Joint Modeling of VIX and SPX options at a single and common maturity with risk management applications*

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Abstract

A double gamma model is proposed for the VIX. The VIX is modeled as gamma distributed with a mean and variance that respond to a gamma distributed realized variance over the preceeding month. Conditionally on VIX and the realized variance, the logarithm of the stock is variance gamma distributed with affine conditional drift and quadratic variation. The joint density for the triple, realized variance, VIX and the SPX is in closed form. Calibrating the model jointly to SPX and VIX options a risk management application illustrates a hedge for realized volatility options.

1 Introduction

The S&P 500 index along with options written on the index is now coupled with the VIX index and options on this index. The two are not unrelated as the square of the VIX index is the price of a one month variance swap paying the annualized one month realized variance of returns on the S&P 500 index. The variance swap rate itself is typically synthesized from S&P 500 options using

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procedures described for example in Carr and Lee (2009). In fact this is how the VIX is now computed. There is then a demand and an interest in jointly and consistently modeling options on these two indices. For recent work in this direction we cite Cont and Kokholm (2010), Gatheral (2008), Bergomi (2004, 2005, 2008), Broadie and Jain (2008) and Buehler (2006).

Though the VIX may be and is computed every day, it is by construction eventually the price of a different asset, as the variance of the S&P 500 index over disjoint months along with the price of such a payout are potentially unrelated. Of course they may be linked by assuming a model relating them. There are various levels of consistency that may be asked for and modeled. For example one may ask for a consistent modeling of both sets of options across all strikes and maturities through calendar time. Alternatively one may seek to model consistently across strike and maturity at a single point of time, the calibration date. Restricting further one may jointly model across both sets of strikes at a single maturity. The focus of this paper is on the last of these alternatives.

In a partial defense of such a modeling strategy we note that Lévy processes like the variance gamma model (Madan and Seneta (1990), Madan, Carr and Chang (1998)) synthesized option prices just across a single maturity, while an effective parsimonious synthesis across maturities using a one dimensional Markov model came later in the Sato process of Carr, Geman, Madan and Yor (2007). An effective synthesis across calendar time has not yet been attained for just the S&P 500 index itself as most models are recalibrated continuously. Hence we focus attention here on just two smiles at a common maturity.

From a risk management perspective one may seek to determine a static portfolio of relatively short maturity options on the S&P 500 index and the VIX with a view to covering a comparable maturity risk exposure in a realized volatility swap. For such an application one seeks a joint and consistent modeling of the two smiles. We apply the model developed here to hedge such an exposure.

The outline of the rest of the paper is as follows. Section 2 sets out the joint model for the two smiles. Section 3 describes the calibration procedure followed for a single calibration date. Section 4 illustrates with a risk management application. Section 5 concludes.

2 The Joint Model

We formulate a joint model for the logarithm of the index at a fixed, and say, unit maturity that we denote by s and the square of the VIX at the same time denoted by v. A third variable of interest is the realized variance on the index that we shall take to be proportional to x. Options trade on the exponential of s and the square root of v, while x will be a hidden variable whose conditional law given $s.v$ will be inferred for a risk management exercise. We shall in fact formulate a joint law for the triple (s, x, v).

Various models were attempted for the law of v or its square root and we mention in passing the gamma distribution, or a general power of a gamma distributed variable. It was found that these models failed to match out of the money call option prices. In line with Gatheral's (Gatheral (2008)) suggestion of a double log normal we develop here a double gamma given the relative tractability of the associated Laplace transforms. Since the variable x is scaled in all its occurences we take it to be our first gamma variable with unit mean and variance $1/\gamma$. The marginal density for x is then

$$f(x) = \frac{\gamma^{\gamma}}{\Gamma(\gamma)} x^{\gamma-1} e^{-\gamma x}, \quad x > 0,$$

where $\Gamma(x)$ is the gamma function.

Next we specify the conditional density of v given x and we take this to be gamma distributed with scale coefficient α and shape coefficient $\beta + \delta x$. The mean and variance of v respond linearly to x and this specification models the response of the square of the VIX at unit time to the proxy for realized variance x to that time. The joint density for x, v is then

$$g(x,v) = \frac{\gamma^{\gamma} x^{\gamma-1} e^{-\gamma x}}{\Gamma(\gamma)} \frac{\alpha^{\beta+\delta x} v^{\beta+\delta x-1} e^{-\alpha v}}{\Gamma(\beta+\delta x)}, \quad x, v > 0.$$

The marginal density for v would require an integration over x and is not available in closed form but it may be accessed by transform methods from the joint characteristic function that is easily seen to be

$$\begin{aligned}
\Lambda(\kappa,\lambda) &= E\left[\exp\left(i\kappa x + i\lambda v\right)\right] \\
&= E\left[\exp\left(i\kappa x + (\beta+\delta x)\ln\left(\frac{\alpha}{\alpha - i\lambda}\right)\right)\right] \\
&= \left(\frac{\alpha}{\alpha - i\lambda}\right)^{\beta} E\left[\exp\left(i\left(\kappa - i\delta\ln\left(\frac{\alpha}{\alpha - i\lambda}\right)\right)x\right)\right] \\
&= \left(\frac{\alpha}{\alpha - i\lambda}\right)^{\beta} \left(\frac{\gamma}{\gamma - i\left(\kappa - i\delta\ln\left(\frac{\alpha}{\alpha - i\lambda}\right)\right)}\right)^{\gamma} \\
&= \left(\frac{\alpha}{\alpha - i\lambda}\right)^{\beta} \left(\frac{c}{c - i\kappa - \delta\ln\left(\frac{\alpha}{\alpha - i\lambda}\right)}\right)^{\gamma}
\end{aligned}$$

Conditional on x, v we suppose that the logarithm of the stock is variance gamma distributed X_{CGM} with parameters CGM. The characteristic function for X_{CGM} is

$$E\left[\exp\left(iuX_{CGM}\right)\right] = \exp\left(C\ln\left(\frac{GM}{(M-iu)(G+iu)}\right)\right).$$

The density for X_{CGM} is from Madan, Carr and Chang (1998) on transformation to the CGM parametreization as per Carr, Geman, Madan and Yor (2002)
,

$$f_{CGM}(x) = \frac{(GM)^{C}}{2^{C-1}\Gamma(C)\sqrt{2\pi}\left(\frac{G+M}{2}\right)^{C-1/2}} \exp\left(\frac{G-M}{2}x\right)|x|^{C-1/2} K_{C-1/2}\left(\frac{G+M}{2}|x|\right)$$

where $K_\nu(x)$ is the modified Bessel function.

The quadratic variation of a variance gamma process with density in CGM parameterization is

$$C\left(\frac{1}{M^2}+\frac{1}{G^2}\right).$$

We suppose the conditional density for the logarithm of the stock has a conditional drift adapted to x, v as

$$r+\eta x+\zeta v$$

with quadratic variation modeled to respond to the level of x, v as

$$C=k+ax+bv.$$

The parameters G, M are constants.

The conditional drift for the stock is organized by writing

$$s=r+\eta x+\zeta v+(k+ax+bv)\ln\left(\frac{(M-1)(G+1)}{GM}\right)+X_{CGM}+\omega$$

where the constant ω is chosen to set the unconditional drift to be the interest rate.

The conditional expectation of the exponential of s given x, v is

$$E\left[e^s|x,v\right]=\exp\left(r+\eta x+\zeta v+\omega\right)$$

Hence we set

$$\omega=-\ln E\left[\exp\left(\eta x+\zeta v\right)\right]$$

and on evaluation of the joint characteristic function of x, v at $-i\eta, -i\zeta$ we infer that

$$\omega=\beta\ln\left(\frac{\alpha-\zeta}{\alpha}\right)+\gamma\ln\left(\frac{\gamma-\eta-\delta\ln\left(\frac{\alpha}{\alpha-\zeta}\right)}{\gamma}\right).$$

The joint characteristic function of the triple s, x, v is then

$$\begin{aligned}
E\left[\exp\left(ius+i\kappa x+i\lambda v\right)\right] &= \exp(iu\left(r+\omega+k\ln\left(\frac{(M-1)(G+1))}{GM}\right)\right)\times \\
& E\left[\begin{array}{c} iu\left(\eta+a\ln\left(\frac{(M-1)(G+1))}{GM}\right)\right)x \\ +iu\left(\zeta+b\ln\left(\frac{(M-1)(G+1))}{GM}\right)\right)v \\ +i\kappa x+i\lambda v \\ +(k+ax+bv)\ln\left(\frac{GM}{(M-iu)(G+iu)}\right) \end{array}\right] \\
&= \exp\left(iu\left(r+\omega+k\ln\left(\frac{(M-1)(G+1))}{GM}\right)\right)+k\ln\left(\frac{GM}{(M-iu)(G+iu)}\right)\right)\times \\
& \Lambda\left(\begin{array}{c} \kappa+u\left(\eta+a\ln\left(\frac{(M-1)(G+1))}{GM}\right)\right)-ia\ln\left(\frac{GM}{(M-iu)(G+iu)}\right), \\ \lambda+u\left(\zeta+b\ln\left(\frac{(M-1)(G+1))}{GM}\right)\right)-ib\ln\left(\frac{GM}{(M-iu)(G+iu)}\right) \end{array}\right)
\end{aligned}$$

There are 11 parameters in the model and they are $\gamma, \alpha, \beta, \delta, k, a, b, \eta, \zeta, G$ and M.

The joint density for the triple (s, x, v) is given by

$$
\begin{aligned}
h(s,x,v) &= g(x,v) \times \frac{(GM)^C}{2^{C-1}\sqrt{2\pi}\Gamma(C)\left(\frac{G+M}{2}\right)^{C-.5}} \exp\left(\frac{G-M}{2}w\right)|w|^{C-.5} K_{C-.5}\left(\frac{G+M}{2}|w|\right) \\
w &= s - r - \left(\eta + a\ln\left(\frac{(M-1)(G+1))}{GM}\right)\right)x \\
&\quad - \left(\zeta + b\ln\left(\frac{(M-1)(G+1))}{GM}\right)\right)v - \omega - k\ln\left(\frac{(M-1)(G+1))}{GM}\right) \\
C &= k + ax + bv \\
g(x,v) &= \frac{1}{\Gamma(\gamma)}c^{\gamma}x^{\gamma-1}e^{-cx}\frac{\alpha^{\beta+\delta x}}{\Gamma(\beta+\delta x)}v^{\beta+\delta x-1}e^{-\alpha v} \\
\omega &= \beta\ln\left(\frac{\alpha-\zeta}{\alpha}\right) + \gamma\ln\left(\frac{c-\eta-\delta\ln\left(\frac{\alpha}{\alpha-\zeta}\right)}{c}\right)
\end{aligned}
$$

We term this model $sxvvgwadqv$ for modeling the triple s, x, v and specifying s as vg conditional on x, v with $affine\ drift$ and $quadratic\ variation$.

3 Calibration of the Joint Model for the triple s, x, v

We may organize the parameters into two groups, $\gamma, \alpha, \beta, \delta$ of the double gamma model for the square of the VIX. Followed by $k, a, b, \eta, \zeta, G, M$ for the VG in CGM format. Conditional on x, v the parameter C is modeled by the affine function $k+ax+bv$. Similarly conditional on x, v the stock drift is $\eta x+\zeta v$. The first step is to find the double gamma parameters for the square of the VIX. Unfortunately there are no quoted options on the square of the VIX but just options on the VIX. One could build options on the square of the VIX from VIX options but the range of traded strikes on the VIX may be too narrow for evaluating the price of the tail of the square. Instead we proceed by first fitting the double gamma model to the VIX even though it is a model for the square.

We next use these parameters to generate prices for calls on VIX over a wide range of strikes and we then use these prices to determine prices on calls for the square of the VIX. For a call option on the square of the VIX with strike k the call price $C(k)$ is obtained in terms of call prices $c(x)$ on the VIX for strike x

and density $f(x)$ by

$$\begin{aligned} C(k) &= e^{-rt}\int_{\sqrt{k}}^{\infty}\left(x^2-k\right)f(x)dx \\ &= e^{-rt}\int_{\sqrt{k}}^{\infty}2x(1-F(x))dx \\ &= \int_{\sqrt{k}}^{\infty}2x\left(-c'(x)\right)dx. \end{aligned} \tag{1}$$

The procedure may also be reversed with density $g(k)$ for the squared VIX by writing

$$\begin{aligned} c(x) &= e^{-rt}\int_{x^2}^{\infty}(\sqrt{k}-x)g(k)dk \\ &= e^{-rt}\int_{x^2}^{\infty}\frac{1}{2\sqrt{k}}(1-G(k))dk \\ &= \int_{x^2}^{\infty}\frac{1}{2\sqrt{k}}\left(-C'(k)\right)dk \end{aligned} \tag{2}$$

Equation 1 is employed to generate option prices on the squared VIX to get starting values for the double gamma model for the squared VIX after fitting this model to the VIX. Equation 2 is then used in calibrating VIX options for a model on the squared VIX.

We illustrate the procedure for data on October 20, 2008 a month after the Lehman collapse. Fitting the double gamma model to VIX options yielded the following parameter values.

$$\begin{aligned} \gamma &= 0.6617 \\ \alpha &= 0.5474 \\ \beta &= 18.7080 \\ \delta &= 5.8241 \end{aligned}$$

Figure 1 presents a graph of the fit of the double gamma model to VIX options.

The double gamma parameter values fitted to VIX options are then used to generate a set of call prices on the square of the VIX and the double gamma model is then fit to the generated squared VIX call prices. The resulting parameter values for the double gamma squared VIX model are

$$\begin{aligned} \gamma &= 0.198797 \\ \alpha &= 0.002392 \\ \beta &= 3.583938 \\ \delta &= 1.753731 \end{aligned}$$

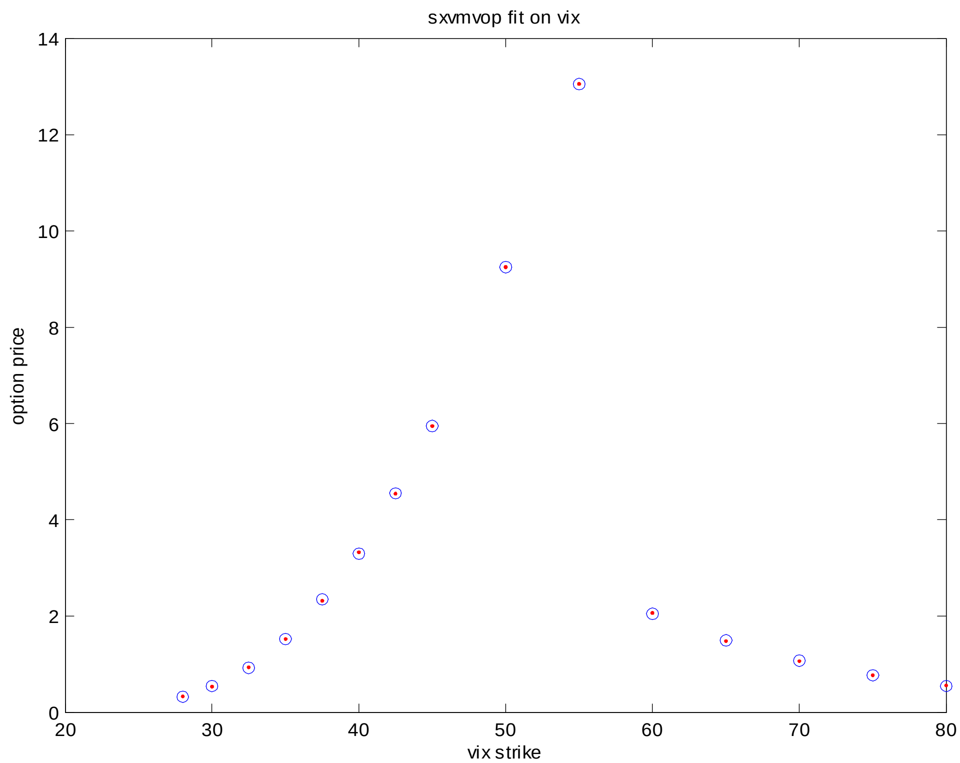

Figure 1: Fit of double gamma model to VIX option directly on October 20, 2008. Market prices are represented by circles while model prices are shown as dots.

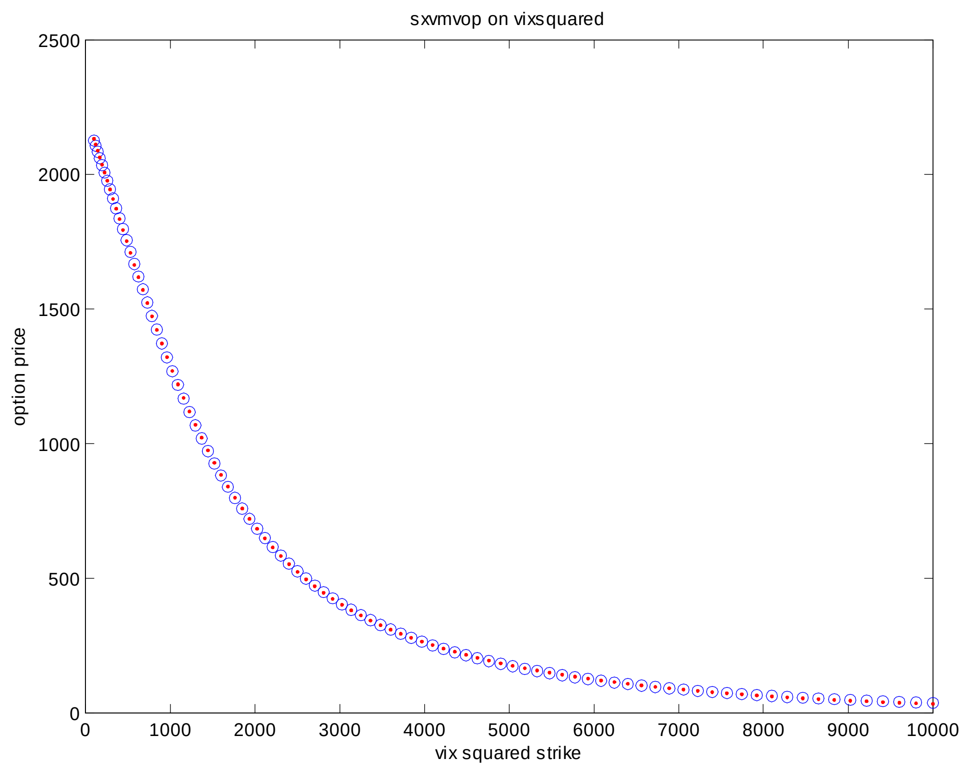

Figure 2: Fit of the double gamma model to call option prices for the squared VIX. Circles represent the call prices generated from the double gamma fit to the VIX. The double gamma model for the VIX squared are shown as dots.

Figure 2 shows the fit of the double gamma model to call prices on the squared VIX.

The eleven parameter model *sxvvgwadqv* is then fit simultaneously to options on the SPX and VIX. The resulting parameter values for October 20, 2008 were

$$\begin{aligned}
\gamma &= 0.157211 \\
\alpha &= 0.002421 \\
\beta &= 3.728842 \\
\delta &= 1.802546 \\
k &= 0.144401 \\
a &= 0.007435 \\
b &= 0.000228 \\
\eta &= 0.052668 \\
\zeta &= 0.000110 \\
G &= 8.758678 \\
M &= 20.20196
\end{aligned}$$

Figures 3 and 4 present graphs of the fit to VIX and SPX options on October 20, 2008.

4 A Risk Management Application

By way of a risk management application we construct a portfolio of options on the S&P 500 index and VIX options with a view to earning the ask price on the liability of a call option on realized volatility struck at a volatility of 60. This is a little out of the money as the variance swap quote for a month was estimated from the S&P 500 index option data at 51.08.

The mean of the hidden variable x is unity and we let it represent realized variance by scaling it by the square of the variance swap quote. This scale factor was 2609.47. The level of the SPX was 984.41. The cash flow on a realized volatility call struck at 60 is then modeled as

$$c = \left(\sqrt{2609.47 * x} - 60\right)^{+}$$

The ask price for this cash flow seen as a liability is evaluated using the methods of Cherny and Madan (2010), Carr, Madan and Vicente Alvarez (2011) as the negative of the distorted expectation of $-c$, where we distort the conditional distribution $p(x|s,v)$ of x given s,v. The conditional distribution is obtained from the joint density of the triple by Bayes rule. The distortion used is *minmaxvar* at the stress level of 0.25.

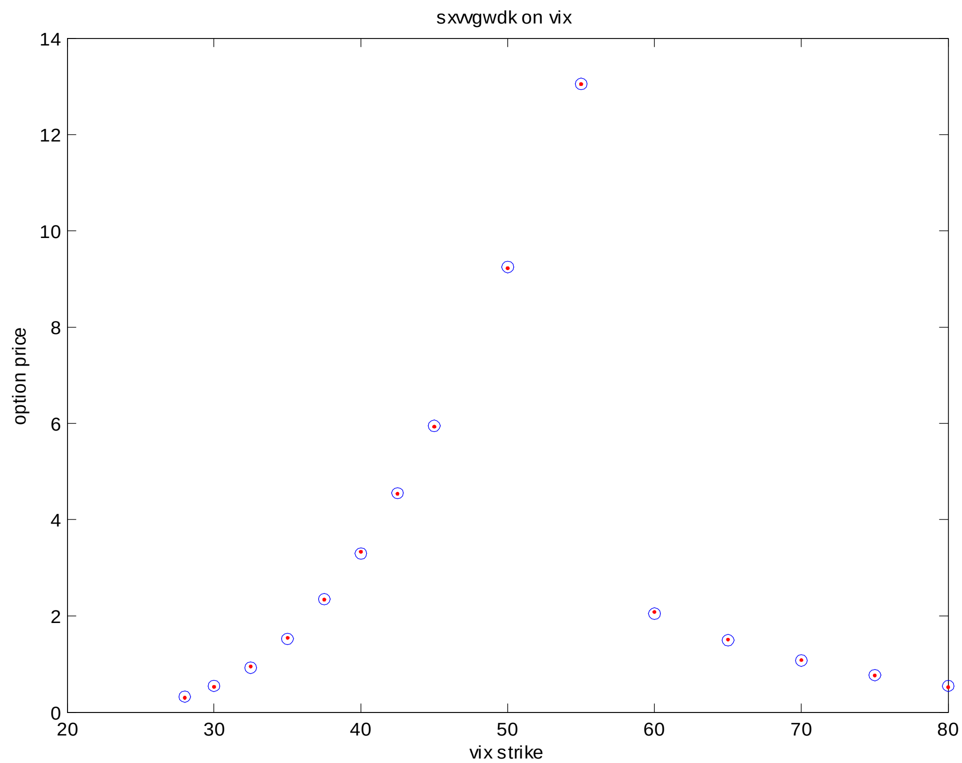

Figure 3: Fit of sxvvgwadqv to VIX options on October 20, 2008. Market prices are represented by circles while model prices are shown as dots.

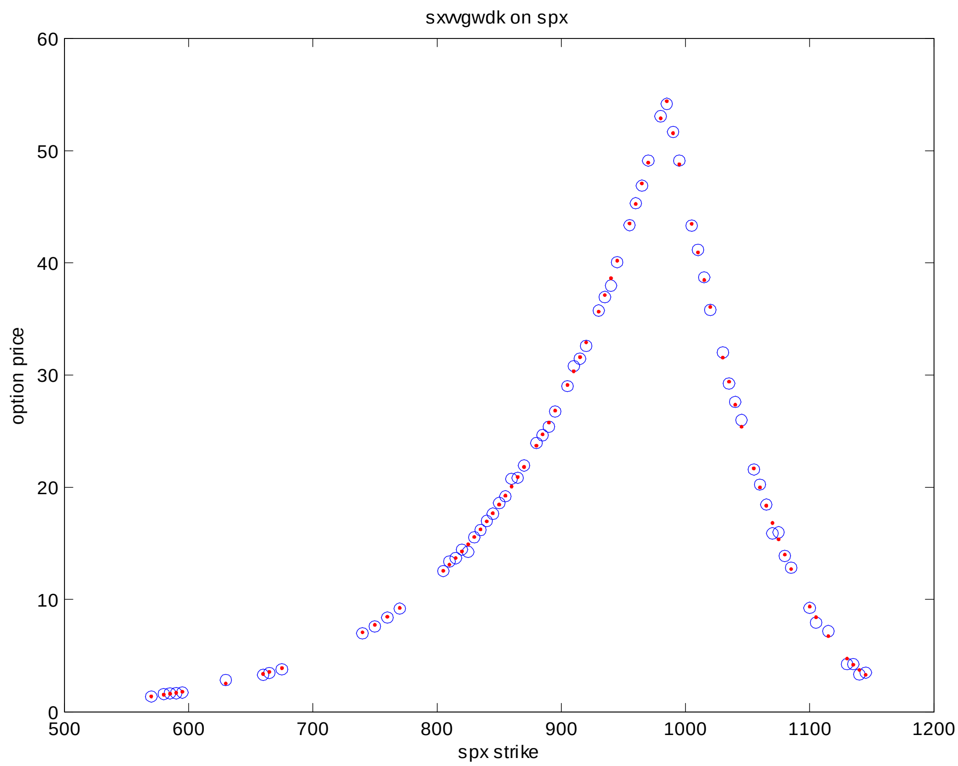

Figure 4: Fit of sxvvgwadqv to SPX options on October 20, 2008. Market prices are represented by circles while model prices are shown as dots.

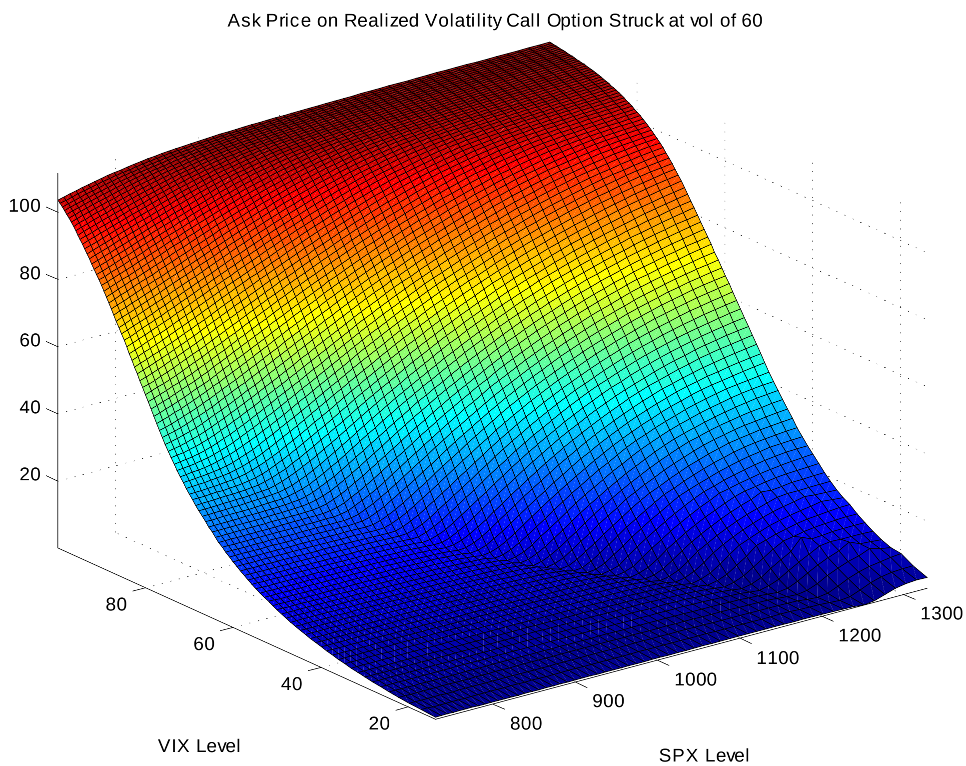

Figure 5: Ask Price for a call option on realized volatility struck at a volatility of 60 constructed from the conditional distribution of x given the SPX and the VIX by minmaxvar distortion at stress level 0.25.

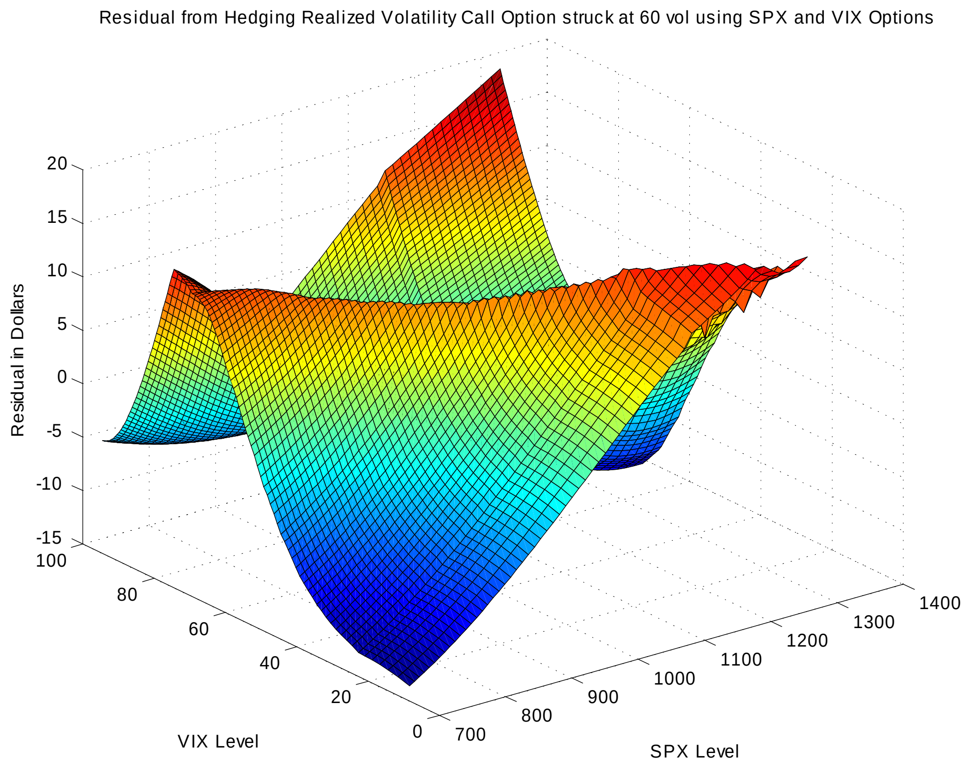

Figure 6: Difference between the hedge cash flow and the target cash flow attained on minimizing the least squares distance between the two cash flows.

Figure 5 presents a graph of this ask price as function of the the level of the SPX and the VIX.

We construct a portfolio of positions in the money market, the SPX, the VIX and options thereon to minimize by least squares the gap between the hedge cash flow and the target cash flow given by the ask price function displayed in Figure 5. Figure 6 presents the gap between the hedge cash flow and the target cash flow. The strikes used for the option positions are the same as those displayed in the model calibration.

The cash flows contingent on the level of the SPX and VIX held as hedges are presented in Figures 7 and 8.

5 Conclusion

A double gamma model is proposed for the VIX where the mean and variance of the VIX respond to a proxy viewed as the realized variance over the preceeding

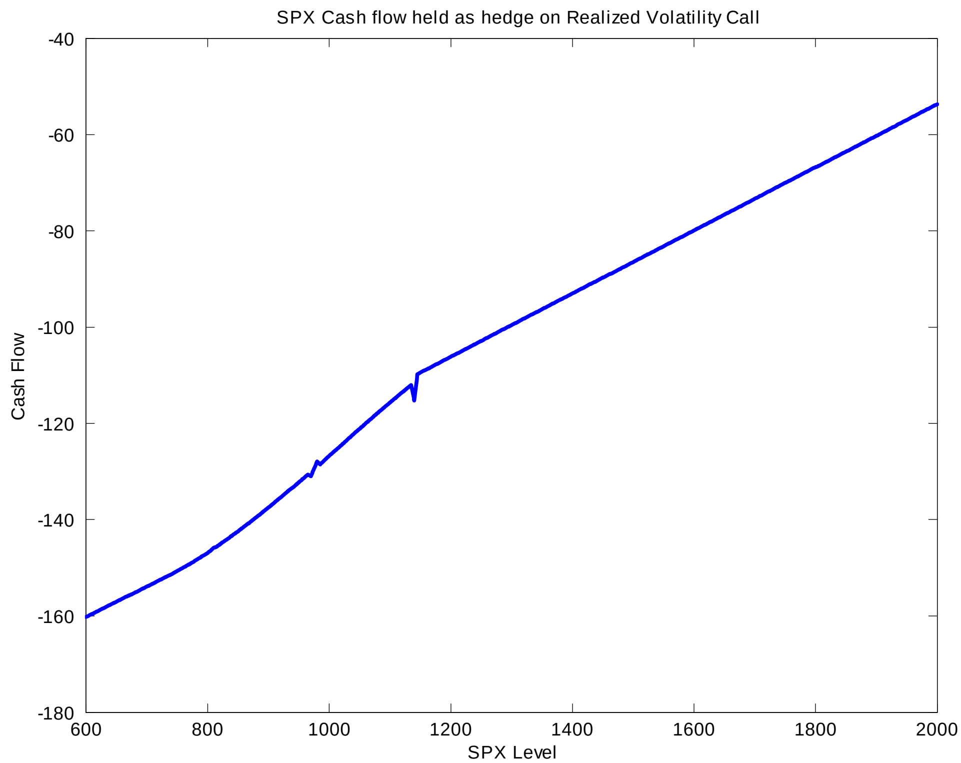

Figure 7: Cash flow contingent on the level of the SPX held as a hedge for the realized volatility call.

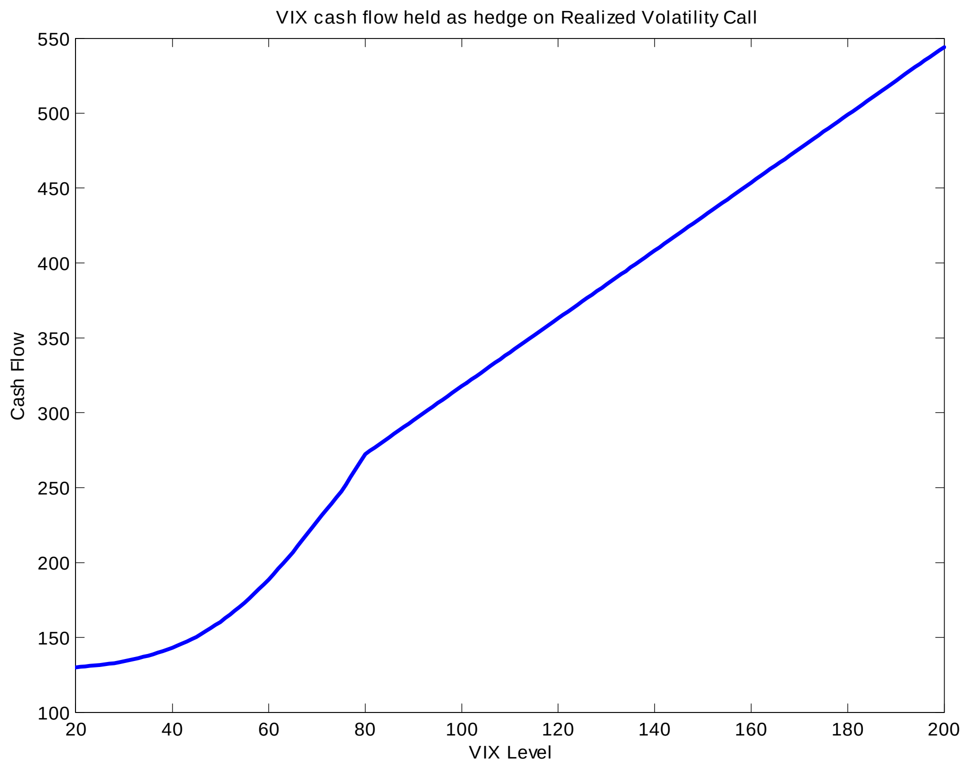

Figure 8: Cash flow contingent on the level of the VIX held as a hedge for the realized volatility call.

month. Conditional on the realized variance proxy and the VIX the logarithm of the stock is modeled as conditionally variance gamma distributed with affine conditional drift and quadratic variation. The resulting model for the triple, i) realized variance over the month, ii) the VIX at month end and iii) the S&P 500 index at month end, is a closed form joint density with eleven parameters. The model is calibrated jointly to SPX and VIX options for the data as at October 20, 2008. A risk management application hedging an ask price for a call on realized volatility struck at a volatility of 60 illustrates a model application.

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