati ODE for which classical existence theorems do not apply. By using *a priori* estimates obtained thanks to optimal control tools, we nevertheless prove an existence and uniqueness result for the latter ODE, and then deduce a verification theorem that provides a rigorous solution to the execution problem. Using numerical methods we eventually illustrate our results and discuss their implications. In particular, we show how our model can be used to build statistical arbitrage strategies.

Key words: Optimal execution, Statistical arbitrage, Stochastic optimal control, Riccati equations.

1 Introduction

When executing large blocks of assets, financial agents need to control their overall trading costs by finding the optimal balance between trading rapidly to minimize the market price risk and trading slowly to minimize execution costs and market impact. Building on the first rigorous approaches introduced by Bertsimas and Lo in [11] and Almgren and Chriss in [6] and [7], many models for the optimal execution of large orders have been proposed in the last two decades. Subsequently, almost all practitioners today slice their large orders into small (child) orders according to optimized trading schedules inspired by the academic literature.

The basic Almgren-Chriss model is a discrete-time model where the agent posts market orders (MOs) to maximize a mean-variance objective function. Many extensions of this seminal model have been proposed. Regarding the framework, (Forsyth and Kennedy, [17]) examines the use of quadratic variation rather than variance in the objective function, (Schied and Schöneborn, [33]) uses stochastic control tools to characterize and find optimal strategies for a Von Neumann–Morgenstern investor, and (Guéant, [21]) provides results for optimal liquidation within a Von Neumann-Morgenstern expected utility framework with general market impact functions and derives subsequent results for block trade pricing. As for the model parameters, (Almgren, [3]) studies the case of random execution costs, (Almgren, [4, 5]) addresses stochastic liquidity and volatility, (Lehalle, [28]) discusses how to take into account statistical aspects of the

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variability of estimators of the main exogenous variables such as volumes or volatilities in the optimization phase, and (Cartea and Jaimungal, [14]) provides a closed-form strategy incorporating order flows from all agents. Furthermore, numerous market impact and limit order book (LOB) models have also been studied. For instance, (Obizhaeva and Wang, [32]), later generalized in (Alfonsi, Fruth and Schied, [1]), proposes a single-asset market impact model where price dynamics are derived from a dynamic LOB model with resilience, (Alfonsi and Schied, [2]) derives explicit optimal execution strategies in a discrete-time LOB model with general shape functions and an exponentially decaying price impact, (Gatheral, [19]) uses the no-dynamic-arbitrage principle to address the viability of market impact models, and (Gatheral, Schied and Slynko, [20]) obtains explicit optimal strategies with a transient market impact in an expected cost minimization setup. As for order and execution strategy types, the Almgren-Chriss framework focuses on orders of the Implementation Shortfall (IS) type with MOs only. Other execution strategies have been studied in the literature, like Volume-Weighted Average Price (VWAP) orders in (Konishi, [26]), (Frei and Westray, [18]) and (Guéant and Royer, [25]), but also Target Close (TC) orders and Percentage of Volume (POV) orders, in (Guéant, [22]). Besides, several models focusing on optimal execution with limit orders have been proposed, as in (Bayraktar and Ludkovski, [9]), but also in (Guéant, Lehalle, and Fernandez-Tapia, [24]) and (Guéant and Lehalle, [23]). Regarding the existence of several venues, the case of optimal splitting of orders across different liquidity pools has been addressed in (Laruelle, Lehalle, and Pages, [27]), in (Cartea, Jaimungal, and Penalva, [15]), and more recently in (Baldacci and Manziuk, [8]).

Another recent and important stream of the optimal execution literature deals with adding predictive signals of future price changes.[1] Typical examples of these signals include order book imbalances, forecasts of the future order flow of market participants, and other price-based technical indicators. The usual formalism in the literature with predictive signals is to consider Brownian or Black-Scholes dynamics, along with independent mean-reverting Markov signals. The case of Ornstein-Uhlenbeck-type signals is of special interest as it usually leads to closed-form formulas. For the interested reader, we refer to (Belak and Muhle-Karbe, [10]) where the authors consider optimal execution with general Markov signals and an application to "target zone models", and to (Lehalle and Neuman, [30]) and (Neuman and Voß, [31]) in which the authors provide an optimal trading framework incorporating Markov signals and a transient market impact.

In practice, operators routinely face the problem of having to execute simultaneously large orders regarding various assets, such as in block trading for funds facing large subscriptions or withdrawals, or when considering multi-asset trades in statistical arbitrage trading strategies. More generally, banks and market makers manage their (il)liquidity and market risk, when it comes to executing trades, in the context of a central risk book; hence the need for multi-asset models. However, in contrast to the single-asset case, the existing literature on the joint execution scheduling of large orders in multiple assets, or a single asset inside a multi-asset portfolio, is rather limited. Besides, most papers simply consider correlated Brownian motions when modelling the joint dynamics of prices. The problem of using single-asset models or unrealistic multivariate models for portfolio trading is that they do not balance execution and market impact with price risk at the portfolio or strategy level, and the resulting trading curves of individual assets usually turn out to be suboptimal.

The first paper presenting a way to build multi-asset trading curves in an optimized way is (Almgren and Chriss, [7]). Almgren and Chriss consider indeed, in an appendix of their seminal paper, a multi-asset extension of their discrete-time model – see Appendix A for a solution of the classical portfolio execution problem in a continuous-time setting with correlated Brownian dynamics for prices. Several extensions to this model have been proposed since then. (Lehalle, [29]) considers adding an inventory constraint to balance the different portfolio lines during the portfolio execution process. (Schied and Schöneborn, [34]) shows, under general continuous-time multidimensional price and market impact dynamics and for an exponential utility objective function, that deterministic strategies are optimal. In (Cartea, Jaimungal, and Penalva, [15]), the authors use stochastic control tools to derive optimal execution strategies for basic multi-asset trading algorithms such as optimal entry/exit times and cointegration-based statistical arbitrage. (Bismuth, Guéant, and Pu, [12]) addresses optimal portfolio liquidation (along with other portfolio related problems) by coupling Bayesian learning and stochastic control to derive optimal strategies under uncertainty on model parameters in the Almgren-Chriss framework. Regarding the literature around the addition of predictive signals, (Emschwiller, Petit, and Bouchaud, [16]) extends optimal trading with Markovian predictors to the multi-asset case, with linear trading costs, using a mean-field approach that reduces the problem to a single-asset one.

[1] We consider this stream of the literature to be closely related to our topic of multi-asset optimal execution. Indeed, when trading an asset, the dynamics of another asset within or outside the portfolio can be regarded as a predictive signal that can enhance the execution process.

A notable model for the multivariate dynamics of financial variables that goes beyond the simple one where prices diffuse like correlated Brownian motions is the multivariate Ornstein-Uhlenbeck (multi-OU) model. It is especially attractive because it is parsimonious, and yet general enough to cover a wide spectrum of multi-dimensional dynamics. Multi-OU dynamics offer a large coverage since particular cases include correlated Brownian motions but also cointegrated dynamics which are heavily used in statistical arbitrage. (Cartea, Gan, and Jaimungal, [13]) is, to our knowledge, the pioneering paper in the use of the multi-OU model for the price dynamics in a multi-asset optimal execution problem. Indeed, the authors proposed an interesting model where the asset prices have multi-OU dynamics and the agent maximizes an objective function given by the expectation of the P&L minus a running penalty related to the instantaneous variance of the portfolio. In their approach, the problem boils down to a system of ODEs involving a Matrix Riccati ODE for which the classical existence theorems related to linear-quadratic control theory apply.

In this paper, we propose a model similar to the one in [13], but where the objective function is of the Von Neumann-Morgenstern type: an expected exponential utility of the P&L.[2] By using classical stochastic optimal control tools we show that the problem boils down to solving a system of ODEs involving a Matrix Riccati ODE. However, unlike what happens in [13], the use of an expected exponential utility framework to account for the risk leads to a Matrix Riccati ODE for which classical existence theorems do not apply. By using *a priori* estimates obtained thanks to optimal control tools, we nevertheless prove an existence and uniqueness result for the latter ODE, and then deduce a verification theorem that provides a rigorous solution to the execution problem.

The main contribution of this paper is therefore to propose a model for multi-asset portfolio execution under multi-OU price dynamics in an expected utility framework that accounts for the overall risk associated with the execution process. We focus on the problem where an agent is in charge of unwinding a large portfolio, but also illustrate the use of our results for multi-asset statistical arbitrage purposes.

The remainder of this paper is organized as follows. In Section 2 we present the optimal execution problem in the form of a stochastic optimal control problem and show that solving the associated Hamilton-Jacobi-Bellman (HJB) equation boils down to solving a system of ODEs involving a Matrix Riccati ODE. We then prove a global existence result for that ODE and eventually provide a solution to the initial stochastic optimal control problem thanks to a verification argument. In Section 3, we then illustrate our results with numerical approximations of the optimal strategies and numerical simulations of prices. Our examples focus on optimal liquidation but we also illustrate and discuss the use of our results for building statistical arbitrage strategies. The core of the paper is followed by two appendices: one dedicated to the special case where the multi-OU dynamics reduces to a simple correlated Brownian dynamics and another dedicated to some form of limit case where execution costs are ignored – the latter case being useful to obtain *a priori* estimates for our general problem.

2 The optimal liquidation problem

2.1 Modelling framework and notations

In this paper, we consider a filtered probability space $\left(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}\right)$ satisfying the usual conditions. We assume this probability space to be large enough to support all the processes we introduce.

We consider a market with $d \in \mathbb{N}^*$ assets, and a trader wishing to liquidate her portfolio over a period of time $[0,T]$, with $T > 0$. Her inventory process[3] $(q_t)_{t\in[0,T]} = \left(q^1_t, \ldots, q^d_t\right)^\intercal_{t\in[0,T]}$ evolves as

$$dq_t = v_t dt, \tag{1}$$

with $q_0 \in \mathbb{R}^d$ given, where $(v_t)_{t\in[0,T]} = (v^1_t, \ldots, v^d_t)^\intercal_{t\in[0,T]}$ represents the trading rate of the trader for each asset.

[2] Our model accounts therefore for the risk in a different manner than the model presented in [15]. Comparisons are difficult to carry out as risk aversion parameters in the two models have different meanings.

[3] The superscript $^\intercal$ designates the transpose operator. It transforms here a line vector into a column vector.

The prices of the d assets are modelled as a d-dimensional Ornstein-Uhlenbeck process $(S_t)_{t\in[0,T]} = \left(S_t^1, \ldots, S_t^d\right)^\intercal_{t\in[0,T]}$:[4]

$$dS_t = R(\overline{S} - S_t)dt + V dW_t, \tag{2}$$

with $S_0 \in \mathbb{R}^d$ given, where $\overline{S} \in \mathbb{R}^d$, $R \in \mathcal{M}_d(\mathbb{R})$, $V \in \mathcal{M}_{d,k}(\mathbb{R})$, and $(W_t)_{t\in[0,T]} = \left(W_t^1, \ldots, W_t^k\right)^\intercal_{t\in[0,T]}$ is a k-dimensional standard Brownian motion (with independent coordinates) for some $k \in \mathbb{N}^*$. For what follows, we introduce $\Sigma = VV^\intercal$.

Finally, the process $(X_t)_{t\in[0,T]}$ modelling the trader's cash account has the dynamics

$$dX_t = -v_t^\intercal S_t dt - L(v_t)dt, \tag{3}$$

with $X_0 \in \mathbb{R}$ given, where $L : \mathbb{R}^d \to \mathbb{R}_+$ is a function representing the temporary market impact or execution costs of the trader. In this paper, we mainly consider the case where L is a positive-definite quadratic form (see below).

The trader aims at maximizing the expected utility of her wealth at the end of the trading window $[0, T]$. This wealth is the sum of the amount X_T on the cash account at time T and the value of the remaining inventory evaluated here as $q_T^\intercal S_T - \ell(q_T)$, where the term $\ell(q_T)$ is a discount applied to the Mark-to-Market (MtM) value that proxies liquidity and market price risk and penalizes any terminal non-zero position. In what follows, we mainly consider the case where the penalty function $\ell : \mathbb{R}^d \to \mathbb{R}_+$ is a positive-semidefinite quadratic form (see below).

Mathematically, the trader therefore wants to solve the dynamic optimization problem

$$\sup_{v\in\mathcal{A}} \mathbb{E}\left[-e^{-\gamma\left(X_T + q_T^\intercal S_T - \ell(q_T)\right)}\right], \tag{4}$$

where $\gamma > 0$ is the absolute risk aversion parameter of the trader, and $\mathcal{A}$ is the set of admissible controls, to be defined below.

To define the set of admissible controls $\mathcal{A}$, we first introduce a notion of "linear growth" relevant in our context.

Definition 1. *Let $t \in [0, T]$. An $\mathbb{R}^d$-valued, $\mathbb{F}$-adapted process $(\zeta_s)_{s\in[t,T]}$ is said to satisfy a linear growth condition on $[t, T]$ with respect to $(S_s)_{s\in[t,T]}$ if there exists a constant $C_T > 0$ such that for all $s \in [t, T]$,*

$$\|\zeta_s\| \leq C_T \left(1 + \sup_{\tau\in[t,s]} \|S_\tau\|\right)$$

almost surely.

We then define for all $t \in [0, T]$:

$$\mathcal{A}_t = \left\{(v_s)_{s\in[t,T]},\ \mathbb{R}^d\text{-valued},\ \mathbb{F}\text{-adapted, satisfying a linear growth condition with respect to } (S_s)_{s\in[t,T]}\right\}, \tag{5}$$

and take $\mathcal{A} := \mathcal{A}_0$.

It is natural to use the tools of stochastic optimal control to solve the above dynamic optimization problem. Let us define the value function of the problem $u : [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ as

$$u(t, x, q, S) = \sup_{v\in\mathcal{A}_t} \mathbb{E}\left[-e^{-\gamma\left(X_T^{t,x,S,v} + (q_T^{t,q,v})^\intercal S_T^{t,S} - \ell(q_T^{t,q,v})\right)}\right], \tag{6}$$

where for $(t, x, q, S) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ and $v \in \mathcal{A}_t$, the processes $(q_s^{t,q,v})_{s\in[t,T]}$, $(S_s^{t,S})_{s\in[t,T]}$, and $(X_s^{t,x,S,v})_{s\in[t,T]}$ have respective dynamics

$$dq_s^{t,q,v} = v_s ds,$$
$$dS_s^{t,S} = R(\overline{S} - S_s^{t,S})ds + V dW_s,$$

and

$$dX_s^{t,x,S,v} = -v_s^\intercal S_s^{t,S} ds - L(v_s)ds,$$

with $S_t^{t,S} = S$, $q_t^{t,q,v} = q$, and $X_t^{t,x,S,v} = x$.

[4]The generalization with a permanent impact component is straightforward.

2.2 Hamilton-Jacobi-Bellman equation

The HJB equation associated with the problem (4) is given by

$$\begin{aligned}0 = \partial_t w(t,x,q,S) &+ \sup_{v\in\mathbb{R}^d} \left(-(v^\intercal S + L(v))\partial_x w(t,x,q,S) + v^\intercal \nabla_q w(t,x,q,S)\right)\\ &+ (\overline{S}-S)^\intercal R^\intercal \nabla_S w(t,x,q,S) + \frac{1}{2}\mathrm{Tr}\left(\Sigma D^2_{SS} w(t,x,q,S)\right),\end{aligned} \tag{7}$$

for all $(t,x,q,S)\in[0,T)\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d$ with the terminal condition

$$w(T,x,q,S) = -e^{-\gamma(x+q^\intercal S-\ell(q))} \quad \forall (x,q,S)\in\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d. \tag{8}$$

In order to study (7), we are going to use the following ansatz:

$$w(t,x,q,S) = -e^{-\gamma(x+q^\intercal S+\theta(t,q,S))} \quad \forall (t,x,q,S)\in[0,T]\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d. \tag{9}$$

The interest of this ansatz is based on the following proposition:

Proposition 1. *Let $\tau < T$. If there exists $\theta : [\tau,T]\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ solution to*

$$\begin{aligned}0 = \partial_t\theta(t,q,S) &+ \sup_{v\in\mathbb{R}^d}\left(v^\intercal\nabla_q\theta(t,q,S) - L(v)\right) + \frac{1}{2}Tr\left(\Sigma D^2_{SS}\theta(t,q,S)\right)\\ &-\frac{\gamma}{2}(q+\nabla_S\theta(t,q,S))^\intercal\Sigma(q+\nabla_S\theta(t,q,S)) + (\overline{S}-S)^\intercal R^\intercal(\nabla_S\theta(t,q,S)+q)\end{aligned} \tag{10}$$

on $[\tau,T)\times\mathbb{R}^d\times\mathbb{R}^d$, with terminal condition

$$\theta(T,q,S) = -\ell(q) \quad \forall (q,S)\in\mathbb{R}^d\times\mathbb{R}^d, \tag{11}$$

then the function $w : [\tau,T]\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ defined by

$$w(t,x,q,S) = -e^{-\gamma(x+q^\intercal S+\theta(t,q,S))} \quad \forall (t,x,q,S)\in[\tau,T]\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d$$

is a solution to (7) on $[\tau,T)\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d$ with terminal condition (8).

Proof. Let $\theta : [\tau,T]\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ be a solution to (10) on $[\tau,T)\times\mathbb{R}^d\times\mathbb{R}^d$ with terminal condition (11), then we have for all $(t,x,q,S)\in[\tau,T)\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d$:

$$\begin{aligned}&\partial_t w(t,x,q,S) + \sup_{v\in\mathbb{R}^d}\left(-(v^\intercal S+L(v))\partial_x w(t,x,q,S) + v^\intercal\nabla_q w(t,x,q,S)\right)\\ &+(\overline{S}-S)^\intercal R^\intercal\nabla_S w(t,x,q,S) + \frac{1}{2}\mathrm{Tr}\left(\Sigma D^2_{SS}w(t,x,q,S)\right)\\ =\ & -\gamma\partial_t\theta(t,q,S)w(t,x,q,S) + \sup_{v\in\mathbb{R}^d}\left(\gamma(v^\intercal S+L(v))w(t,x,q,S) - \gamma v^\intercal(\nabla_q\theta(t,q,S)+S)w(t,x,q,S)\right)\\ &+\frac{\gamma^2}{2}\mathrm{Tr}\left(\Sigma(q+\nabla_S\theta(t,q,S))(q+\nabla_S\theta(t,q,S))^\intercal w(t,x,q,S)\right)\\ &-\gamma(\overline{S}-S)^\intercal R^\intercal(\nabla_S\theta(t,q,S)+q)w(t,x,q,S) - \frac{1}{2}\mathrm{Tr}\left(\gamma\Sigma D^2_{SS}\theta(t,q,S)w(t,x,q,S)\right)\\ =\ & -\gamma w(t,x,q,S)\Bigg(\partial_t\theta(t,q,S) + \sup_{v\in\mathbb{R}^d}\left(v^\intercal\nabla_q\theta(t,q,S)-L(v)\right) + \frac{1}{2}\mathrm{Tr}\left(\Sigma D^2_{SS}\theta(t,q,S)\right)\\ &-\frac{\gamma}{2}(q+\nabla_S\theta(t,q,S))^\intercal\Sigma(q+\nabla_S\theta(t,q,S)) + (\overline{S}-S)^\intercal R^\intercal(\nabla_S\theta(t,q,S)+q)\Bigg)\\ =\ & 0.\end{aligned}$$

As it is straightforward to verify that w satisfies the terminal condition (8), the result is proved. □

Assumption 1. *From now on, we assume that the functions L and ℓ are of the form $L(v) = v^\intercal\eta v$ and $\ell(q) = q^\intercal\Gamma q$, for some $\eta\in\mathcal{S}^{++}_d(\mathbb{R})$ and $\Gamma\in\mathcal{S}^{+}_d(\mathbb{R})$.*

With the above assumption, the Legendre-Fenchel transform of L writes

$$H : p \in \mathbb{R}^d \mapsto \sup_{v\in\mathbb{R}^d} v^\intercal p - v^\intercal \eta v = \frac{1}{4}p^\intercal \eta^{-1} p, \tag{12}$$

as the supremum is reached at $v^* = \frac{1}{2}\eta^{-1}p$.

Consequently, we get the following HJB equation for θ:

$$\begin{aligned} 0 = {}& \partial_t\theta(t,q,S) + \frac{1}{4}\nabla_q\theta(t,q,S)^\intercal\eta^{-1}\nabla_q\theta(t,q,S) + \frac{1}{2}\mathrm{Tr}\left(\Sigma D^2_{SS}\theta(t,q,S)\right) \\ & - \frac{\gamma}{2}(q+\nabla_S\theta(t,q,S))^\intercal\Sigma(q+\nabla_S\theta(t,q,S)) + (\overline{S}-S)^\intercal R^\intercal(\nabla_S\theta(t,q,S)+q), \end{aligned} \tag{13}$$

with terminal condition

$$\theta(T,q,S) = -q^\intercal\Gamma q \quad \forall (q,S)\in\mathbb{R}^d\times\mathbb{R}^d. \tag{14}$$

To further study (13), we introduce a second ansatz and look for a solution θ of the following form:

$$\theta(t,q,S) = q^\intercal A(t)q + q^\intercal B(t)S + S^\intercal C(t)S + D(t)^\intercal q + E(t)^\intercal S + F(t) \quad \forall (t,q,S)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^d. \tag{15}$$

The interest of this ansatz is stated in the following proposition:

Proposition 2. *Let $\tau < T$. Assume there exist $A\in C^1([\tau,T],\mathcal{S}_d(\mathbb{R}))$, $B\in C^1([\tau,T],\mathcal{M}_d(\mathbb{R}))$, $C\in C^1([\tau,T],\mathcal{S}_d(\mathbb{R}))$, $D\in C^1\left([\tau,T],\mathbb{R}^d\right)$, $E\in C^1\left([\tau,T],\mathbb{R}^d\right)$, $F\in C^1([\tau,T],\mathbb{R})$ satisfying the system of ODEs*

$$\begin{cases} A'(t) = \frac{\gamma}{2}(B(t)+I_d)\Sigma(B(t)^\intercal+I_d) - A(t)\eta^{-1}A(t) \\ B'(t) = (B(t)+I_d)R + 2\gamma(B(t)+I_d)\Sigma C(t) - A(t)\eta^{-1}B(t) \\ C'(t) = R^\intercal C(t) + C(t)R + 2\gamma C(t)\Sigma C(t) - \frac{1}{4}B(t)^\intercal\eta^{-1}B(t) \\ D'(t) = -(B(t)+I_d)R\overline{S} + \gamma(B(t)+I_d)\Sigma E(t) - A(t)\eta^{-1}D(t) \\ E'(t) = -2C(t)R\overline{S} + R^\intercal E(t) + 2\gamma C(t)\Sigma E(t) - \frac{1}{2}B(t)^\intercal\eta^{-1}D(t) \\ F'(t) = -\overline{S}^\intercal R^\intercal E(t) - Tr(\Sigma C(t)) + \frac{\gamma}{2}E(t)^\intercal\Sigma E(t) - \frac{1}{4}D(t)^\intercal\eta^{-1}D(t), \end{cases} \tag{16}$$

where I_d denotes the identity matrix in $\mathcal{M}_d(\mathbb{R})$, with terminal conditions

$$A(T) = -\Gamma, \quad B(T) = C(T) = D(T) = E(T) = F(T) = 0. \tag{17}$$

Then the function θ defined by (15) *satisfies* (13) *on $[\tau,T)\times\mathbb{R}^d\times\mathbb{R}^d$ with terminal condition* (14)*.*

Proof. Let us consider $A\in C^1([\tau,T],\mathcal{S}_d(\mathbb{R}))$, $B\in C^1([\tau,T],\mathcal{M}_d(\mathbb{R}))$, $C\in C^1([\tau,T],\mathcal{S}_d(\mathbb{R}))$, $D\in C^1\left([\tau,T],\mathbb{R}^d\right)$, $E\in C^1\left([\tau,T],\mathbb{R}^d\right)$, $F\in C^1([\tau,T],\mathbb{R})$ verifying (16) on $[\tau,T)$ with terminal condition (17). Let us consider $\theta : [\tau,T]\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ defined by (15). Then we obtain for all $(t,q,S)\in[\tau,T)\times\mathbb{R}^d\times\mathbb{R}^d$:

$$\begin{aligned} & \partial_t\theta(t,q,S) + \frac{1}{4}\nabla_q\theta(t,q,S)^\intercal\eta^{-1}\nabla_q\theta(t,q,S) + \frac{1}{2}\mathrm{Tr}\left(\Sigma D^2_{SS}\theta(t,q,S)\right) \\ & - \frac{\gamma}{2}(q+\nabla_S\theta(t,q,S))^\intercal\Sigma(q+\nabla_S\theta(t,q,S)) + (\overline{S}-S)^\intercal R^\intercal(\nabla_S\theta(t,q,S)+q), \\ = \quad & q^\intercal A'(t)q + q^\intercal B'(t)S + S^\intercal C'(t)S + D'(t)^\intercal q + E'(t)^\intercal S + F'(t) \\ & + q^\intercal A(t)\eta^{-1}A(t)q + q^\intercal A(t)\eta^{-1}B(t)S + \frac{1}{4}S^\intercal B(t)^\intercal\eta^{-1}B(t)S \\ & + D(t)^\intercal\eta^{-1}A(t)q + \frac{1}{2}\left(D(t)\right)^\intercal\eta^{-1}\left(B(t)\right)S + \frac{1}{4}D(t)^\intercal\eta^{-1}D(t) \\ & + Tr(\Sigma C(t)) - \frac{\gamma}{2}\left(q + B(t)^\intercal q + 2C(t)S + E(t)\right)^\intercal\Sigma\left(q + B(t)^\intercal q + 2C(t)S + E(t)\right) \\ & + \overline{S}^\intercal R^\intercal q + \overline{S}^\intercal R^\intercal\left(B(t)^\intercal q + 2C(t)S + E(t)\right) - S^\intercal R^\intercal q - S^\intercal R^\intercal\left(B(t)^\intercal q + 2C(t)S + E(t)\right) \\ = \quad & q^\intercal\left(A'(t) - \frac{\gamma}{2}(B(t)+I_d)\Sigma(B(t)^\intercal+I_d) + \frac{1}{4}\left(2A(t)\right)\eta^{-1}\left(2A(t)\right)\right)q \end{aligned}$$

$$
\begin{aligned}
&+ q^\intercal \left(B'(t) - (I_d + B(t))R - 2\gamma(B(t)+I_d)\Sigma C(t) + A(t)\eta^{-1}B(t)\right) S \\
&+ S^\intercal \left(C'(t) - R^\intercal C(t) - C(t)R - 2\gamma C(t)\Sigma C(t) + \frac{1}{4}B(t)^\intercal \eta^{-1} B(t)\right) S \\
&+ \left(D'(t) + (B(t)+I_d)R\overline{S} - \gamma(B(t)+I_d)\Sigma E(t) + A(t)\eta^{-1}D(t)\right)^\intercal q \\
&+ \left(E'(t) + 2C(t)R\overline{S} - R^\intercal E(t) - 2\gamma C(t)\Sigma E(t) + \frac{1}{2}B(t)^\intercal \eta^{-1} D(t)\right)^\intercal S \\
&+ \left(F'(t) + \overline{S}^\intercal R^\intercal E(t) + Tr(\Sigma C(t)) - \frac{\gamma}{2}E(t)^\intercal \Sigma E(t) + \frac{1}{4}D(t)^\intercal \eta^{-1} D(t)\right) \\
= \quad & 0.
\end{aligned}
$$

As it is straightforward to verify that θ satisfies the terminal condition (14), the result is proved. □

Remark 1. *Two remarks can be made on the system of ODEs* (16)*:*

- *This system of ODEs can clearly be decomposed into three groups of equations: the first three ODEs for A, B, and C are independent of the others and can be solved as a first step; once we know A, B, and C we can solve the linear ODEs for D and E, and finally F can be obtained with a simple integration;*
- *When $R = 0$ (i.e. in the case where the prices S of the d assets are correlated arithmetic Brownian motions), there is a trivial solution to the last five equations which is $B = C = D = E = F = 0$. A can then be found as shown in Appendix A.*

It is noteworthy that the first system, i.e.

$$
\begin{cases}
A'(t) = \frac{\gamma}{2}(B(t)+I_d)\Sigma(B(t)^\intercal + I_d) - A(t)\eta^{-1}A(t) \\
B'(t) = (B(t)+I_d)R + 2\gamma(B(t)+I_d)\Sigma C(t) - A(t)\eta^{-1}B(t) \\
C'(t) = R^\intercal C(t) + C(t)R + 2\gamma C(t)\Sigma C(t) - \frac{1}{4}B(t)^\intercal \eta^{-1} B(t)
\end{cases}
\tag{18}
$$

boils down to a Matrix Riccati ODE. Indeed, defining $P : [0,T] \to \mathcal{S}_{2d}(\mathbb{R})$ as

$$
P(t) = \begin{pmatrix} A(t) & \frac{1}{2}B(t) \\ \frac{1}{2}B(t)^\intercal & C(t) \end{pmatrix},
\tag{19}
$$

we see that (18) with terminal condition $A(T) = -\Gamma$ and $B(T) = C(T) = 0$ is equivalent to

$$
P'(t) = Q + Y^\intercal P(t) + P(t)Y + P(t)UP(t),
\tag{20}
$$

with terminal condition

$$
P(T) = \begin{pmatrix} -\Gamma & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{S}_{2d}(\mathbb{R}),
\tag{21}
$$

where

$$
Q = \frac{1}{2}\begin{pmatrix} \gamma\Sigma & R \\ R^\intercal & 0 \end{pmatrix} \in \mathcal{S}_{2d}(\mathbb{R}), \qquad Y = \begin{pmatrix} 0 & 0 \\ \gamma\Sigma & R \end{pmatrix} \in \mathcal{M}_{2d}(\mathbb{R}), \qquad U = \begin{pmatrix} -\eta^{-1} & 0 \\ 0 & 2\gamma\Sigma \end{pmatrix} \in \mathcal{S}_{2d}(\mathbb{R}).
$$

When compared to the Matrix Riccati ODEs arising in the linear-quadratic optimal control literature, the distinctive aspect of our equation is that the matrix U characterizing the quadratic term in the Riccati equation has both positive and negative eigenvalues. In particular, we cannot rely on existing results coming from linear-quadratic control theory to prove that there exists a solution to (20) with terminal condition (21). In this paper, we address the existence of a solution by using *a priori* estimates for the value function.

Regarding the set of equations (18), there exists a unique local solution by Cauchy-Lipschitz theorem. In the following section, we therefore first state a verification theorem that solves the problem when on an interval $[\tau, T]$, and use that very result to address global existence and uniqueness of a solution on $[0, T]$.

2.3 Main mathematical results

Theorem 1. *Let $\tau < T$. Let $A \in C^1([\tau,T], \mathcal{S}_d(\mathbb{R}))$, $B \in C^1([\tau,T], \mathcal{M}_d(\mathbb{R}))$, $C \in C^1([\tau,T], \mathcal{S}_d(\mathbb{R}))$, $D \in C^1\left([\tau,T], \mathbb{R}^d\right)$, $E \in C^1\left([\tau,T], \mathbb{R}^d\right)$, $F \in C^1([\tau,T], \mathbb{R})$ be a solution to the system (16) on $[\tau, T)$ with terminal condition (17), and consider the function θ defined by (15) and the associated function w defined by (9).*

Then for all $(t,x,q,S) \in [\tau,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ and $v = (v_s)_{s\in[t,T]} \in \mathcal{A}_t$, we have

$$\mathbb{E}\left[-e^{-\gamma\left(X_T^{t,x,S,v} + \left(q_T^{t,q,v}\right)^\intercal S_T^{t,S} - \ell(q_T^{t,q,v})\right)}\right] \le w(t,x,q,S). \tag{22}$$

Moreover, equality is obtained in (22) by taking the optimal control $(v_s^*)_{s\in[t,T]} \in \mathcal{A}_t$ given by the closed-loop feedback formula

$$v_s^* = \frac{1}{2}\eta^{-1}\left(2A(s)q_s^{t,q,v} + B(s)S_s^{t,S} + D(s)\right). \tag{23}$$

In particular, $w = u$ on $[\tau,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$.

Proof. Let $t \in [\tau, T)$, we first prove that $(v_s^*)_{s\in[t,T]} \in \mathcal{A}_t$ (i.e., $(v_s^*)_{s\in[t,T]}$ is well-defined and admissible). Let us consider the Cauchy initial value problem

$$\forall s \in [t,T], \quad \frac{d\tilde{q}_s}{ds} = \frac{1}{2}\eta^{-1}\left(2A(s)\tilde{q}_s + B(s)S_s^{t,S} + D(s)\right), \qquad \tilde{q}_t = q.$$

The unique solution of that Cauchy problem writes

$$\tilde{q}_s = \exp\left(\int_t^s \phi(\varrho)d\varrho\right)\left(q + \int_t^s \psi\left(\varrho, S_\varrho^{t,S}\right)\exp\left(-\int_t^\varrho \phi(\varsigma)d\varsigma\right)d\varrho\right), \tag{24}$$

where ϕ and ψ are defined by

$$\begin{aligned}
&\phi : s \in [t,T] \mapsto \eta^{-1}A(s),\\
&\psi : (s,S) \in [t,T] \times \mathbb{R}^d \mapsto \frac{1}{2}\eta^{-1}\left(B(s)S + D(s)\right).
\end{aligned}$$

Then v^* can be written as

$$v_s^* = \frac{d\tilde{q}_s}{ds} = \phi(s)\exp\left(\int_t^s \phi(\varrho)d\varrho\right)\left(q + \int_t^s \psi\left(\varrho, S_\varrho^{t,S}\right)\exp\left(-\int_t^\varrho \phi(\varsigma)d\varsigma\right)d\varrho\right) + \psi\left(s, S_s^{t,S}\right). \tag{25}$$

We see from the definition of ϕ and the affine form of ψ in S that v^* satisfies a linear growth condition, and is therefore in $\mathcal{A}_t$.

Let us consider $(t,x,q,S) \in [\tau,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ and $v = (v_s)_{s\in[t,T]} \in \mathcal{A}_t$. We now prove that

$$\mathbb{E}\left[w(T, X_T^{t,x,S,v}, q_T^{t,q,v}, S_T^{t,S})\right] \leqslant w(t,x,q,S). \tag{26}$$

We use the following notations for readability:

$$\begin{aligned}
&\forall s \in [t,T], \quad w(s, X_s^{t,x,S,v}, q_s^{t,q,v}, S_s^{t,S}) = w_s^{t,x,q,S,v},\\
&\forall s \in [t,T], \quad \theta(s, q_s^{t,q,v}, S_s^{t,S}) = \theta_s^{t,q,S,v}.
\end{aligned}$$

By Itô's formula, we have $\forall s \in [\tau, T]$

$$dw_s^{t,x,q,S,v} = \mathcal{L}^v w_s^{t,x,q,S,v} ds + \left(\nabla_S w_s^{t,x,q,S,v}\right)^\intercal V dW_s, \tag{27}$$

where

$$\begin{aligned}
\mathcal{L}^v w_s^{t,x,q,S,v} = {}& \partial_t w_s^{t,x,q,S,v} - (v^\intercal S + v^\intercal \eta v)\partial_x w_s^{t,x,q,S,v} + v^\intercal \nabla_q w_s^{t,x,q,S,v}\\
&+ (\overline{S} - S)^\intercal R^\intercal \nabla_S w_s^{t,x,q,S,v} + \frac{1}{2}\mathrm{Tr}\left(\Sigma D_{SS}^2 w_s^{t,x,q,S,v}\right).
\end{aligned} \tag{28}$$

From (9) and (15) we have

$$\begin{aligned}\nabla_S w_s^{t,x,q,S,v} &= -\gamma w_s^{t,x,q,S,v}\left(q_s^{t,q,v} + \nabla_S \theta_s^{t,q,S,v}\right)\\ &= -\gamma w_s^{t,x,q,S,v}\left(q_s^{t,q,v} + B(s)^\intercal q_s^{t,q,v} + 2C(s)S_s^{t,S} + E(s)\right).\end{aligned} \tag{29}$$

We define $\forall s \in [t,T]$,

$$\kappa_s^{q,S,v} = -\gamma\left(q_s^{t,q,v} + B(s)^\intercal q_s^{t,q,v} + 2C(s)S_s^{t,S} + E(s)\right), \tag{30}$$

$$\xi_{t,s}^{q,S,v} = \exp\left(\int_t^s \kappa_\varrho^{q,S,v\intercal} V dW_\varrho - \frac{1}{2}\int_t^s \kappa_\varrho^{q,S,v\intercal}\Sigma\kappa_\varrho^{q,S,v} d\varrho\right). \tag{31}$$

We then have

$$d\left(w_s^{t,x,q,S,v}\left(\xi_{t,s}^{q,S,v}\right)^{-1}\right) = \left(\xi_{t,s}^{q,S,v}\right)^{-1}\mathcal{L}^v w_s^{t,x,q,S,v} ds. \tag{32}$$

By definition of w, $\mathcal{L}^v w_s^{t,x,q,S,v} \leqslant 0$. Moreover, equality holds for the control reaching the sup in (12). The sup is reached for the unique value

$$v_s = \frac{1}{2}\eta^{-1}\nabla_q \theta_s^{t,q,S,v} \tag{33}$$

$$= \frac{1}{2}\eta^{-1}\left(2A(s)q_s^{t,q,v} + B(s)S_s^{t,S} + D(s)\right), \tag{34}$$

which corresponds to the case $(v_s)_{s\in[t,T]} = (v_s^*)_{s\in[t,T]}$.

As a consequence, $\left(w_s^{t,x,q,S,v}\left(\xi_{t,s}^{q,S,v}\right)^{-1}\right)_{s\in[t,T]}$ is nonincreasing and therefore

$$w(T, X_T^{t,x,S,v}, q_T^{t,q,v}, S_T^{t,S}) \leqslant w(t,x,q,S)\xi_{t,T}^{q,S,v}, \tag{35}$$

with equality when $(v_s)_{s\in[t,T]} = (v_s^*)_{s\in[t,T]}$.

Taking expectations we get

$$\mathbb{E}\left[w\left(T, X_T^{t,x,S,v}, q_T^{t,q,v}, S_T^{t,S}\right)\right] \leqslant w(t,x,q,S)\mathbb{E}\left[\xi_{t,T}^{q,S,v}\right]. \tag{36}$$

We proceed to prove that $\mathbb{E}\left[\xi_{t,T}^{q,S,v}\right]$ is equal to 1. To do so, we use that $\xi_{t,t}^{q,S,v} = 1$ and prove that $(\xi_{t,s}^{q,S,v})_{s\in[t,T]}$ is a martingale under $\left(\mathbb{P}; \mathbb{F} = (\mathcal{F}_s)_{s\in[t,T]}\right)$.

We know that $(q_s^{t,q,v})_{s\in[t,T]}$ satisfies a linear growth condition with respect to $(S_s^{t,S})_{s\in[t,T]}$ since v is an admissible control. Given the form of κ, there exists a constant C such that, almost surely,

$$\sup_{s\in[t,T]} \| \kappa_s^{q,S,v} \|^2 \leqslant C\left(1 + \sup_{s\in[t,T]} \| W_s - W_t \|^2\right). \tag{37}$$

By using classical properties of the Brownian motion, we prove that

$$\exists \epsilon > 0, \forall s \in [t,T], \quad \mathbb{E}\left[\exp\left(\frac{1}{2}\int_s^{(s+\epsilon)\wedge T}\left(\kappa_\varrho^{q,S,v}\right)^\intercal \Sigma \kappa_\varrho^{q,S,v} d\varrho\right)\right] < +\infty. \tag{38}$$

From Novikov condition, we see that $(\xi_{t,s}^{q,S,v})_{s\in[t,T]}$ is a martingale under $\left(\mathbb{P}; \mathbb{F} = (\mathcal{F}_s)_{s\in[t,T]}\right)$.

We obtain

$$\mathbb{E}\left[w(T, X_T^{t,x,S,v}, q_T^{t,q,v}, S_T^{t,S})\right] \leqslant w(t,x,q,S), \tag{39}$$

with equality when $(v_s)_{s\in[t,T]} = (v^*_s)_{s\in[t,T]}$.

We conclude that

$$u(t,x,q,S) = \sup_{(v_s)_{s\in[t,T]}\in\mathcal{A}_t} \mathbb{E}\left[-\exp\left(-\gamma\left(X_T^{t,x,S,v} + \left(q_T^{t,q,v}\right)^\intercal S_T^{t,S} - \ell(q_T^{t,q,v})\right)\right)\right] \quad (40)$$

$$= \mathbb{E}\left[-\exp\left(-\gamma\left(X_T^{t,x,S,v^*} + \left(q_T^{t,q,v^*}\right)^\intercal S_T^{t,S} - \ell(q_T^{t,q,v^*})\right)\right)\right] \quad (41)$$

$$= w(t,x,q,S). \quad (42)$$

□

We will next proceed to prove existence and uniqueness of a solution to the system of ODEs (16) on $[0,T]$ with terminal condition (17), or equivalently to (20) with terminal condition (21).[5]

Theorem 2. *There exists a unique solution* $A \in C^1([0,T],\mathcal{S}_d(\mathbb{R}))$, $B \in C^1([0,T],\mathcal{M}_d(\mathbb{R}))$, $C \in C^1([0,T],\mathcal{S}_d(\mathbb{R}))$, $D \in C^1\left([0,T],\mathbb{R}^d\right)$, $E \in C^1\left([0,T],\mathbb{R}^d\right)$, $F \in C^1([0,T],\mathbb{R})$ *to the system of ODEs* (16) *on* $[0,T]$ *with terminal condition* (17).

Proof. To prove Theorem 2, it is enough, as explained in Remark 1, to show existence and uniqueness for $A \in C^1([0,T],\mathcal{S}_d(\mathbb{R}))$, $B \in C^1([0,T],\mathcal{M}_d(\mathbb{R}))$, and $C \in C^1([0,T],\mathcal{S}_d(\mathbb{R}))$, or equivalently, existence and uniqueness on $[0,T]$ of a solution $P \in C^1([0,T],\mathcal{S}_{2d}(\mathbb{R}))$ to (20) with terminal condition (21).

First, by Cauchy-Lipschitz theorem, there exists a unique maximal solution[6] (A,B,C) to the system of ODEs (18) with terminal condition (17) defined on an open interval $(t_{\min},t_{\max}) \ni T$.

We now show that $t_{\min} = -\infty$, which implies our theorem.

By contradiction, let us assume that $t_{\min} \in (-\infty,T)$ and let $\tau \in (t_{\min},T)$.

Starting from values $(t,x,q,S) \in [\tau,T]\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d$, let us consider the suboptimal strategy $v = (0)_{s\in[t,T]} \in \mathcal{A}_t$ for which $\forall s \in [t,T], q_s^{t,q,v} = q$ and

$$\mathbb{E}\left[-\exp\left(-\gamma\left(X_T^{t,x,S,v} + (q_T^{t,q,v})^\intercal S_T^{t,S} - \ell(q_T^{t,q,v})\right)\right)\right] = \mathbb{E}\left[-\exp\left(-\gamma\left(x + q^\intercal S + q^\intercal\left(S_T^{t,S} - S\right) - q^\intercal\Gamma q\right)\right)\right]. \quad (43)$$

Since $(S_s^{t,S})_{s\in[t,T]}$ follows multivariate Ornstein-Uhlenbeck dynamics, we know that

$$S_T^{t,S} - S = \left(I - e^{-R(T-t)}\right)\left(\overline{S} - S\right) + \int_t^T e^{-R(T-s)} V dW_s.$$

Then $S_T^{t,S} - S \sim \mathcal{N}\left(\left(I - e^{-R(T-t)}\right)\left(\overline{S} - S\right), \Sigma_t\right)$, where the covariance matrix is defined by

$$\Sigma_t = \int_t^T e^{-R(T-s)} \Sigma e^{-R^\intercal(T-s)} ds.$$

Then,

$$\begin{aligned}&\mathbb{E}\left[-\exp\left(-\gamma\left(X_T^{t,x,S,v} + (q_T^{t,q,v})^\intercal S_T^{t,S} - \ell(q_T^{t,q,v})\right)\right)\right]\\ =\ & -\exp\left(-\gamma(x + q^\intercal S)\right)\exp\left(-\gamma\left(q^\intercal\left(I - e^{-R(T-t)}\right)\left(\overline{S} - S\right) - q^\intercal\Gamma q - \frac{1}{2}\gamma q^\intercal\Sigma_t q\right)\right).\end{aligned} \quad (44)$$

Since the strategy is sub-optimal, if we consider θ defined as in (15), we have by Theorem 1

$$-\exp\left(-\gamma\left(x + q^\intercal S + \theta(t,q,S)\right)\right) \geq -\exp\left(-\gamma(x + q^\intercal S)\right)\exp\left(-\gamma\left(q^\intercal\left(I - e^{-R(T-t)}\right)\left(\overline{S} - S\right) - q^\intercal\Gamma q - \frac{1}{2}\gamma q^\intercal\Sigma_t q\right)\right). \quad (45)$$

[5] The result in fact holds on $(-\infty,T]$ as the initial time plays no role.

[6] The fact that A and C are symmetric is itself a consequence of Cauchy-Lipschitz theorem since (A,B,C) and $(A^\intercal,B,C^\intercal)$ are solution of the same Cauchy problem.

We conclude that for all $(t,q,S) \in [\tau,T] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\begin{aligned}\theta(t,q,S) &= \begin{pmatrix} q \\ S \end{pmatrix}^\intercal P(t) \begin{pmatrix} q \\ S \end{pmatrix} + D(t)^\intercal q + E(t)^\intercal S + F(t) \\ &\geq \begin{pmatrix} q \\ S \end{pmatrix}^\intercal \begin{pmatrix} -\frac{\gamma}{2}\Sigma_t - \Gamma & -\frac{1}{2}\left(I - e^{-R(T-t)}\right) \\ -\frac{1}{2}\left(I - e^{-R^\intercal(T-t)}\right) & 0 \end{pmatrix} \begin{pmatrix} q \\ S \end{pmatrix} + \overline{S}^\intercal \left(I - e^{-R^\intercal(T-t)}\right) q,\end{aligned}$$

where $P(t) = \begin{pmatrix} A(t) & \frac{1}{2}B(t) \\ \frac{1}{2}B(t)^\intercal & C(t) \end{pmatrix}$.

We therefore necessarily have, for the natural order on symmetric matrices,[7]

$$\forall t \in [\tau,T], \quad P(t) \geq \begin{pmatrix} -\frac{\gamma}{2}\Sigma_t - \Gamma & -\frac{1}{2}\left(I - e^{-R(T-t)}\right) \\ -\frac{1}{2}\left(I - e^{-R^\intercal(T-t)}\right) & 0 \end{pmatrix}.$$

Now, for $(t,x,q,S) \in [\tau,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$, we have

$$\sup_{v \in \mathcal{A}_t} \mathbb{E}\left[-\exp\left(-\gamma\left(X_T^{t,x,S,v} + (q_T^{t,q,v})^\intercal S_T^{t,S} - (q_T^{t,q,v})^\intercal \Gamma q_T^{t,q,v}\right)\right)\right] \tag{46}$$

$$= \sup_{v \in \mathcal{A}_t} \mathbb{E}\left[-\exp\left(-\gamma\left(x + q^\intercal S + \int_t^T (q_s^{t,q,v})^\intercal dS_s - \int_t^T L(v_s)ds - (q_T^{t,q,v})^\intercal \Gamma q_T^{t,q,v}\right)\right)\right]$$

$$\leq \exp\left(-\gamma\left(x + q^\intercal S\right)\right) \sup_{v \in \mathcal{A}_t} \mathbb{E}\left[-\exp\left(-\gamma\left(\int_t^T (q_s^{t,q,v})^\intercal dS_s\right)\right)\right], \tag{47}$$

If $(v_s)_{s\in[t,T]} \in \mathcal{A}_t$, it is straightforward to see that the process $(q_s^{t,q,v})_{s\in[t,T]}$ is in the space of admissible controls $\mathcal{A}_t^{Merton}$ defined in (53) in Appendix B (in which we study a Merton problem that can be regarded as a limit case of ours when the execution costs and terminal costs vanish). Therefore,

$$\sup_{v \in \mathcal{A}_t} \mathbb{E}\left[-\exp\left(-\gamma\left(X_T^{t,x,S,v} + (q_T^{t,q,v})^\intercal S_T^{t,S} - (q_T^{t,q,v})^\intercal \Gamma q_T^{t,q,v}\right)\right)\right]$$

$$\leq \exp\left(-\gamma\left(x + q^\intercal S\right)\right) \sup_{q \in \mathcal{A}_t^{Merton}} \mathbb{E}\left[-\exp\left(-\gamma\left(\int_t^T q_s^\intercal dS_s\right)\right)\right]. \tag{48}$$

As shown in Appendix B, inequality (48) writes

$$-\exp\left(-\gamma\left(x + q^\intercal S + \theta(t,q,S)\right)\right) \leq -\exp\left(-\gamma\left(x + q^\intercal S + \hat{\theta}(t,S)\right)\right),$$

where $\hat{\theta}(t,S) = S^\intercal \hat{C}(t) S + \hat{E}(t)^\intercal S + \hat{F}(t)$ with $\hat{C} \in C^1\left([\tau,T], \mathcal{S}_d(\mathbb{R})\right)$, $\hat{E} \in C^1\left([\tau,T], \mathbb{R}^d\right)$, $\hat{F} \in C^1\left([\tau,T], \mathbb{R}\right)$ defined by

$$\begin{cases} \hat{C}(t) = (T-t)\frac{1}{2\gamma} R^\intercal \Sigma^{-1} R, \\ \hat{E}(t) = (T-t)\frac{1}{\gamma} R^\intercal \Sigma^{-1} R \overline{S}, \\ \hat{F}(t) = \frac{1}{4\gamma}(T-t)^2 \mathrm{Tr}\left(R^\intercal \Sigma^{-1} R \Sigma\right) + (T-t)\frac{1}{2\gamma} \overline{S}^\intercal R^\intercal \Sigma^{-1} R. \end{cases}$$

We conclude that for all $(t,q,S) \in [\tau,T] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\begin{aligned}\theta(t,q,S) &= \begin{pmatrix} q \\ S \end{pmatrix}^\intercal P(t) \begin{pmatrix} q \\ S \end{pmatrix} + D(t)^\intercal q + E(t)^\intercal S + F(t) \\ &\leq \begin{pmatrix} q \\ S \end{pmatrix}^\intercal \begin{pmatrix} 0 & 0 \\ 0 & \hat{C}(t) \end{pmatrix} \begin{pmatrix} q \\ S \end{pmatrix} + \hat{E}(t)^\intercal S + \hat{F}(t).\end{aligned}$$

[7] For $\underline{M}, \overline{M} \in \mathcal{S}_d(\mathbb{R})$, $\underline{M} \leq \overline{M}$ if and only if $\overline{M} - \underline{M} \in \mathcal{S}_d^+(\mathbb{R})$.

Therefore,

$$\forall t \in [\tau, T], \quad P(t) \leq \begin{pmatrix} 0 & 0 \\ 0 & \hat{C}(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & (T-t)\frac{1}{2\gamma} R^\intercal \Sigma^{-1} R \end{pmatrix}.$$

We have therefore $\forall \tau \in (t_{\min}, T)$, $\forall t \in [\tau, T]$:

$$\begin{pmatrix} -\frac{\gamma}{2}\Sigma_t - \Gamma & -\frac{1}{2}\left(I - e^{-R(T-t)}\right) \\ -\frac{1}{2}\left(I - e^{-R^\intercal(T-t)}\right) & 0 \end{pmatrix} \leq P(t) \leq \begin{pmatrix} 0 & 0 \\ 0 & (T-t)\frac{1}{2\gamma} R^\intercal \Sigma^{-1} R \end{pmatrix}.$$

As $t_{\min}$ is supposed to be finite, from the continuity of the lower and upper bounds, we conclude that there exists $\underline{M}, \overline{M} \in \mathcal{S}_d(\mathbb{R})$ with $\underline{M} \leq \overline{M}$ such that $\forall t \in [t_{\min}, T]$, $P(t)$ stays in the compact set $\{M \in \mathcal{S}_d(\mathbb{R}) \mid \underline{M} \leq M \leq \overline{M}\}$. This contradicts the maximality of the solution, hence $t_{\min} = -\infty$.

□

Theorem 2 implies that Theorem 1 can be applied with $\tau = 0$. In particular, our optimal execution problem is solved and the optimal strategy is given by the closed-loop feedback control (23). In the next section, we illustrate our results with simulations of prices and numerical approximations of the optimal strategies.

3 Numerical results

3.1 Single-asset case

In this section, we study the case of a trader dealing with a single asset S with the following parameters:

- Initial price: $S_0 = \$100$,
- Mean-reversion parameter: $R = 0$ day^{-1}, $R = 1$ day^{-1} or $R = 10$ day^{-1} (see below in the examples),
- Long-term average: $\overline{S} = \$100$,
- Volatility: $\sigma = 5\ \$ \cdot \text{day}^{-\frac{1}{2}}$,
- Temporary impact: $L(v) = \eta v^2$, with $\eta = 1 \cdot 10^{-3}\ \$ \cdot \text{day}$.

Figure 1 represents trajectories of the price process $(S_t)_{t\in[0,T]}$ for different values of R, using the same Brownian paths.

We consider a trader wishing to unwind a portfolio with $q_0 = 1000$ assets over the time interval $[0, T]$ where $T = 1$ day. In order to enforce almost complete liquidation, we set $\Gamma = 12$ \$.

We consider the case where the absolute risk aversion parameter is $\gamma = 1 \cdot 10^{-3}\ \$^{-1}$. For the three price trajectories of Figure 1, we plot the optimal execution strategy and the corresponding inventory process in Figure 2 and Figure 3 respectively.

An interesting observation can be made here: the higher the mean-reversion parameter, the lower the influence of price risk on the execution strategy. In particular, when R is large, the trader acts almost as if she was performing a VWAP/TWAP execution plus a simple mean-reverting statistical arbtrage strategy: the average level of $(v_t)_{t\in[0,T]}$ in the case where $R = 10$ day^{-1} is indeed driven by the total number of assets to sell and its oscillations are highly correlated to those of $(S_t)_{t\in[0,T]}$: the trader sells faster when the price is above $\overline{S}$ and slower when it is below $\overline{S}$.

Given the above observation, it is natural to illustrate how our model can be used to build a statistical arbitrage strategy by setting $q_0 = 0$ and $\Gamma = 0$: the trader has no initial inventory and just wants to maximize the expected utility of the MtM value of her portfolio at time T.

We chose $R = 10$ day^{-1} to focus on mean reversion and extend the trading window by setting $T = 9$ days. A trajectory of the price process $(S_t)_{t\in[0,T]}$ is plotted in Figure 4.

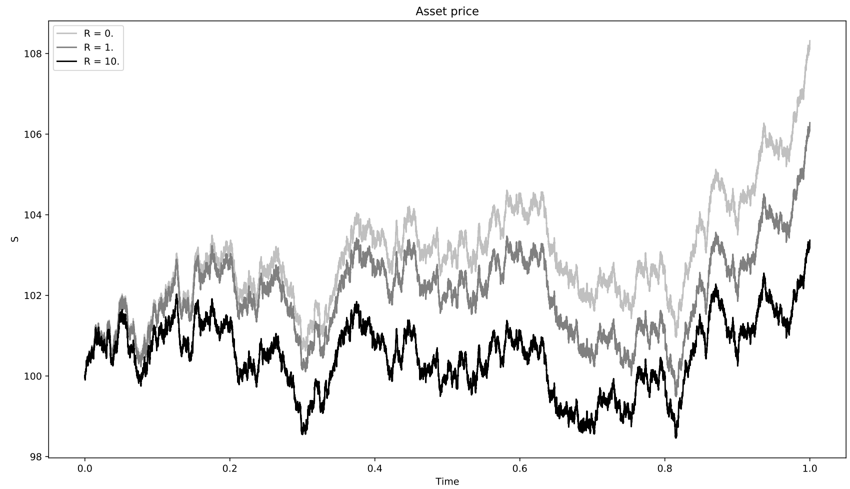

Figure 1: Trajectory of the asset price for different values of R.

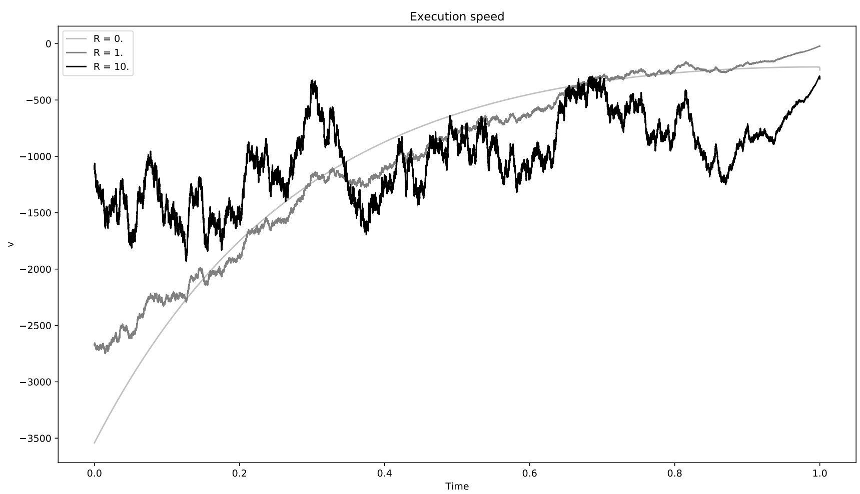

Figure 2: Execution speed $(v_t)_{t\in[0,T]}$ for different values of R ($\gamma = 1 \cdot 10^{-3}$ $\$^{-1}$).

We then plot in Figure 5 the optimal execution strategy $(v_t)_{t\in[0,T]}$ for different values of the risk aversion parameter γ: $\gamma = 1 \cdot 10^{-1}$ $\$^{-1}$, $\gamma = 1 \cdot 10^{-2}$ $\$^{-1}$ and $\gamma = 1 \cdot 10^{-7}$ $\$^{-1}$.

We observe that, as expected, the optimal strategies look highly correlated to the price trajectory. In Figure 6, we plot the corresponding inventory of the trader as a function of time for the different values of γ.
In all three cases, as expected, the trader sells the asset when the price is above $\overline{S}$, and starts buying when it goes below $\overline{S}$: her inventory is mean-reverting toward 0. Of course, the higher the risk aversion, the closer to 0 her inventory remains.

We finally perform 1500 Monte-Carlo simulations and plot, in Figures 7, 8, and 9 the distributions of the MtM value at time T for $\gamma = 1 \cdot 10^{-1}$ $\$^{-1}$, $\gamma = 1 \cdot 10^{-2}$ $\$^{-1}$ and $\gamma = 1 \cdot 10^{-7}$ $\$^{-1}$ respectively.

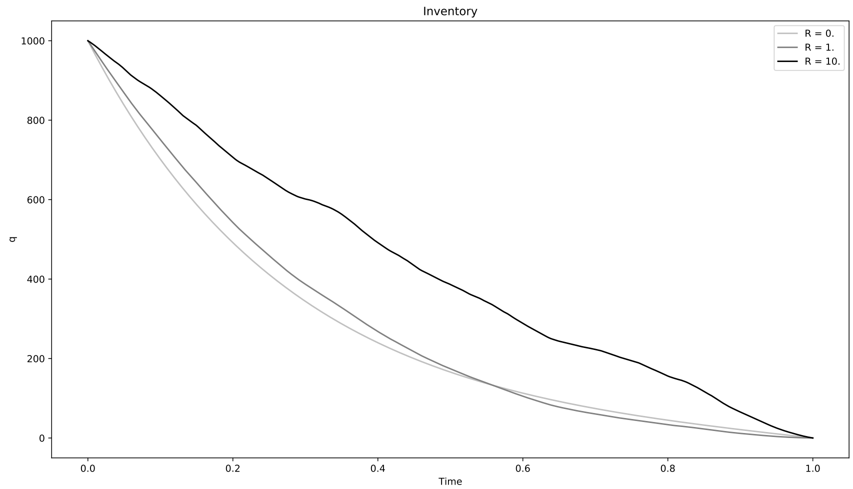

Figure 3: Trajectory of the inventory $(q_t)_{t\in[0,T]}$ for different values of R ($\gamma = 1 \cdot 10^{-3}$ $\$^{-1}$).

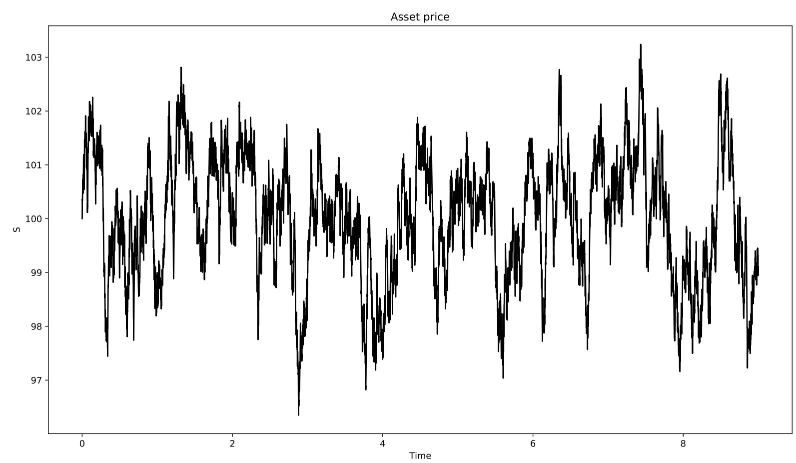

Figure 4: Asset price $(S_t)_{t\in[0,T]}$ for $R = 10$ day^{-1}.

We see that our strategy allows to make money on average by taking advantage of the mean reversion. In the first case (Figure 7), we get an average MtM value of \$1986 and a standard deviation of \$266. In the second case (Figure 8), we get an average MtM value of \$2538 and a standard deviation of \$373. In the third case (Figure 9), we get an average MtM value of \$2708 and a standard deviation of \$920.

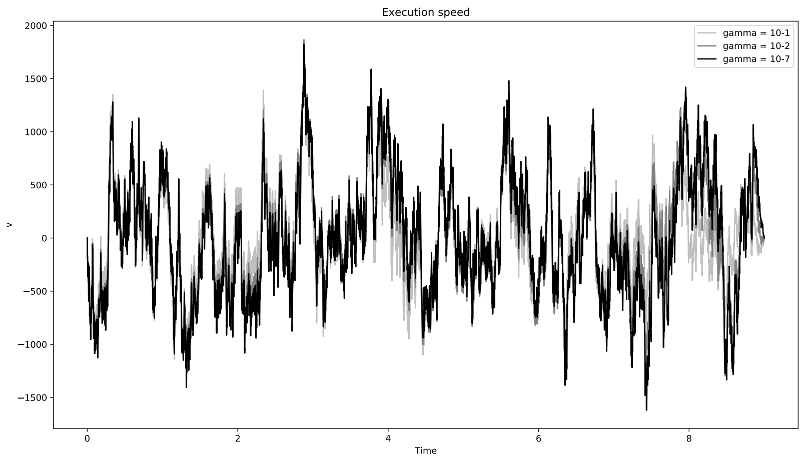

Figure 5: Execution speed $(v_t)_{t\in[0,T]}$ for different values of γ ($R = 10$ day^{-1}).

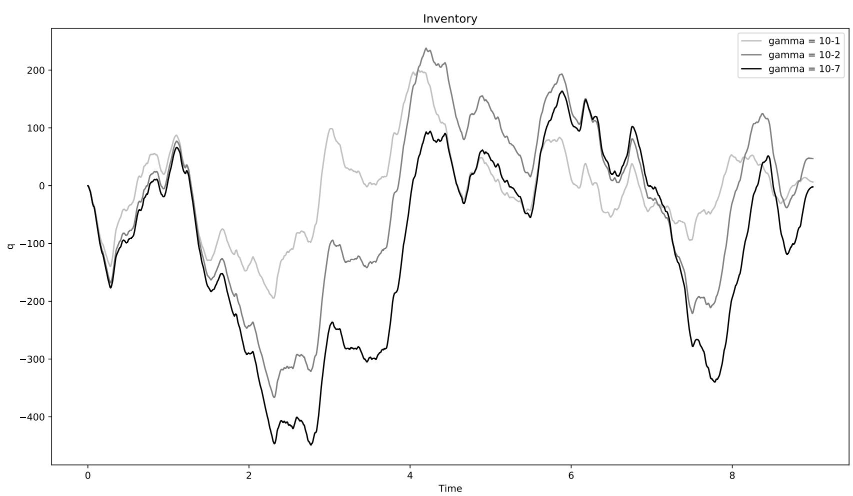

Figure 6: Trajectory of the inventory $(q_t)_{t\in[0,T]}$ for different values of γ ($R = 10$ day^{-1}).

3.2 Multi-asset case

We now study the case of a trader in charge of 2 assets S^1, S^2 with the following parameters:

- Initial price: $S^1_0 = S^2_0 = \$100$,
- Mean-reversion matrix: $R = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ (no-cointegration case) or $R = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$ (cointegration case),
- Long-term average: $\overline{S} := \left(\overline{S}^1, \overline{S}^2\right) = (\$100, \$100)$,
- Quadratic covariation matrix: $\Sigma = \begin{pmatrix} 25 & 7.5 \\ 7.5 & 25 \end{pmatrix}$ (which corresponds to an arithmetic volatility of 5 $\$ \cdot \text{day}^{-\frac{1}{2}}$ for

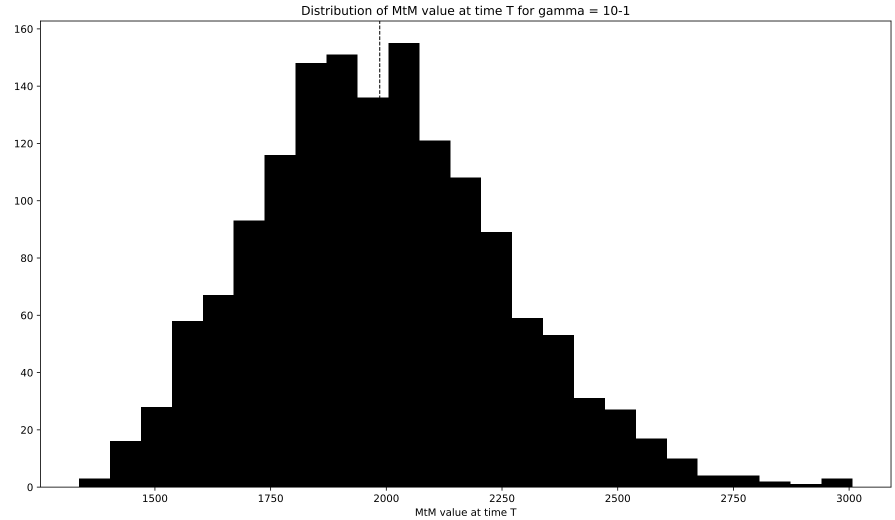

Figure 7: Distribution of MtM value at time T for $\gamma = 1 \cdot 10^{-1}$ $\$^{-1}$ ($R = 10$ day^{-1}).

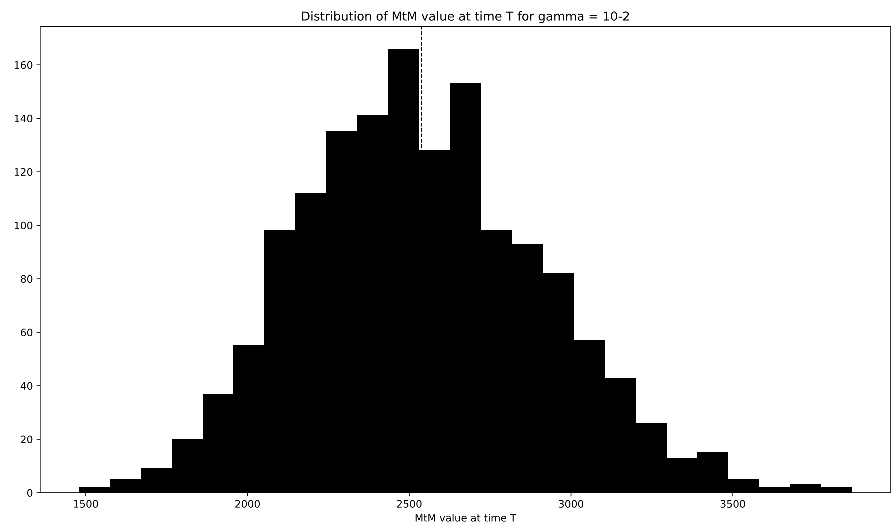

Figure 8: Distribution of MtM value at time T for $\gamma = 1 \cdot 10^{-2}$ $\$^{-1}$ ($R = 10$ day^{-1}).

the two assets, and a correlation of 0.3),

- Temporary impact: $L(v) = v^\intercal \eta v$, with $\eta = \left(1 \cdot 10^{-3}\ \$ \cdot \text{day}\right) \times I_2$.

We assume that the trader has an initial inventory $q_0 = (1000, 1000)$ and that she wants to liquidate within $T = 2$ days. Her risk aversion is given by $\gamma = 2 \cdot 10^{-3}$ $\$^{-1}$. We penalize the remaining inventory with the matrix $\Gamma = 12 \times I_2$.

Let us first consider that the matrix R is given by $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ (the no-cointegration case). We simulate in Figure 10 a

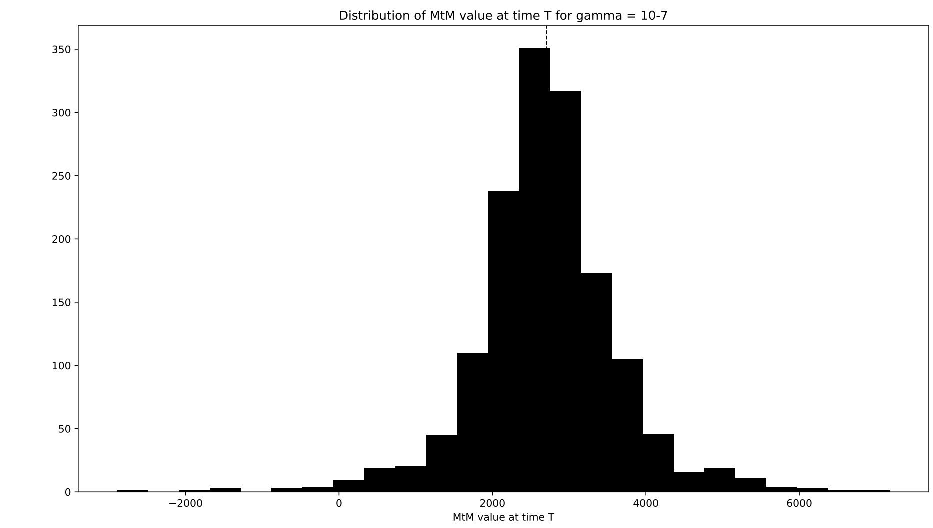

Figure 9: Distribution of MtM value at time T for $\gamma = 1 \cdot 10^{-7}$ $\$^{-1}$ ($R = 10$ day^{-1}).

corresponding trajectory for the prices of the two assets. We then plot in Figures 11 and 12 the optimal strategy and the associated inventories, respectively.

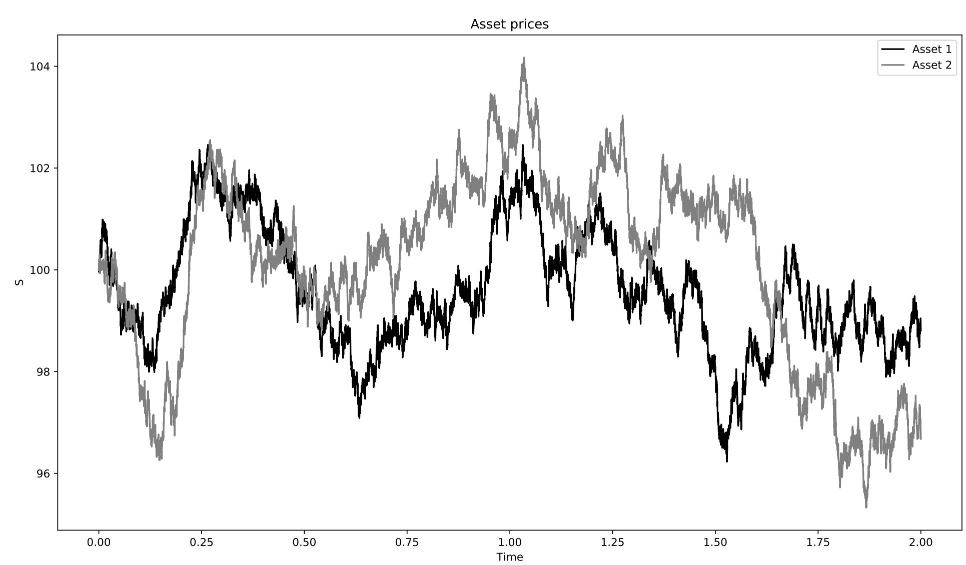

Figure 10: Asset prices $(S_t^1)_{t\in[0,T]}$ and $(S_t^2)_{t\in[0,T]}$ in the no-cointegration case.

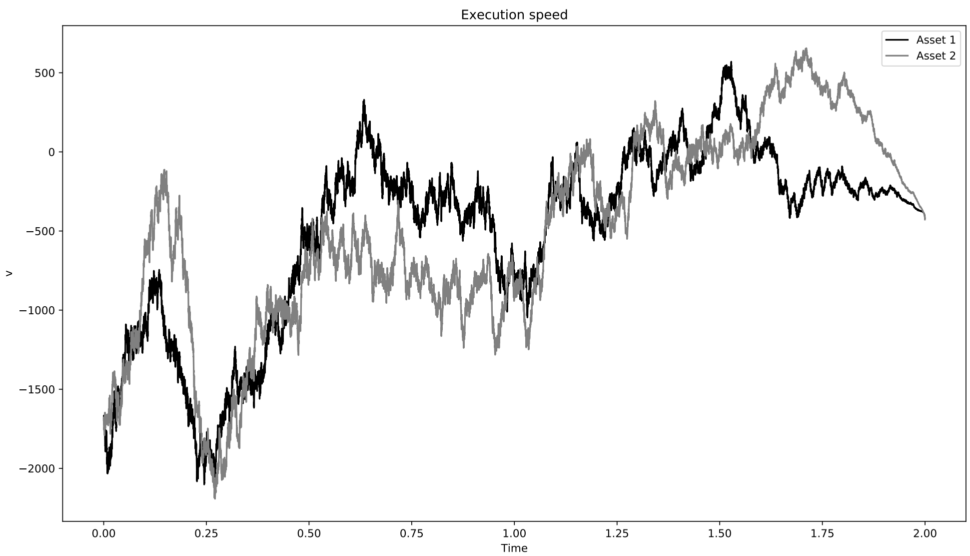

Figure 11: Execution speeds $(v_t^1)_{t\in[0,T]}$ and $(v_t^2)_{t\in[0,T]}$ in the no-cointegration case.

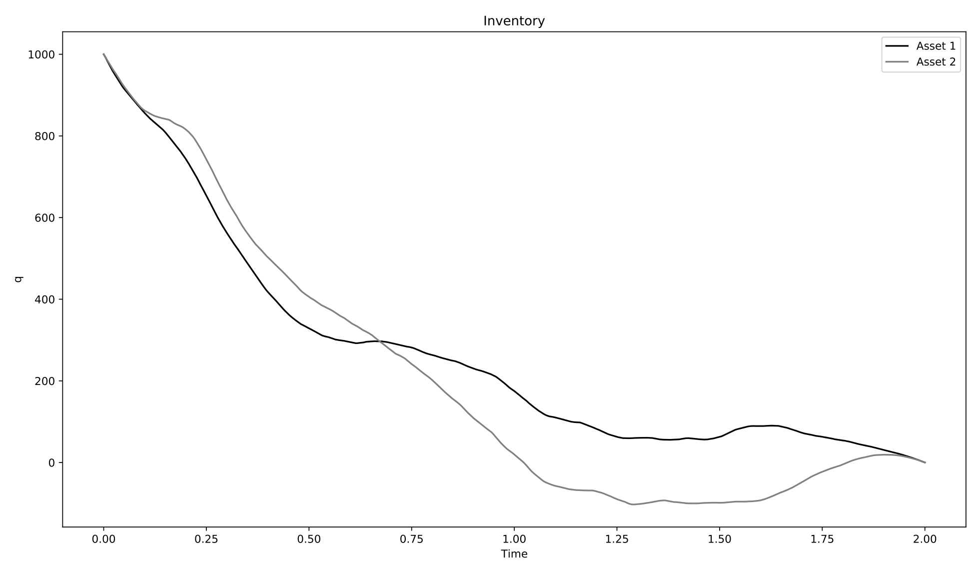

Figure 12: Trajectory of the inventories $(q_t^1)_{t\in[0,T]}$ and $(q_t^2)_{t\in[0,T]}$ in the no-cointegration case.

We now qualitatively compare these results with those obtained in the cointegration case where the matrix R is given by $\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$. We simulate in Figure 13 a corresponding trajectory for the prices of the two assets. As before, we plot in Figures 14 and 15 the optimal strategy and the associated inventories, respectively.

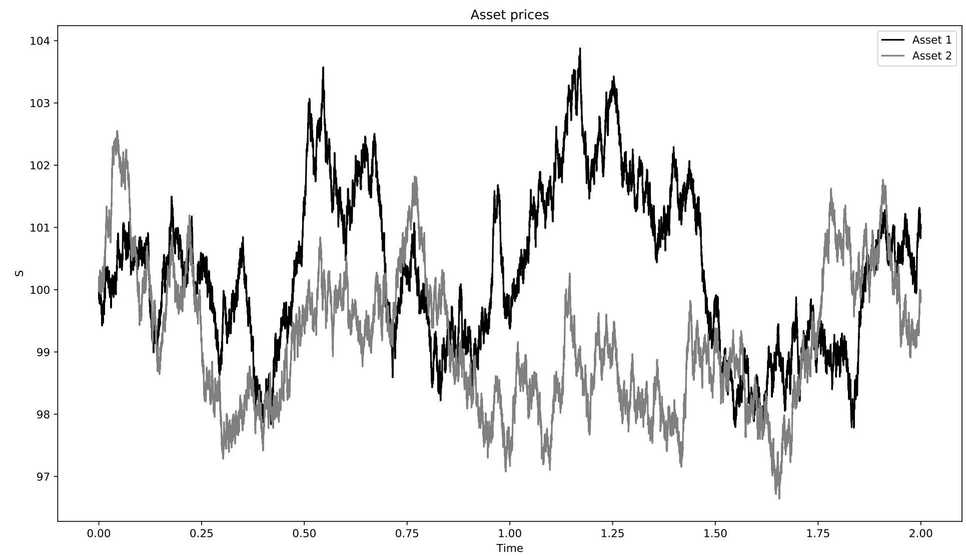

Figure 13: Asset prices $(S_t^1)_{t\in[0,T]}$ and $(S_t^2)_{t\in[0,T]}$ in the case of cointegration.

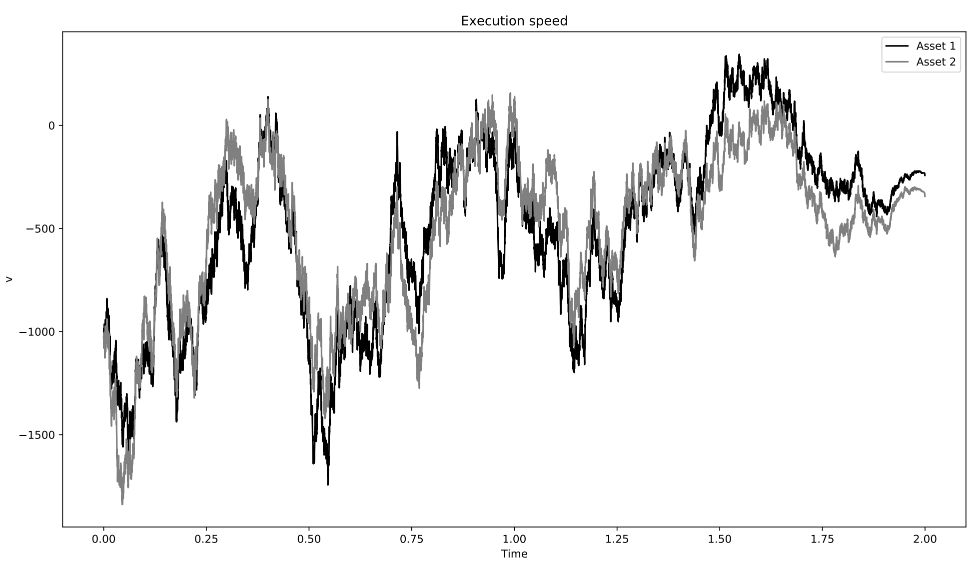

Figure 14: Execution speeds $(v_t^1)_{t\in[0,T]}$ and $(v_t^2)_{t\in[0,T]}$ in the case of cointegration.

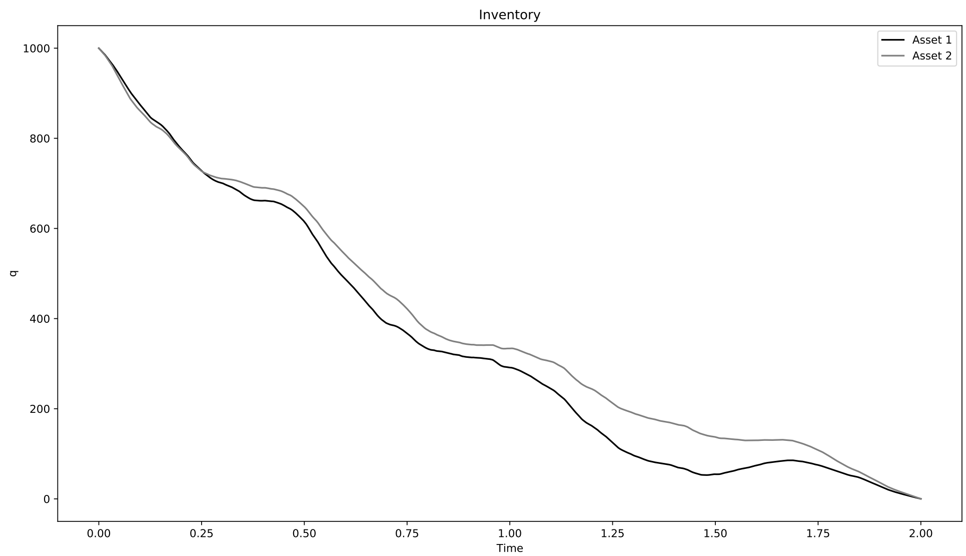

Figure 15: Trajectory of the inventories $(q_t^1)_{t\in[0,T]}$ and $(q_t^2)_{t\in[0,T]}$ in the case of cointegration.

As expected, the comparison of Figures 12 and 15 yields that, in the presence of cointegration, the trader tends to execute slower.

Conclusion

In this paper, we have shown how to account for cross-asset co-movements when executing trades in multiple assets. In our model, the agent has an exponential utility and the prices have multivariate Ornstein-Uhlenbeck dynamics, capturing the complex cross-asset dynamics of prices better than Brownian motions only. The advantage of our approach is twofold: (i) it accurately accounts for risk at the portfolio level, and (ii) it is versatile and can be used for basket execution, exogenous signal incorporation, and statistical arbitrage.

Our simulations show that considering cross-asset relations leads to different execution strategies. In particular, the presence of cointegration is exploited by the optimal strategy and usually leads to a reduction in the execution speed since the global variance of the portfolio is reduced.

The advantages for practitioners are numerous. Considering asset execution within a portfolio allows to manage risk across a wider basket of assets rather than considering only the risk of a single trade. Agents can hold securities on their balance sheets for longer, reducing market impact and execution costs. Moreover, from a regulation point of view, multivariate optimal execution models that naturally offset risks in a portfolio are of great interest. In fact, the new FRTB (Fundamental Review of the Trading Book) regulation will lead practitioners to assess liquidity risks within a centralized risk book for capital requirements. In this context, our model can reduce the liquidity risk of the execution process by taking into account the joint dynamics of the assets.

A Appendix - Multi-asset optimal execution with correlated Brownian motions and execution costs

We consider in this appendix the problem of multi-asset optimal execution in the case where prices are correlated arithmetic Brownian motions. This problem is a special case of that presented in this paper, corresponding to $R = 0$ in the dynamics (2) of the asset prices. Therefore, the results presented in the paper apply. However, when $R = 0$, as mentioned in Remark 1, the system of ODEs (16) simplify since a trivial solution to the last five equations is $B = C = D = E = F = 0$. Therefore, the problem boils down to finding $A \in C^1([0,T], \mathcal{S}_d(\mathbb{R}))$ solution of the following terminal value problem:

$$\begin{cases} A'(t) &= \frac{\gamma}{2}\Sigma - A(t)\eta^{-1}A(t) \\ A(T) &= -\Gamma. \end{cases} \tag{49}$$

In this appendix we show that, when $\Sigma \in S_d^{++}(\mathbb{R})$, A can be found in closed form.

For that purpose, we introduce the change of variables

$$a(t) = \eta^{-\frac{1}{2}}A(t)\eta^{-\frac{1}{2}} \quad \forall t \in [0,T].$$

and notice that (49) is equivalent to the terminal value problem

$$\begin{cases} a'(t) = & \hat{A}^2 - a(t)^2 \\ a(T) = & -C, \end{cases} \tag{50}$$

where $\hat{A} = \sqrt{\frac{\gamma}{2}}\left(\eta^{-\frac{1}{2}}\Sigma\eta^{-\frac{1}{2}}\right)^{\frac{1}{2}} \in \mathcal{S}_d^{++}(\mathbb{R})$ and $C = \eta^{-\frac{1}{2}}\Gamma\eta^{-\frac{1}{2}} \in \mathcal{S}_d^{+}(\mathbb{R})$.

To solve (50) we use a classical trick for Riccati equations, shown in the following Proposition:

Proposition 3. *Let $\xi : [0,T] \to \mathcal{S}_d(\mathbb{R})$ defined as*

$$\xi(t) = -\frac{\hat{A}^{-1}}{2}\left(I - e^{-2\hat{A}(T-t)}\right) - e^{-\hat{A}(T-t)}\left(C + \hat{A}\right)^{-1}e^{-\hat{A}(T-t)} \tag{51}$$

be the unique solution of the linear ODE

$$\begin{cases} \xi'(t) = \hat{A}\xi(t) + \xi(t)\hat{A} + I_d \\ \xi(T) = -\left(C + \hat{A}\right)^{-1}. \end{cases} \tag{52}$$

Then $\forall t \in [0,T]$, $\xi(t)$ is invertible and $a : t \in [0,T] \to \hat{A} + \xi(t)^{-1} \in \mathcal{S}_d(\mathbb{R})$ is the unique solution of (50).

Proof. First, we easily verify that ξ, defined in (51), is solution of the linear ODE (52). We see that, for all t in $[0,T]$, $\xi(t)$ is the sum of $-\frac{\hat{A}^{-1}}{2}\left(I - e^{-2\hat{A}(T-t)}\right) \in \mathcal{S}_d^{--}(\mathbb{R})$ and $-e^{-\hat{A}(T-t)}\left(C + \hat{A}\right)^{-1}e^{-\hat{A}(T-t)} \in \mathcal{S}_d^{--}(\mathbb{R})$, so $\xi(t) \in \mathcal{S}_d^{--}(\mathbb{R})$ and is invertible.

We also note that

$$a'(t) = -\xi(t)^{-1}\xi'(t)\xi(t)^{-1} = -\xi(t)^{-1}\hat{A} - \hat{A}\xi(t)^{-1} - \xi(t)^{-2} = \hat{A}^2 - \left(\hat{A} + \xi(t)^{-1}\right)^2 = \hat{A}^2 - a(t)^2$$

and $a(T) = -C$, hence the result. □

We deduce the following corollary:

Corollary 1.

$$\forall t \in [0,T], \quad A(t) = \eta^{\frac{1}{2}}\left(\hat{A} - \left(\frac{\hat{A}^{-1}}{2}\left(I - e^{-2\hat{A}(T-t)}\right) + e^{-\hat{A}(T-t)}\left(C + \hat{A}\right)^{-1}e^{-\hat{A}(T-t)}\right)^{-1}\right)\eta^{\frac{1}{2}}.$$

B Appendix - Merton portfolio optimization problem under Ornstein-Uhlenbeck dynamics and exponential utility

B.1 Modelling framework

We study in this appendix a Merton model where prices have multivariate Ornstein-Uhlenbeck dynamics. It is closely related to our model and can be seen as some form of limit case corresponding to no execution costs (i.e. $L = 0$) and no terminal penalty (i.e. $\ell = 0$).

The results obtained in this appendix are essential in our proof of existence of a solution to the system of ODEs (16) on $[0, T]$ with terminal condition (17) (see Theorem 2).

As in the paper, we consider a model with d assets, whose prices are modelled by a d-dimensional stochastic process $(S_t)_{t\in[0,T]} = \left(S_t^1, \ldots, S_t^d\right)^\intercal_{t\in[0,T]}$ with dynamics

$$dS_t = R(\overline{S} - S_t)dt + V dW_t,$$

where $\overline{S} \in \mathbb{R}^d$, $R \in \mathcal{M}_d(\mathbb{R})$, $V \in \mathcal{M}_{d,k}(\mathbb{R})$, and $(W_t)_{t\in[0,T]} = \left(W_t^1, \ldots, W_t^k\right)^\intercal_{t\in[0,T]}$ is a k-dimensional standard Brownian motion (with independent coordinates), for some $k \in \mathbb{N}^*$. As before, we write $\Sigma = VV^\intercal$.

We consider a trader optimizing her portfolio over the period $[0, T]$ by controlling at each time the number of each asset in her portfolio, i.e. she controls the d-dimensional process $(q_t)_{t\in[0,T]} = \left(q_t^1, \ldots, q_t^d\right)^\intercal_{t\in[0,T]}$, where q_t^i denotes the number of assets i in the portfolio at time t, for each $i \in \{1, \ldots, d\}$ ($t \in [0, T]$). The process $(q_t)_{t\in[0,T]}$ lies in the space of admissible controls $\mathcal{A}_0^{Merton}$, where for $t \in [0, T]$, the set $\mathcal{A}_t^{Merton}$ is defined as

$$\mathcal{A}_t^{Merton} := \left\{(q_s)_{s\in[t,T]},\ \mathbb{R}^d\text{-valued},\ \mathbb{F}\text{-adapted, satisfying a linear growth condition with respect to } (S_s)_{s\in[t,T]}\right\}. \tag{53}$$

We introduce the process $(\mathcal{V}_t)_{t\in[0,T]}$ modelling the MtM value of the trader's portfolio, i.e.

$$\forall t \in [0, T], \quad \mathcal{V}_t = \mathcal{V}_0 + \int_0^t q_s^\intercal dS_s, \qquad \mathcal{V}_0 \in \mathbb{R} \text{ given}$$

For a given $\gamma > 0$, the trader aims at maximizing the following objective function:

$$\mathbb{E}\left[-e^{-\gamma \mathcal{V}_T}\right], \tag{54}$$

over the set of admissible controls $(q_t)_{t\in[0,T]} \in \mathcal{A}_0^{Merton}$. We define her value function $\hat{u} : [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ as

$$\hat{u}(t, \mathcal{V}, S) = \sup_{q\in\mathcal{A}_t^{Merton}} \mathbb{E}\left[-e^{-\gamma \mathcal{V}_T^{t,\mathcal{V},S,q}}\right] \qquad \forall (t, \mathcal{V}, S) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d,$$

where $(\mathcal{V}_s^{t,\mathcal{V},S,q})_{s\in[t,T]}$ denotes the process defined by

$$d\mathcal{V}_s^{t,\mathcal{V},S,q} = q_s^\intercal dS_s^{t,S}, \quad \mathcal{V}_t^{t,\mathcal{V},S,q} = \mathcal{V}$$

with

$$dS_s^{t,S} = R(\overline{S} - S_s^{t,S})ds + V dW_s, \quad S_t^{t,S} = S.$$

B.2 HJB equation

The HJB equation associated with Problem (54) is given by

$$\begin{aligned} 0 = &\partial_t \hat{w}(t, \mathcal{V}, S) + \nabla_S \hat{w}(t, \mathcal{V}, S)^\intercal R(\overline{S} - S) + \frac{1}{2}\mathrm{Tr}\left(\Sigma D^2_{SS}\hat{w}(t, \mathcal{V}, S)\right) \\ &+ \sup_{q\in\mathbb{R}^d}\left\{\partial_\mathcal{V}\hat{w}(t, \mathcal{V}, S) q^\intercal R(\overline{S} - S) + \frac{1}{2}\partial^2_{\mathcal{V}\mathcal{V}}\hat{w}(t, \mathcal{V}, S) q^\intercal\Sigma q + \partial_\mathcal{V}\nabla_S\hat{w}(t, \mathcal{V}, S)^\intercal\Sigma q\right\} \end{aligned} \tag{55}$$

for all $(t, \mathcal{V}, S) \in [0,T) \times \mathbb{R} \times \mathbb{R}^d$, with terminal condition

$$\hat{w}(T, \mathcal{V}, S) = -e^{-\gamma \mathcal{V}} \quad \forall (\mathcal{V}, S) \in \mathbb{R} \times \mathbb{R}^d. \tag{56}$$

To solve the above HJB equation, we use the ansatz

$$\hat{w}(t, \mathcal{V}, S) = -e^{-\gamma\left(\mathcal{V} + \hat{\theta}(t,S)\right)}. \tag{57}$$

Indeed, we have the following proposition:

Proposition 4. *If there exists $\hat{\theta} : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ solution to*

$$0 = \partial_t \hat{\theta}(t,S) + \frac{1}{2} Tr\left(\Sigma D^2_{SS} \hat{\theta}(t,S)\right) + \frac{1}{2\gamma} (\overline{S} - S)^\intercal R^\intercal \Sigma^{-1} R (\overline{S} - S) \tag{58}$$

on $[0,T) \times \mathbb{R}^d$, with terminal condition

$$\hat{\theta}(T, S) = 0 \qquad \forall S \in \mathbb{R}^d, \tag{59}$$

then the function $\hat{w} : [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ defined by

$$\hat{w}(t, \mathcal{V}, S) = -e^{-\gamma\left(\mathcal{V} + \hat{\theta}(t,S)\right)} \quad \forall (t, \mathcal{V}, S) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$$

is a solution to (55) *on $[0,T) \times \mathbb{R} \times \mathbb{R}^d$ with terminal condition* (56)*.*

Proof. Let $\hat{\theta} : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be a solution to (58) on $[0,T) \times \mathbb{R}^d$ with terminal condition (59), then we have for all $(t, \mathcal{V}, S) \in [0,T) \times \mathbb{R} \times \mathbb{R}^d$:

$$\begin{aligned}
&\partial_t \hat{w}(t,\mathcal{V},S) + \nabla_S \hat{w}(t,\mathcal{V},S)^\intercal R(\overline{S} - S) + \frac{1}{2}\mathrm{Tr}\left(\Sigma D^2_{SS} \hat{w}(t,\mathcal{V},S)\right) \\
&+ \sup_{q \in \mathbb{R}^d} \left\{ \partial_{\mathcal{V}} \hat{w}(t,\mathcal{V},S) q^\intercal R(\overline{S} - S) + \frac{1}{2} \partial^2_{\mathcal{V}\mathcal{V}} \hat{w}(t,\mathcal{V},S) q^\intercal \Sigma q + \partial_{\mathcal{V}} \nabla_S \hat{w}(t,\mathcal{V},S)^\intercal \Sigma q \right\} \\
= & -\gamma \partial_t \hat{\theta}(t,S) \hat{w}(t,\mathcal{V},S) - \gamma \nabla_S \hat{\theta}(t,S) R(\overline{S} - S) \hat{w}(t,\mathcal{V},S) - \gamma \hat{w}(t,\mathcal{V},S) \frac{1}{2}\mathrm{Tr}\left(\Sigma D^2_{SS} \hat{\theta}(t,S)\right) \\
&+ \frac{\gamma^2}{2} \hat{w}(t,\mathcal{V},S) \nabla_S \hat{\theta}(t,S)^\intercal \Sigma \nabla_S \hat{\theta}(t,S) \\
&+ \sup_{q \in \mathbb{R}^d} \left\{ -\gamma \hat{w}(t,\mathcal{V},S) q^\intercal R(\overline{S} - S) + \frac{\gamma^2}{2} \hat{w}(t,\mathcal{V},S) q^\intercal \Sigma q + \gamma^2 \hat{w}(t,\mathcal{V},S) \nabla_S \hat{\theta}(t,\mathcal{V},S)^\intercal \Sigma q \right\} \\
= & -\gamma \hat{w}(t,\mathcal{V},S) \Bigg(\partial_t \hat{\theta}(t,S) + \nabla_S \hat{\theta}(t,S) R(\overline{S} - S) + \frac{1}{2}\mathrm{Tr}\left(\Sigma D^2_{SS} \hat{\theta}(t,S)\right) - \frac{\gamma}{2} \nabla_S \hat{\theta}(t,S)^\intercal \Sigma \nabla_S \hat{\theta}(t,S) \\
&+ \sup_{q \in \mathbb{R}^d} \left\{ q^\intercal \left(R(\overline{S} - S) - \gamma \Sigma \nabla_S \hat{\theta}(t,S) \right) - \frac{\gamma}{2} q^\intercal \Sigma q \right\} \Bigg).
\end{aligned}$$

The supremum in the last line is reached at

$$q^*(t,S) = \frac{1}{\gamma} \Sigma^{-1} R(\overline{S} - S) - \nabla_S \hat{\theta}(t,S),$$

and we obtain after simplifications:

$$\begin{aligned}
&\partial_t \hat{w}(t,\mathcal{V},S) + \nabla_S \hat{w}(t,\mathcal{V},S)^\intercal R(\overline{S} - S) + \frac{1}{2}\mathrm{Tr}\left(\Sigma D^2_{SS} \hat{w}(t,\mathcal{V},S)\right) \\
&+ \sup_{q \in \mathbb{R}^d} \left\{ \partial_{\mathcal{V}} \hat{w}(t,\mathcal{V},S) q^\intercal R(\overline{S} - S) + \frac{1}{2} \partial^2_{\mathcal{V}\mathcal{V}} \hat{w}(t,\mathcal{V},S) q^\intercal \Sigma q + \partial_{\mathcal{V}} \nabla_S \hat{w}(t,\mathcal{V},S)^\intercal \Sigma q \right\} \\
= & -\gamma \hat{w}(t,\mathcal{V},S) \left(\partial_t \hat{\theta}(t,S) + \frac{1}{2}\mathrm{Tr}\left(\Sigma D^2_{SS} \hat{\theta}(t,S)\right) + \frac{1}{2\gamma} (\overline{S} - S)^\intercal R^\intercal \Sigma^{-1} R(\overline{S} - S) \right) \\
= & \, 0.
\end{aligned}$$

As $\hat{w}$ satisfies the terminal condition (56), the result is proved. ∎

We now use a second ansatz and look for a function $\hat{\theta}$ solution to (58) on $[0,T)\times\mathbb{R}^d$ with terminal condition (59) of the following form:

$$\hat{\theta}(t,S) = S^\intercal \hat{C}(t) S + \hat{E}(t)^\intercal S + \hat{F}(t), \tag{60}$$

We have indeed the following proposition:

Proposition 5. *Assume there exists $\hat{C}\in C^1([0,T],\mathcal{S}_d(\mathbb{R}))$, $\hat{E}\in C^1\left([0,T],\mathbb{R}^d\right)$, $\hat{F}\in C^1([0,T],\mathbb{R})$ satisfying the system of ODEs*

$$\begin{cases} \hat{C}'(t) &= -\frac{1}{2\gamma} R^\intercal \Sigma^{-1} R \\ \hat{E}'(t) &= \frac{1}{\gamma} R^\intercal \Sigma^{-1} R \overline{S} \\ \hat{F}'(t) &= -Tr\left(\hat{C}(t)\Sigma\right) - \frac{1}{2\gamma}\overline{S}^\intercal R^\intercal \Sigma^{-1} R \overline{S}, \end{cases} \tag{61}$$

with terminal condition

$$\hat{C}(T) = \hat{E}(T) = \hat{F}(T) = 0. \tag{62}$$

Then the function $\hat{\theta}$ defined by (60) *satisfies* (58) *on $[0,T)\times\mathbb{R}^d$ with terminal condition* (59)*.*

Proof. Let us consider $\hat{C}\in C^1([0,T],\mathcal{S}_d(\mathbb{R}))$, $\hat{E}\in C^1\left([0,T],\mathbb{R}^d\right)$, $\hat{F}\in C^1([0,T],\mathbb{R})$ verifying (61) on $[0,T)$ with terminal condition (62). Let us consider $\hat{\theta}:[0,T]\times\mathbb{R}^d\to\mathbb{R}$ defined by (60). Then we obtain for all $(t,S)\in[0,T)\times\mathbb{R}^d$:

$$\begin{aligned} & \partial_t\hat{\theta}(t,S) + \frac{1}{2}\mathrm{Tr}\left(\Sigma D^2_{SS}\hat{\theta}(t,S)\right) + \frac{1}{2\gamma}(\overline{S}-S)^\intercal R^\intercal \Sigma^{-1} R(\overline{S}-S) \\ = \; & S^\intercal \hat{C}'(t) S + \hat{E}'(t)^\intercal S + \hat{F}'(t) + \mathrm{Tr}\left(\hat{C}(t)\Sigma\right) + \frac{1}{2\gamma}(\overline{S}-S)^\intercal R^\intercal \Sigma^{-1} R(\overline{S}-S) \\ = \; & 0. \end{aligned}$$

As it is straightforward to verify that $\hat{\theta}$ satisfies the terminal condition (59), the result is proved. □

It is straightforward to see that there exists a unique solution $\hat{C}\in C^1([0,T],\mathcal{S}_d(\mathbb{R}))$, $\hat{E}\in C^1\left([0,T],\mathbb{R}^d\right)$, $\hat{F}\in C^1([0,T],\mathbb{R})$ to (61) with terminal condition (62). We can then prove the following verification theorem.

Theorem 3. *We consider the functions $\hat{C}\in C^1([0,T],\mathcal{S}_d(\mathbb{R}))$, $\hat{E}\in C^1\left([0,T],\mathbb{R}^d\right)$, $\hat{F}\in C^1([0,T],\mathbb{R})$ solutions to* (61) *with terminal condition*

$$\hat{C}(T) = \hat{E}(T) = \hat{F}(T) = 0,$$

i.e. for all $t\in[0,T]$,

$$\begin{cases} \hat{C}(t) = (T-t)\frac{1}{2\gamma} R^\intercal \Sigma^{-1} R, \\ \hat{E}(t) = (T-t)\frac{1}{\gamma} R^\intercal \Sigma^{-1} R\overline{S}, \\ \hat{F}(t) = \frac{1}{4\gamma}(T-t)^2\, Tr\left(R^\intercal \Sigma^{-1} R \Sigma\right) + (T-t)\frac{1}{2\gamma}\overline{S}^\intercal R^\intercal \Sigma^{-1} R. \end{cases}$$

We consider the function $\hat{\theta}$ defined by

$$\hat{\theta}(t,S) = S^\intercal \hat{C}(t) S + \hat{E}(t)^\intercal S + \hat{F}(t),$$

and the associated function $\hat{w}$ defined by

$$\hat{w}(t,\mathcal{V},S) = -e^{-\gamma\left(\mathcal{V}+\hat{\theta}(t,S)\right)}.$$

For all $(t,\mathcal{V},S)\in[0,T]\times\mathbb{R}\times\mathbb{R}^d$ and $q=(q_s)_{s\in[t,T]}\in\mathcal{A}_t^{Merton}$, we have

$$\mathbb{E}\left[-e^{-\gamma\left(\mathcal{V}_T^{t,\mathcal{V},S,q}\right)}\right] \le \hat{w}(t,\mathcal{V},S). \tag{63}$$

Moreover, equality is obtained in (63) *by taking the optimal control $(q^*_s)_{s\in[t,T]}\in\mathcal{A}_t^{Merton}$ given by the closed-loop feedback formula*

$$q^*_s = \frac{1}{\gamma}\Sigma^{-1}R(\overline{S}-S_s^{t,S}) - \hat{C}(s)S_s^{t,S} - \hat{E}(s). \tag{64}$$

In particular, $\hat{w}=\hat{u}$.

Proof. It is obvious that $(q^*_s)_{s\in[t,T]} \in \mathcal{A}_t^{Merton}$ (i.e., $(q^*_s)_{s\in[t,T]}$ is well-defined and admissible):

$$\exists C_T > 0, \forall s \in [t,T], \qquad \| q^*_s \| \leqslant C_T \left(1 + \sup_{\tau\in[t,s]} \| S_\tau \| \right). \tag{65}$$

Let us consider $(t, \mathcal{V}, S) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$ and $q = (q_s)_{s\in[t,T]} \in \mathcal{A}_t^{Merton}$. We now prove that

$$\mathbb{E}\left[\hat{w}(T, \mathcal{V}_T^{t,\mathcal{V},S}, S_T^{t,S}) \right] \leqslant \hat{w}(t, \mathcal{V}, S). \tag{66}$$

We use the following notations for readability

$$\forall s \in [t,T], \quad \hat{w}(s, \mathcal{V}_s^{t,\mathcal{V},S,q}, S_s^{t,S}) = \hat{w}_s^{t,\mathcal{V},S,q}, \tag{67}$$

$$\forall s \in [t,T], \quad \hat{\theta}(s, S_s^{t,S}) = \hat{\theta}_s^{t,S}. \tag{68}$$

By Itô's formula, we have $\forall s \in [0,T]$

$$d\hat{w}_s^{t,\mathcal{V},S,q} = \mathcal{L}^q \hat{w}_s^{t,\mathcal{V},S,q} ds + \left(\partial_\mathcal{V} \hat{w}_s^{t,\mathcal{V},S,q} q_s + \nabla_S \hat{w}_s^{t,\mathcal{V},S,q} \right)^\intercal V dW_s, \tag{69}$$

where

$$\begin{aligned} \mathcal{L}^q \hat{w}_s^{t,\mathcal{V},S,q} =& \partial_t w_s^{t,\mathcal{V},S,q} + \left(\nabla_S \hat{w}_s^{t,\mathcal{V},S,q} \right)^\intercal R(\overline{S} - S) + \partial_\mathcal{V} \hat{w}_s^{t,\mathcal{V},S,q} q_s^\intercal R(\overline{S} - S) + \frac{1}{2} \mathrm{Tr}\left(\Sigma D^2_{SS} \hat{w}_s^{t,\mathcal{V},S,q} \right) \\ &+ \frac{1}{2} \partial^2_{\mathcal{V}\mathcal{V}} \hat{w}_s^{t,\mathcal{V},S,q} q_s^\intercal \Sigma q_s + \left(\partial_\mathcal{V} \nabla_S \hat{w}_s^{t,\mathcal{V},S,q} \right)^\intercal \Sigma q_s. \end{aligned} \tag{70}$$

We have

$$\begin{aligned} \nabla_S \hat{w}_s^{t,\mathcal{V},S,q} &= -\gamma \hat{w}_s^{t,\mathcal{V},S,q} \nabla_S \theta_s^{t,S} \\ &= -\gamma \hat{w}_s^{t,\mathcal{V},S,q} \left(2\hat{C}(s) S_s^{t,S} + \hat{E}(s) \right), \end{aligned} \tag{71}$$

and

$$\partial_\mathcal{V} \hat{w}_s^{t,\mathcal{V},S,q} = -\gamma \hat{w}_s^{t,\mathcal{V},S,q}. \tag{72}$$

We define $\forall s \in [t,T]$

$$\kappa_s^q = -\gamma \left(q_s + 2\hat{C}(s) S_s^{t,S} + \hat{E}(s) \right), \tag{73}$$

$$\xi_{t,s}^q = \exp\left(\int_t^s \kappa_\varrho^{q\intercal} V dW_\varrho - \frac{1}{2} \int_t^s \kappa_\varrho^{q\intercal} \Sigma \kappa_\varrho^q d\varrho \right). \tag{74}$$

We then have

$$d\left(\hat{w}_s^{t,\mathcal{V},S,q} \left(\xi_{t,s}^q \right)^{-1} \right) = \left(\xi_{t,s}^q \right)^{-1} \mathcal{L}^q \hat{w}_s^{t,\mathcal{V},S,q} ds. \tag{75}$$

By definition of $\hat{w}$, $\mathcal{L}^q \hat{w}_s^{t,\mathcal{V},S,q} \leqslant 0$.

Moreover, equality holds for the control reaching the sup in (55). It is easy to see that the sup is reached for the unique value

$$\begin{aligned} q_s &= \frac{1}{\gamma} \Sigma^{-1} R(\overline{S} - S_s^{t,S}) - \nabla_S \hat{\theta}(t, S_s^{t,S}) \qquad (76) \\ &= \frac{1}{\gamma} \Sigma^{-1} R(\overline{S} - S_s^{t,S}) - 2\hat{C}(s) S_s^{t,S} - \hat{E}(s), \qquad (77) \end{aligned}$$

which corresponds to $(q_s)_{s\in[t,T]} = (q^*_s)_{s\in[t,T]}$.

As a consequence, $\left(\hat{w}^{t,\mathcal{V},S,q}_s \left(\xi^q_{t,s}\right)^{-1}\right)_{s\in[t,T]}$ is nonincreasing and therefore

$$\hat{w}(T, \mathcal{V}^{t,\mathcal{V},S,q}_T, S^{t,S}_T) \leqslant \hat{w}(t, \mathcal{V}, S)\xi^q_{t,T}, \tag{78}$$

with equality when $(q_s)_{s\in[t,T]} = (q^*_s)_{s\in[t,T]}$.

Taking expectation we get

$$\mathbb{E}\left[\hat{w}(T, \mathcal{V}^{t,\mathcal{V},S,q}_T, S^{t,S}_T)\right] \leqslant \hat{w}(t, \mathcal{V}, S)\mathbb{E}\left[\xi^q_{t,T}\right]. \tag{79}$$

We proceed to prove that $E\left[\xi^q_{t,T}\right]$ is equal to 1. To do so, we use that $\xi^q_{t,t} = 1$ and prove that $(\xi^q_{t,s})_{s\in[t,T]}$ is a martingale under $\left(\mathbb{P}; \mathbb{F} = (\mathcal{F}_s)_{s\in[t,T]}\right)$.

We know that $(q^{t,q}_s)_{s\in[t,T]}$ satisfies a linear growth condition with respect to $(S^{t,S}_s)_{s\in[t,T]}$. Given the form of κ one can easily show that there exists a constant C such that

$$\sup_{s\in[t,T]} \| \kappa^q_s \|^2 \leqslant C\left(1 + \sup_{s\in[t,T]} \| W_s - W_t \|^2\right). \tag{80}$$

By using classical properties of the Brownian motion, we prove that

$$\exists \epsilon > 0, \forall s \in [t, T], \quad \mathbb{E}\left[\exp\left(\frac{1}{2}\int_s^{(s+\epsilon)\wedge T} \kappa^{q\intercal}_\varrho \Sigma \kappa^q_\varrho d\varrho\right)\right] < +\infty. \tag{81}$$

From Novikov condition, we see that $(\xi^q_{t,s})_{s\in[t,T]}$ is a martingale under $\left(\mathbb{P}; \mathbb{F} = (\mathcal{F}_s)_{s\in[t,T]}\right)$.

We obtain

$$\mathbb{E}\left[\hat{w}\left(T, \mathcal{V}^{t,\mathcal{V},S,q}_T, S^{t,S}_T\right)\right] \leqslant \hat{w}(t, \mathcal{V}, S), \tag{82}$$

with equality when $(q_s)_{s\in[t,T]} = (q^*_s)_{s\in[t,T]}$.

We conclude that

$$\hat{u}(t, \mathcal{V}, S) = \sup_{(q_s)_{s\in[t,T]}\in\mathcal{A}^{Merton}_t} \mathbb{E}\left[-\exp\left(-\gamma V^{t,\mathcal{V},S,q}_T\right)\right] \tag{83}$$

$$= \mathbb{E}\left[-\exp\left(-\gamma V^{t,\mathcal{V},S,q^*}_T\right)\right] \tag{84}$$

$$= \hat{w}(t, \mathcal{V}, S). \tag{85}$$

□

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