On the Joint Calibration of SPX and VIX Options

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Motivation

- VIX options started trading in 2006
- How to build a model for the SPX that jointly calibrates to SPX options, VIX futures, and VIX options?
- Back in 2007, Jim Gatheral was one of the first to investigate this question. Jim showed that a diffusive model (the double mean-reverting model) could approximately match both markets.
- Later, others have argued that jumps in SPX are needed to fit both markets.
- In this talk, I revisit this problem, trying to answer the following questions:

Does there exist a diffusive model on the SPX that jointly calibrates to SPX options, VIX futures, and VIX options?

If so, how to build one such model? If not, why?

Gatheral (2008)

Consistent Modeling of SPX and VIX options

Jim Gatheral

Merrill Lynch

The Fifth World Congress of the Bachelier Finance Society
London, July 18, 2008

Double CEV dynamics

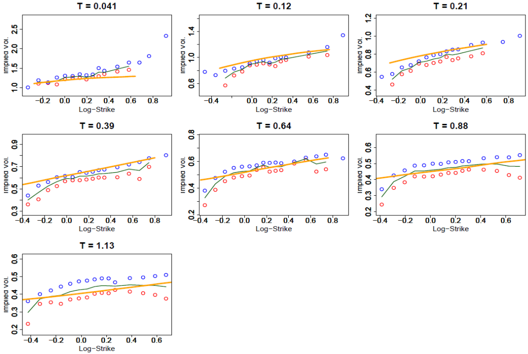
- Buehler's affine variance curve functional is consistent with double mean reverting dynamics of the form:

$$\begin{aligned} \frac{dS}{S} &= \sqrt{v}\, dW \\ dv &= -\kappa\,(v - v')\,dt + \eta_1\, v^{\alpha}\, dZ_1 \\ dv' &= -c\,(v' - z_3)\,dt + \eta_2\, v'^{\beta}\, dZ_2 \end{aligned} \tag{2}$$

  for any choice of α, $\beta \in [1/2, 1]$.
  - We will call the case $\alpha = \beta = 1/2$ *Double Heston*,
  - the case $\alpha = \beta = 1$ *Double Lognormal*,
  - and the general case *Double CEV*.
- All such models involve a short term variance level v that reverts to a moving level v' at rate κ. v' reverts to the long-term level z_3 at the slower rate $c < \kappa$.

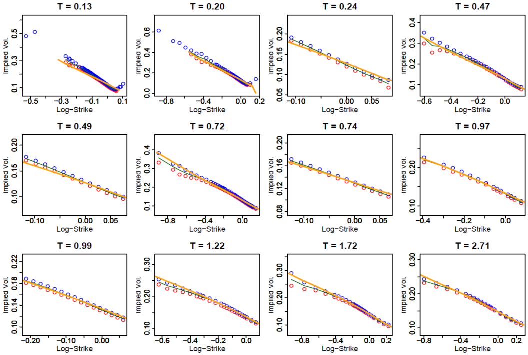
Double CEV fit to VIX options as of 03-Apr-2007

Setting the correlation ρ between volatility factors z_1 and z_2 to its historical average (see later) and iterating on the volatility of volatility parameters ξ_1 and ξ_2 to minimize the differences between model and market VIX option prices, we obtain the following fits (orange lines):

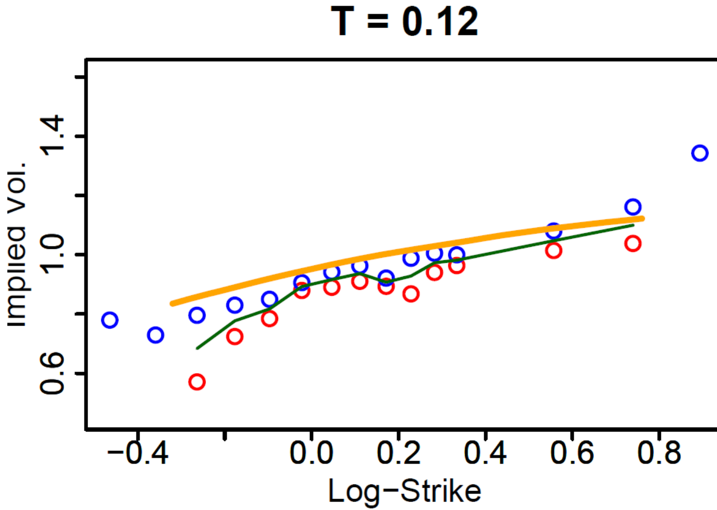


Double CEV fit to SPX options as of 03-Apr-2007

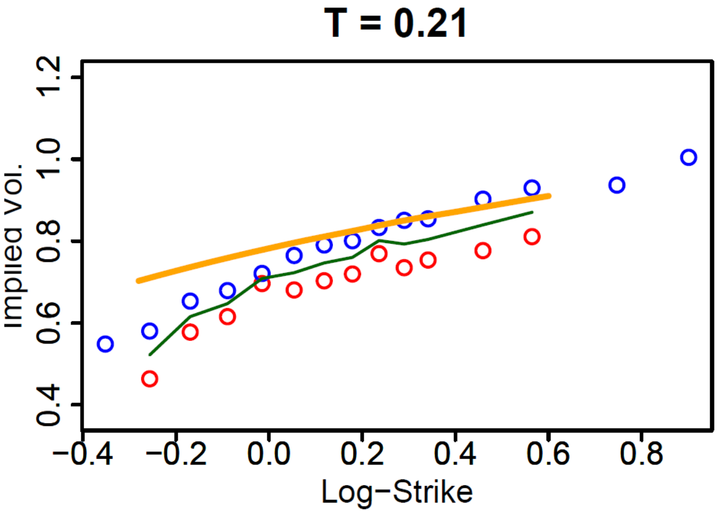
Minimizing the differences between model and market SPX option prices, we find $\rho_1 = -0.9$, $\rho_2 = -0.7$ and obtain the following fits to SPX option prices (orange lines):



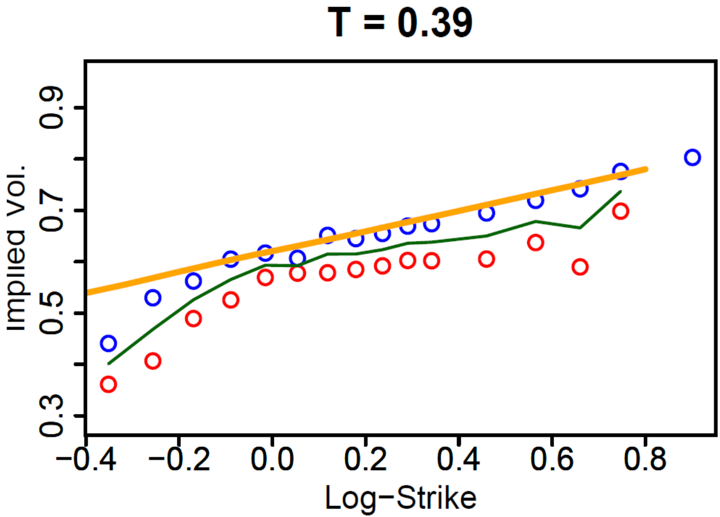
Fit to VIX options



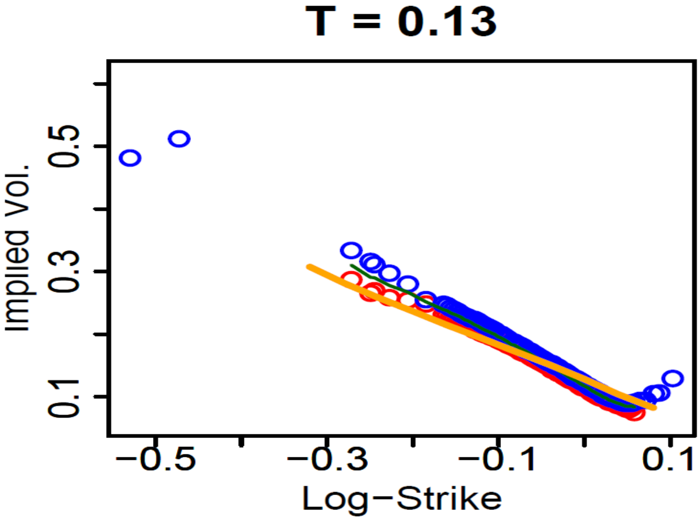
Fit to VIX options



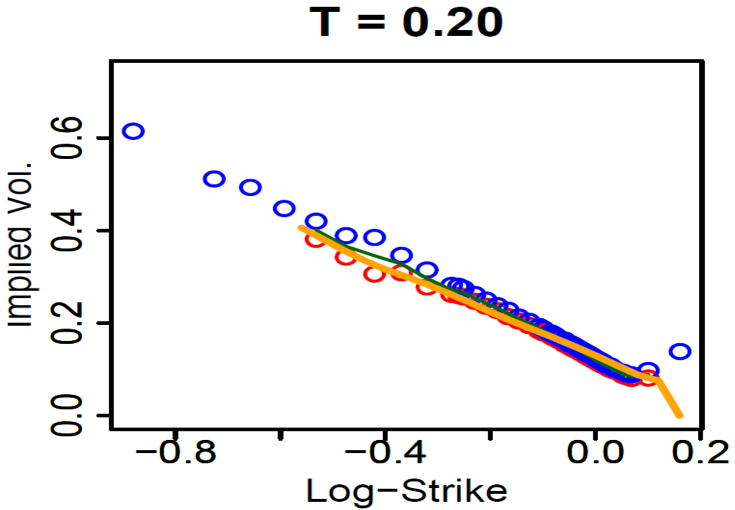
Fit to VIX options



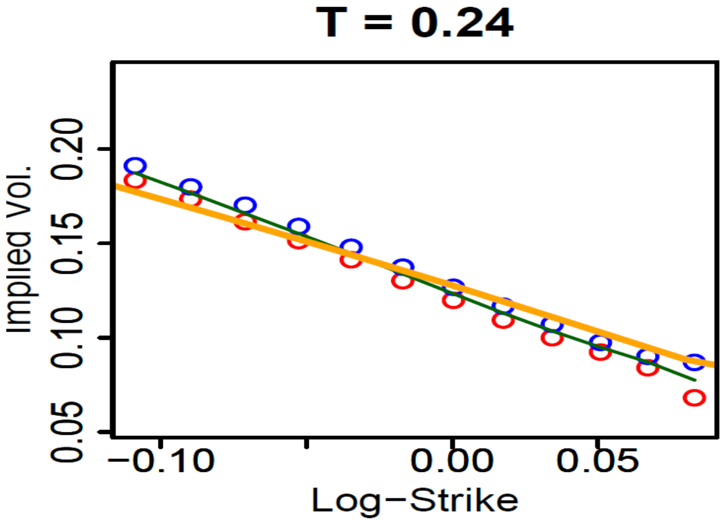
Fit to SPX options



Fit to SPX options



Fit to SPX options



- Joint calibration not so good for short maturities (up to 6 months)
- Unfortunate as these are the most liquid maturities for VIX futures and options
- Vol-of-vol is either too large for VIX market, or too small for SPX market (or both)

Trying with jumps in SPX

Sepp (2012)

Part I. Joint calibration of SPX and VIX skews using jumps

I consider several volatility models to reproduce the volatility skew observed in equity options on the S&P500 index:
Local volatility model (LV)
Jump-diffusion model (JD)
Stochastic volatility model (SV)
Local stochastic volatility model (LSV) with jumps

For each model, I analyze its implied skew for options on the VIX

I show that LV, JD and SV without jumps are not consistent with the implied volatility skew observed in option on the VIX

I show that:
Only the SV model with appropriately chosen jumps can fit the implied VIX skew

Importantly, that only the LSV model with jumps can fit both Equity and VIX option skews

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Baldeaux-Badran (2014)

Consistent Modelling of VIX and Equity Derivatives Using a $3/2$ plus Jumps Model

Jan Baldeaux and Alexander Badran

Abstract

The paper demonstrates that a pure-diffusion $3/2$ model is able to capture the observed upward-sloping implied volatility skew in VIX options. This observation contradicts a common perception in the literature that jumps are required for the consistent modelling of equity and VIX derivatives. The pure-diffusion model, however, struggles to reproduce the smile in the implied volatilities of short-term index options. One remedy to this problem is to augment the model by introducing jumps in the index. The resulting $3/2$ plus jumps model turns out to be as tractable as its pure-diffusion counterpart when it comes to pricing equity, realized variance and VIX derivatives, but accurately captures the smile in implied volatilities of short-term index options.

Keywords: Stochastic volatility plus jumps model, $3/2$ model, VIX derivatives

Baldeaux-Badran (2014)

$$dS_t = S_{t-}\left((r - \lambda\bar{\mu})dt + \rho\sqrt{V_t}dW_t^1 + \sqrt{1-\rho^2}\sqrt{V_t}dW_t^2 + (e^{\xi} - 1)dN_t\right), \quad (3)$$

$$dV_t = \kappa V_t(\theta - V_t)dt + \epsilon(V_t^{3/2})dW_t^1, \quad (4)$$

where we denote by N a Poisson process at constant rate λ, by e^{ξ} the relative jump size of the stock and N is adapted to a filtration $(\mathcal{F}_t)_{t\in[0,T]}$. The distribution of ξ is assumed to be normal with mean μ and variance σ^2. The parameters μ, $\bar{\mu}$, and σ satisfy the following relationship

$$\mu = \log(1+\bar{\mu}) - \frac{1}{2}\sigma^2.$$

Kokholm-Stisen (2015)

$$\frac{dS_t}{S_t} = (r - q - \overline{\mu}\lambda)dt + \sqrt{V_t}dW_t + (e^{J^S} - 1)dN_t \tag{1}$$

$$dV_t = \kappa(\theta - V_t)dt + \eta\sqrt{V_t}dZ_t + J^V dN_t \tag{2}$$

where, W_t and Z_t are Wiener processes correlated with coefficient $\rho \in [-1, 1]$ and $\theta, \kappa, \eta \geq 0$. The price and volatility processes have simultaneous jumps with constant arrival intensity $\lambda \geq 0$. The jumps in volatility are independent and identically exponentially distributed with mean $\mu_v \geq 0$. Conditionally, on the jump in volatility, the jump in the price process is normally distributed with:

$$J^V \sim \exp(\mu_v), \quad J^S \,|\, J^V = y \sim N(\mu_s + \rho_J y, \sigma^2) \tag{3}$$

where $\sigma \geq 0$, $\rho_J \in [-1, 1]$, $\mu_s \in \mathbb{R}$. The martingale condition on the discounted price process imposes that:

$$\overline{\mu} = \frac{e^{\mu_s + \frac{1}{2}\sigma^2}}{1 - \rho_J \mu_v} - 1 \tag{4}$$

Kokholm-Stisen (2015)

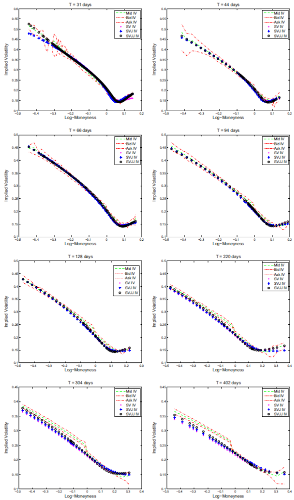

Figure 6. Fit to the SPX option smiles on May 16, 2012, of the SV, SVJ and SVJJ models calibrated to SPX options and VIX derivatives without the Feller condition imposed

Kokholm-Stisen (2015)

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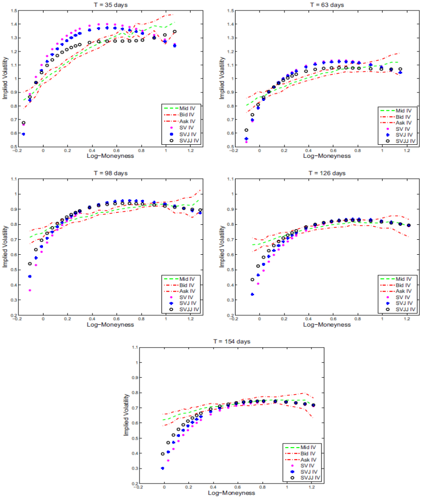


Figure 7.
Fit to the VIX option smiles on May 16, 2012, of the SV, SVJ and SVJJ models calibrated to SPX options and VIX derivatives without the Feller condition imposed

Kokholm-Stisen (2015)

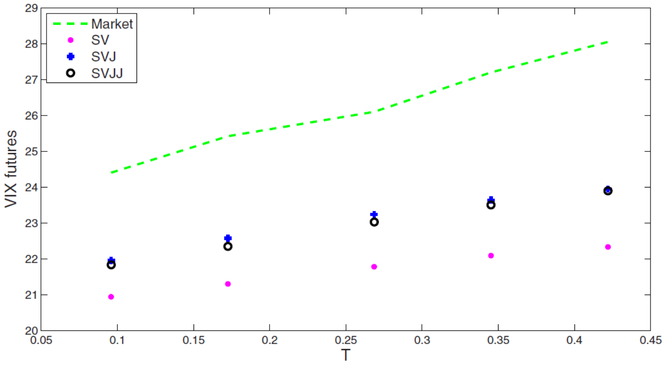


Figure 8. Fit to the VIX futures on May 16, 2012, of the SV, SVJ and SVJJ models calibrated to SPX options and VIX derivatives without the Feller condition imposed

Bardgett-Gourier-Leippold (2015)

$$dY_t = [-\lambda^{Yv}(v_{t-}, m_{t-})(\theta_Z(1,0,0) - 1) - \frac{1}{2}v_{t-}]dt + \sqrt{v_{t-}}dW_t^Y + dJ_t^Y,$$

$$dv_t = \kappa_v(m_{t-} - v_{t-})dt + \sigma_v\sqrt{v_{t-}}dW_t^v + dJ_t^v,$$

$$dm_t = \kappa_m(\theta_m - m_{t-})dt + \sigma_m\sqrt{m_{t-}}dW_t^m + dJ_t^m,$$

Bardgett-Gourier-Leippold (2015)

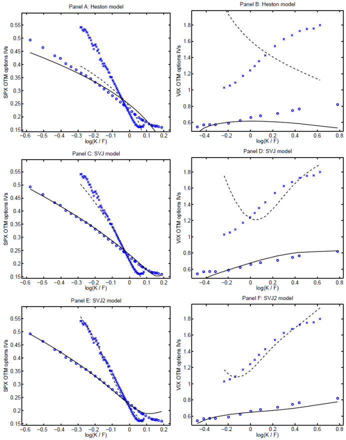


Figure 2: Market and model IVs on May 5, 2010, obtained by a joint calibration on the S&P 500 and VIX option market. Circles represent the market IV for $T = 0.05$ (S&P 500) and $T = 0.04$ (VIX). Crosses represent the market IV for $T = 0.3$ (S&P 500) and $T = 0.36$ (VIX). The dashed line corresponds to the model fit for $T = 0.05$ (S&P 500) and $T = 0.04$ (VIX) while the solid line corresponds to the model fit for $T = 0.3$ (S&P 500) and $T = 0.36$ (VIX). Panels A (S&P 500) and B (VIX) plot the model IVs based on the Heston model. Panels C and D display the corresponding results for the SVJ model, while Panels E and F do so for the SVJ2 model.

Papanicolaou-Sircar (2014)

- Use a regime-switching stochastic volatility model
- Hidden regime θ: continuous time Markov chain

$$
\begin{aligned}
dX_t &= \left(r - \frac{1}{2} f^2(\theta_t) Y_t - \delta \nu(\theta_{t\text{-}})\right) dt + f(\theta_t)\sqrt{Y_t}\, dW_t - \lambda(\theta_t) J_t\, dN_t \ , \\
dY_t &= \kappa(\bar{Y} - Y_t)dt + \gamma\sqrt{Y_t}\, dB_t \ , \\
dN_t &= \mathbb{1}_{[\theta_t \neq \theta_{t-}]},
\end{aligned}
$$

Papanicolaou-Sircar (2014)

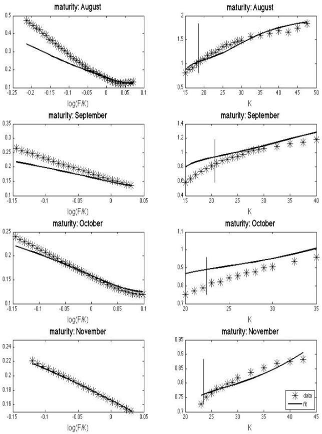


Figure 9: *The implied volatilities of July 27th SPX options (left column) and VIX options (right column), plotted alongside those of a fitted Heston with jumps. The fitted parameter values are given in Table 4. The vertical lines in the plots on the right mark the VIX futures price on the date of maturity.*

Cont-Kokholm (2013)

- Framework *à la* Bergomi:
  1. Model dynamics of forward variances $V_t^{[T_i,T_{i+1}]}$
  2. Given $V_{T_i}^{[T_i,T_{i+1}]}$, model dynamics of SPX
- Simultaneous (Lévy) jumps on forward variances and SPX
- First time a model seems to be able to jointly fit SPX skew and VIX level even for short maturities

Cont-Kokholm (2013)

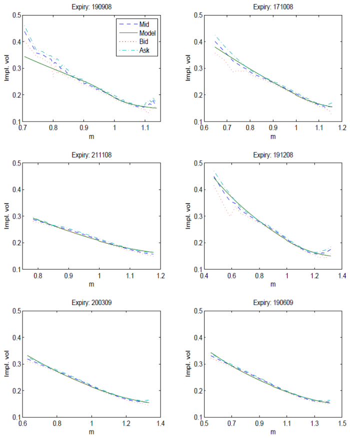


Figure 6: S&P 500 implied volatility smiles on August 20th 2008 for the model with normally distributed jump sizes plotted against moneyness $m = K/S_t$ on the horizontal axis.

Cont-Kokholm (2013)

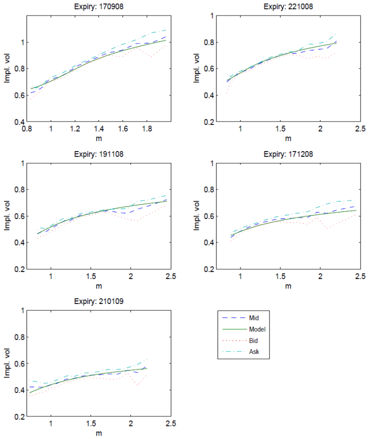


Figure 4: VIX implied volatility smiles on August 20th 2008 for the model with normally distributed jump sizes plotted against moneyness $m = K/VIX_t$ on the horizontal axis.

Pacati-Pompa-Renò (2015)

$$\begin{cases} dx_t = \left[r - q - \lambda\bar{\mu} - \frac{1}{2}\left(\sigma_{1,t}^2 + \phi_t + \sigma_{2,t}^2\right)\right] dt + \sqrt{\sigma_{1,t}^2 + \phi_t}\, dW_{1,t}^S + \sigma_{2,t} dW_{2,t}^S + c_x dN_t \\ d\sigma_{1,t}^2 = \alpha_1(\beta_1 - \sigma_{1,t}^2)dt + \Lambda_1 \sigma_{1,t} dW_{1,t}^\sigma + c_\sigma dN_t + c'_\sigma dN'_t \\ d\sigma_{2,t}^2 = \alpha_2(\beta_2 - \sigma_{2,t}^2)dt + \Lambda_2 \sigma_{2,t} dW_{2,t}^\sigma \end{cases}$$

Pacati-Pompa-Renò (2015)

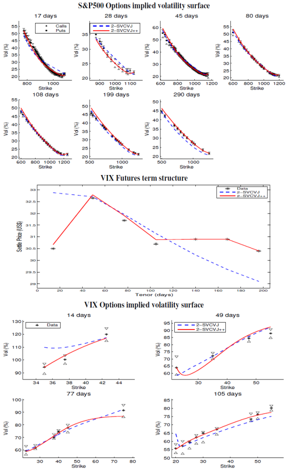


Figure 1: This figure reports market and model implied volatilities for S&P500 (plot at the top) and VIX (plot at the bottom) options, together with the term structure of VIX futures (plot in the middle) on September 2, 2009 obtained calibrating jointly on the three markets the 2-SVCVJ (blue dashed line) and 2-SVCVJ++ (red line). Maturities and tenors are expressed in days and volatilities are in % points and VIX futures settle prices are in US$.

Pacati-Pompa-Renò (2015)

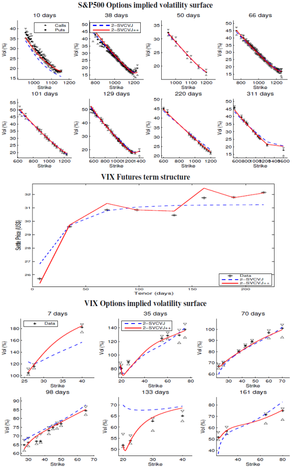


Figure 2: This figure reports market and model implied volatilities for S&P500 (plot at the top) and VIX (plot at the bottom) options, together with the term structure of VIX futures (plot in the middle) on August 11, 2010 obtained calibrating jointly on the three markets the 2-SVCVJ (blue dashed line) and 2-SVCVJ++ (red line). Maturities and tenors are expressed in days and volatilities are in % points and VIX futures settle prices are in US$.

Trying again with no jumps in SPX

Goutte-Ismail-Pham (2017)

- Also use a regime-switching stochastic volatility model
- Hidden regime Z: continuous time Markov chain

$$\begin{cases} dS_t = S_t(r\,dt + \sqrt{V_t}\,dW_t^1), & S_0 = s \\ dV_t = \kappa(Z_t)(\theta(Z_t) - V_t)dt + \xi(Z_t)\sqrt{V_t}\,dW_t^2, & V_0 = v_0. \end{cases}$$

Goutte-Ismail-Pham (2017)

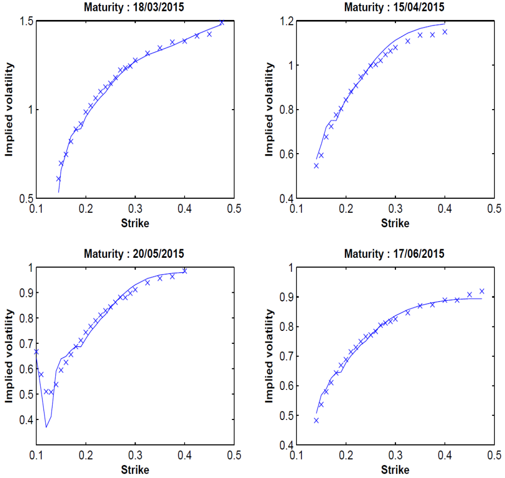


Figure 8: Implied volatilities of February 13, 2015, for VIX call options and the calibrated smile.

Goutte-Ismail-Pham (2017)

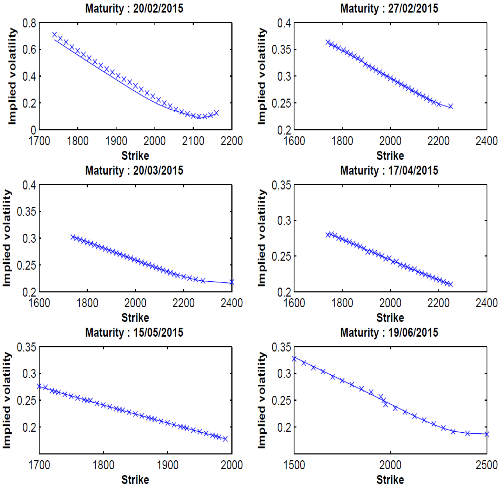


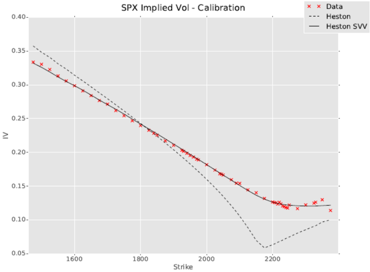
Figure 9: Implied volatilities of February 13, 2015, for S&P 500 call options and the calibrated smile.

Goutte-Ismail-Pham (2017)

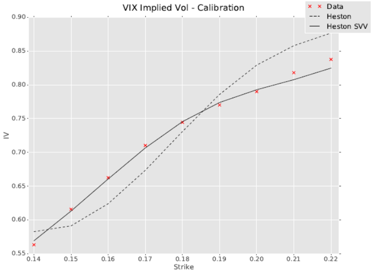
...but problem with SPX market data

Fouque-Saporito (2017)

- Based on Heston model with stochastic vol of vol
- No jumps
- Good fit to both SPX and VIX options... but only for maturities ≥ 4 months



(a) S&P 500



(b) VIX

So does there exist a diffusive model on the SPX that jointly calibrates to SPX options, VIX futures, and VIX options?

No answer yet...

Diffusive model on SPX calibrated to SPX options

- For simplicity, let us assume zero interest rates, repos, and dividends.
- Let $\mathcal{F}_t$ denote the market information available up to time t.
- We consider diffusive models on the SPX index

$$\frac{dS_t}{S_t} = \sigma_t \, dW_t \tag{3.1}$$

that are calibrated to the full SPX smile, i.e., from Gyöngy (1986) and Dupire (1994), that satisfy for all $t \geq 0$

$$\mathbb{E}[\sigma_t^2 | S_t] = \sigma_{\text{loc}}^2(t, S_t). \tag{3.2}$$

- W denotes a standard $(\mathcal{F}_t)$-Brownian motion, (σ_t) an $(\mathcal{F}_t)$-adapted process, and σ_{loc} the local volatility function.

- In such diffusive models, the (idealized) VIX index satisfies, using the notation $\tau = \frac{30}{365} = 30$ days,

$$\mathrm{VIX}_T^2 = \mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau} \sigma_t^2\,dt \middle| \mathcal{F}_T\right] \tag{3.3}$$

- The prices at time 0 of the VIX future and the VIX call options with common maturity T are respectively given by

$$\mathrm{VIX}_0^{\mathrm{model}}(T) = \mathbb{E}\left[\sqrt{\mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau} \sigma_t^2\,dt \middle| \mathcal{F}_T\right]}\right], \tag{3.4}$$

$$C_{\mathrm{VIX}}^{\mathrm{model}}(T,K) = \mathbb{E}\left[\left(\sqrt{\mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau} \sigma_t^2\,dt \middle| \mathcal{F}_T\right]} - K\right)_+\right]. \tag{3.5}$$

- We observe market prices for those instruments, for discrete VIX future maturities T_i, denoted by $\mathrm{VIX}_0^{\mathrm{mkt}}(T_i)$ and $C_{\mathrm{VIX}}^{\mathrm{mkt}}(T_i,K)$, with the most liquid maturities lying below 6 months.
- **Can we find a model satisfying (3.1)-(3.2) and such that for all T_i and K, $\mathrm{VIX}_0^{\mathrm{model}}(T_i) = \mathrm{VIX}_0^{\mathrm{mkt}}(T_i)$ and $C_{\mathrm{VIX}}^{\mathrm{model}}(T_i,K) = C_{\mathrm{VIX}}^{\mathrm{mkt}}(T_i,K)$?**

The case of instantaneous VIX

- $\tau \to 0$: The realized variance over 30 days is then simply replaced by the instantaneous variance, and (3.4)-(3.5) boil down to

$$\mathrm{instVIX}_0^{\mathrm{model}}(T) = \mathbb{E}\left[\sigma_T\right], \quad (4.1)$$

$$C_{\mathrm{instVIX}}^{\mathrm{model}}(T,K) = \mathbb{E}\left[(\sigma_T - K)_+\right]. \quad (4.2)$$

- Reminder: (The distributions of) two random variables X and Y are said to be in convex order if and only if, for any convex function f, $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$. Denoted by $X \leq_c Y$. Both distributions have same mean, but distribution of Y is more "spread" than that of X.
- By conditional Jensen, $\mathbb{E}[\sigma_t^2|S_t] = \sigma_{\mathrm{loc}}^2(t,S_t) \Longrightarrow$ for each t, $X_t := \sigma_{\mathrm{loc}}^2(t,S_t)$ and $Y_t := \sigma_t^2$ are in convex order: $X_t \leq_c Y_t$.

$$X_t := \sigma_{\mathrm{loc}}^2(t, S_t) \qquad \text{and} \qquad Y_t := \sigma_t^2$$

- Conversely, if $X_t \leq_c Y_t$, then there exists a joint distribution π_t of (S_t, σ_t) such that $\mathbb{E}[\sigma_t^2 | S_t] = \sigma_{\mathrm{loc}}^2(t, S_t)$ for all t.
- Indeed, from Strassen's theorem (1965), there exists a joint distribution π'_t of (X_t, Y_t) such that $\mathbb{E}[Y_t | X_t] = X_t$. One then defines π_t as follows: S_t follows the risk-neutral distribution of the SPX for maturity t and, given S_t, $X_t = \sigma_{\mathrm{loc}}^2(t, S_t)$ is known and σ_t^2 is chosen to follow the conditional distribution of Y_t given X_t under π'_t.

- If $\mathrm{instVIX}_0^{\mathrm{mkt}}(t)$ and $C_{\mathrm{instVIX}}^{\mathrm{mkt}}(t, K)$ were accessible, we could imply from the market the distribution of σ_t^2, and compare it to the risk-neutral distribution of $\sigma_{\mathrm{loc}}^2(t, S_t)$.
- A **necessary and sufficient condition** for a jointly calibrating diffusive model on the SPX to exist would then simply be that for each t those two market-implied distributions be in the right convex order:

$$\sigma_{\mathrm{loc}}^2(t, S_t) \leq_c \sigma_t^2$$

- Any process defined by (3.1) where for each t, given S_t, the distribution of σ_t is specified by π_t, is a solution.
- This general construction does not address the issue of the dynamics of (σ_t): σ_t and $\sigma_{t'}$ could be very loosely related for arbitrarily close t and t'.

- In practice, to build a calibrating process, one would discretize time and recursively solve **martingale transport problems**:

$$\mathcal{L}\left(\sigma^2_{\text{loc}}(t_k, S_{t_k})\right) \text{ and } \mathcal{L}\left(\sigma^2_{t_k}\right) \text{given}, \quad \mathbb{E}[\sigma^2_{t_k} | \sigma^2_{\text{loc}}(t_k, S_{t_k})] = \sigma^2_{\text{loc}}(t_k, S_{t_k}). \quad (4.3)$$

- Solutions π'_{t_k} to those martingale transport problems include left- and right-curtains (Beiglböck-Juillet, Henry-Labordère), forward-starting solutions to the Skorokhod embedding problems (Dupire), and the local variance gamma model of Carr.
- (4.3) is a **new type of application of martingale transport to finance**:
  - Usually, the martingality constraint applies to the underlying at two different dates (Henry-Labordère, Beiglböck, Penkner, Nutz, Touzi, Martini, De Marco, Dolinsky, Soner, Obłój, Stebegg, JG...)
  - Here it applies to **two types of instantaneous variances at a single date**, ensuring that the SPX smile is matched.
- It can already be seen in this limiting case that **it might be impossible to build a diffusive model that jointly calibrates to SPX and VIX options**. This happens if (and only if) for some t the market-implied distribution of $\sigma^2_{\text{loc}}(t, S_t)$ is “more spread” than that of the instantaneous VIX squared.

The real case

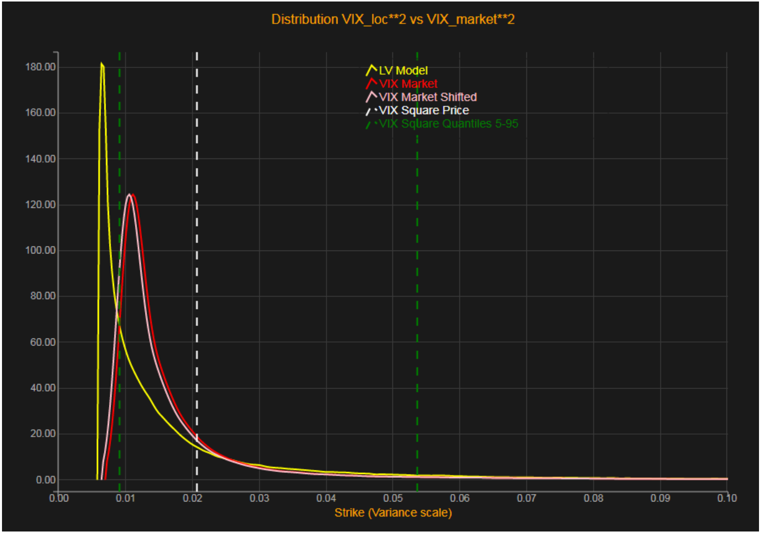
- In reality, squared VIX are not instantaneous variances but the **fair strikes** of **30-day** realized variances.
- Let us look at recent data (Sep 21, 2017). We compare the market distributions of

$$\mathrm{VIX}^2_{\mathrm{loc},T} := \mathbb{E}^{\mathrm{loc}}\left[\frac{1}{\tau}\int_T^{T+\tau} \sigma^2_{\mathrm{loc}}(t,S_t)\,dt \,\middle|\, S_T\right]$$

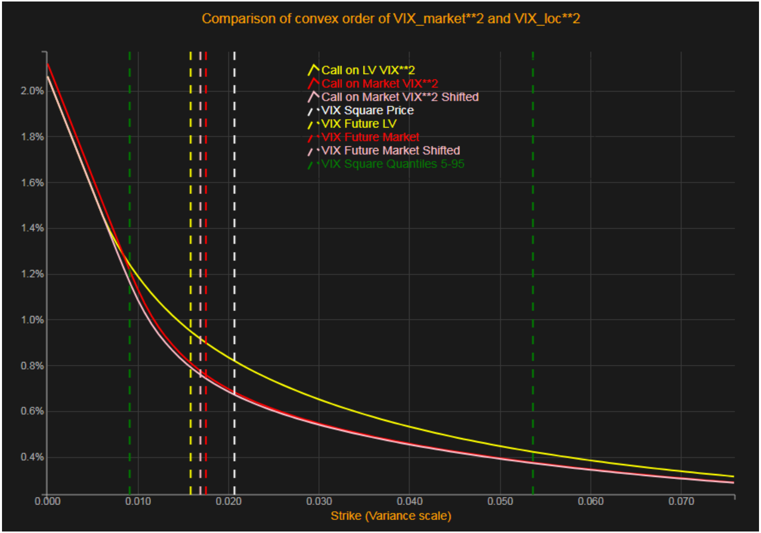
and

$$\mathrm{VIX}^2_{\mathrm{mkt},T} \quad \left(\longleftrightarrow \mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau} \sigma^2_t\,dt \,\middle|\, \mathcal{F}_T\right]\right)$$

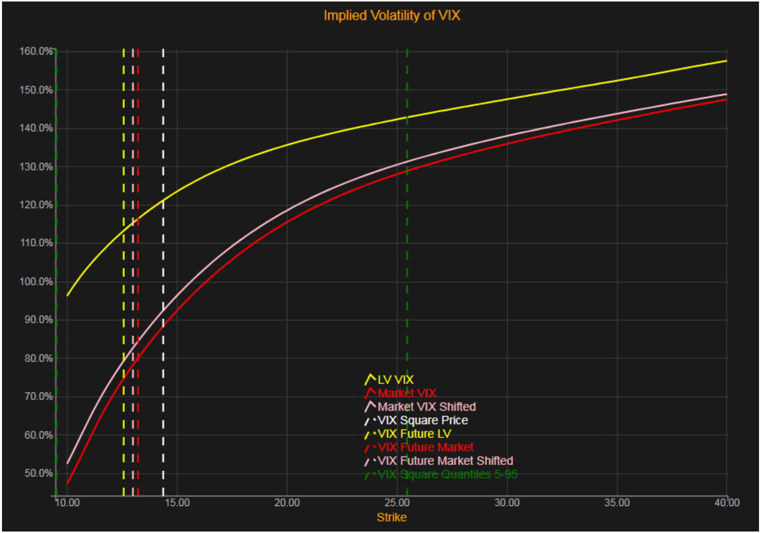
$T = 2$ months


Distribution VIX_loc**2 vs VIX_market**2
LV Model
VIX Market
VIX Market Shifted
VIX Square Price
VIX Square Quantiles 5-95
180.00
160.00
140.00
120.00
100.00
80.00
60.00
40.00
20.00
0.00
0.00
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.10
Strike (Variance scale)

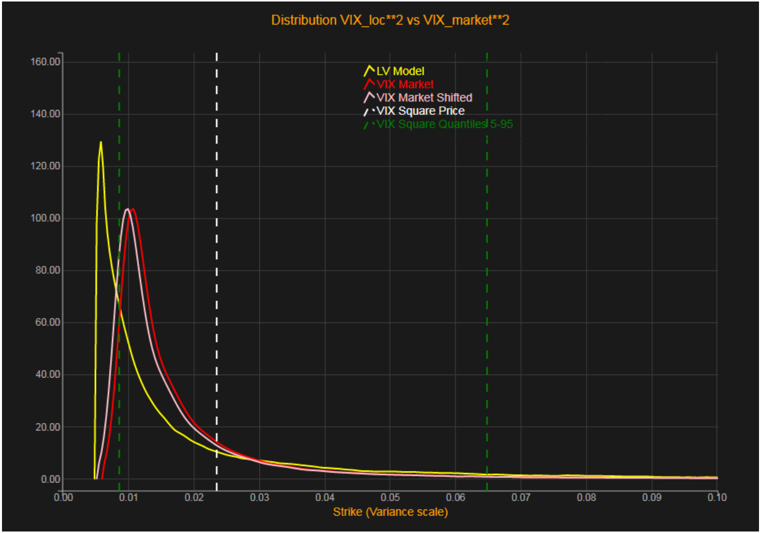
$T = 2$ months


Comparison of convex order of VIX_market**2 and VIX_loc**2
Call on LV VIX**2
Call on Market VIX**2
Call on Market VIX**2 Shifted
VIX Square Price
VIX Future LV
VIX Future Market
VIX Future Market Shifted
VIX Square Quantiles 5-95
2.0%
1.8%
1.6%
1.4%
1.2%
1.0%
0.8%
0.6%
0.4%
0.000
0.010
0.020
0.030
0.040
0.050
0.060
0.070
Strike (Variance scale)

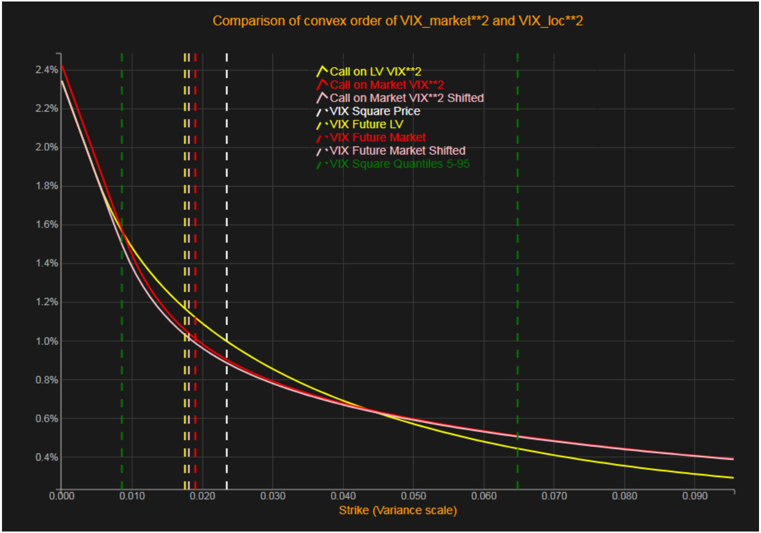
$T = 2$ months


Implied Volatility of VIX
160.0%
150.0%
140.0%
130.0%
120.0%
110.0%
100.0%
90.0%
80.0%
70.0%
60.0%
50.0%
10.00
15.00
20.00
25.00
30.00
35.00
40.00
Strike
LV VIX
Market VIX
Market VIX Shifted
VIX Square Price
VIX Future LV
VIX Future Market
VIX Future Market Shifted
VIX Square Quantiles 5-95

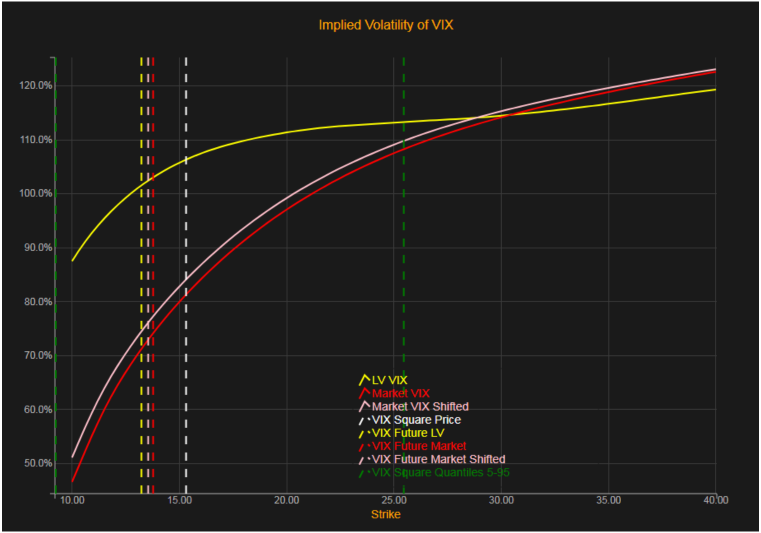
$T = 3$ months


Distribution VIX_loc**2 vs VIX_market**2
LV Model
VIX Market
VIX Market Shifted
VIX Square Price
VIX Square Quantiles 5-95
160.00
140.00
120.00
100.00
80.00
60.00
40.00
20.00
0.00
0.00
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.10
Strike (Variance scale)

$T = 3$ months

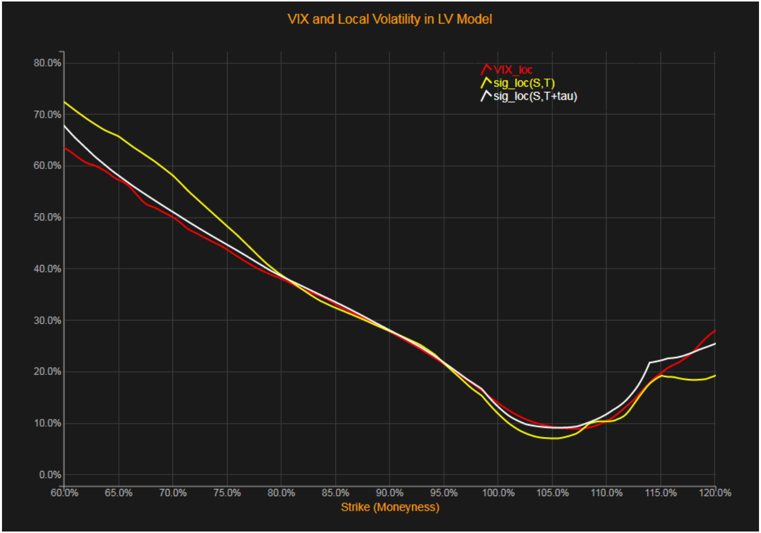

Comparison of convex order of VIX_market**2 and VIX_loc**2
Call on LV VIX**2
Call on Market VIX**2
Call on Market VIX**2 Shifted
VIX Square Price
VIX Future LV
VIX Future Market
VIX Future Market Shifted
VIX Square Quantiles 5-95
2.4%
2.2%
2.0%
1.8%
1.6%
1.4%
1.2%
1.0%
0.8%
0.6%
0.4%
0.000
0.010
0.020
0.030
0.040
0.050
0.060
0.070
0.080
0.090
Strike (Variance scale)

$T = 3$ months



Implied Volatility of VIX
120.0%
110.0%
100.0%
90.0%
80.0%
70.0%
60.0%
50.0%
10.00
15.00
20.00
25.00
30.00
35.00
40.00
Strike
LV VIX
Market VIX
Market VIX Shifted
VIX Square Price
VIX Future LV
VIX Future Market
VIX Future Market Shifted
VIX Square Quantiles 5-95

$T = 3$ months


VIX and Local Volatility in LV Model
VIX_loc
sig_loc(S,T)
sig_loc(S,T+tau)
80.0%
70.0%
60.0%
50.0%
40.0%
30.0%
20.0%
10.0%
0.0%
60.0%
65.0%
70.0%
75.0%
80.0%
85.0%
90.0%
95.0%
100.0%
105.0%
110.0%
115.0%
120.0%
Strike (Moneyness)

$$\mathrm{VIX}^2_{\mathrm{loc},T} := \mathbb{E}^{\mathrm{loc}}\left[\frac{1}{\tau}\int_T^{T+\tau}\sigma^2_{\mathrm{loc}}(t,S_t)\,dt\middle|S_T\right]$$
$$\mathrm{VIX}^2_T = \mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau}\sigma^2_t\,dt\middle|\mathcal{F}_T\right]$$

- In typical market conditions, for short maturities (up to 3-4 months),

$$\mathrm{VIX}^2_{\mathrm{loc},T} \not\leq_c \mathrm{VIX}^2_{\mathrm{mkt},T}$$

  The local volatility model yields a VIX distribution that is "more spread" than the VIX distribution implied from VIX futures and options. Consistent with the fact that so far all the diffusive models calibrated to SPX smile produced market short term VIX implied volatilities that are too large.
- However:

$$\sigma^2_{\mathrm{loc}}(t,S_t) \leq_c \sigma^2_t \quad \not\Longrightarrow \quad \mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau}\sigma^2_{\mathrm{loc}}(t,S_t)\,dt\middle|\mathcal{F}_T\right] \leq_c \mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau}\sigma^2_t\,dt\middle|\mathcal{F}_T\right]$$

  - Serial correlation and $\mathcal{F}_T$ conditioning may undo convex ordering.
  - The fact that $\mathrm{VIX}^2_{\mathrm{loc},T} \not\leq_c \mathrm{VIX}^2_{\mathrm{mkt},T}$ for short maturities (up to 4-5 months usually) does not allow us to conclude that there exists no diffusive model on the SPX that fits both SPX and VIX markets.

Example

- A trivial almost counterexample:

$$Y_0 = X_0 + Z, \qquad Y_1 = X_1 - Z$$

  with $\mathbb{E}[Z|X_0] = \mathbb{E}[Z|X_1] = 0$ (e.g., Z has zero mean and is independent from (X_0, X_1)).
- Y_0 can be much larger than X_0 in the convex order and Y_1 can be much larger than X_1 in the convex order, if Z has large variance.
- However, $Y_0 + Y_1 = X_0 + X_1$.

Example

- $X_0 = W_{t_1}$, $X_1 = -W_{t_2}$, $Y_0 = W_{t_3}$, and $Y_1 = -W_{t_3}$, with $0 < t_1 < t_2 < t_3$.
- $\mathbb{E}[Y_0|X_0] = X_0$, $\mathbb{E}[Y_1|X_1] = X_1$, hence $X_0 \leq_c Y_0$ and $X_1 \leq_c Y_1$, yet $0 = Y_0 + Y_1 <_c X_0 + X_1$.
- In this example, Y_0 and Y_1 (resp. X_0 and X_1) are negatively correlated, and convex order is not preserved under the sum.

Example

- We generalize the previous example: $G = (X_0, Y_0, X_1, Y_1)$ Gaussian vector.
- We assume that $\mathbb{E}[Y_0|X_0] = X_0$ and $\mathbb{E}[Y_1|X_1] = X_1$, and look for necessary and sufficient conditions under which $X_0 + X_1 \le_c Y_0 + Y_1$.[1]
- $m_X := \mathbb{E}[X]$, σ_X std dev of X, ρ_{XY} the correlation between X and Y.
- Since G is Gaussian, $\mathbb{E}[Y_i|X_i] = m_{Y_i} + \rho_{X_iY_i}\frac{\sigma_{Y_i}}{\sigma_{X_i}}(X_i - m_{X_i})$ so

$$m_{X_i} = m_{Y_i} \text{ and } \sigma_{X_i} = \rho_{X_iY_i}\sigma_{Y_i}. \tag{5.1}$$

In particular, $\rho_{X_iY_i} > 0$. As a consequence, $m_{X_0+X_1} = m_{Y_0+Y_1}$, and since $X_0 + X_1$ and $Y_0 + Y_1$ are Gaussian,

$$X_0 + X_1 \le_c Y_0 + Y_1 \iff \mathrm{Var}(X_0 + X_1) \le \mathrm{Var}(Y_0 + Y_1).$$

[1] We ignore trivial cases by assuming that all components of G have positive variance.

Example

- Now, using the second equation in (5.1), we have

$$\begin{aligned}\mathrm{Var}(X_0 + X_1) &= \sigma_{X_0}^2 + \sigma_{X_1}^2 + 2\rho_{X_0X_1}\sigma_{X_0}\sigma_{X_1} \\ &= \rho_{X_0Y_0}^2\sigma_{Y_0}^2 + \rho_{X_1Y_1}^2\sigma_{Y_i}^2 + 2\rho_{X_0X_1}\rho_{X_0Y_0}\sigma_{Y_0}\rho_{X_1Y_1}\sigma_{Y_1}\end{aligned}$$

  so $X_0 + X_1 \leq_c Y_0 + Y_1$ if and only if

$$\sigma_{Y_0}^2(1-\rho_{X_0Y_0}^2) + \sigma_{Y_1}^2(1-\rho_{X_1Y_1}^2) + 2\sigma_{Y_0}\sigma_{Y_1}(\rho_{Y_0Y_1} - \rho_{X_0X_1}\rho_{X_0Y_0}\rho_{X_1Y_1}) \geq 0.$$

  In particular, if $\sigma_{Y_0} = \sigma_{Y_1}$, $\rho_{X_iY_i} \neq 1$ for $i \in \{0,1\}$, and

$$\chi := \frac{\rho_{Y_0Y_1} - \rho_{X_0X_1}\rho_{X_0Y_0}\rho_{X_1Y_1}}{\sqrt{1-\rho_{X_0Y_0}^2}\sqrt{1-\rho_{X_1Y_1}^2}} < -1$$

  then $X_0 + X_1 \not\leq_c Y_0 + Y_1$.

- The conditioning with respect to $\mathcal{F}_T$ may undo convex ordering too.
- Simple counterexample: if $X \leq_c Y$ with X $\mathcal{F}$-measurable and not constant, and Y independent of $\mathcal{F}$, then $\mathbb{E}[Y] = \mathbb{E}[Y|\mathcal{F}] <_c \mathbb{E}[X|\mathcal{F}] = X$.
- In our case, $\frac{1}{\tau}\int_T^{T+\tau} \sigma_t^2\,dt$ is not independent of $\mathcal{F}_T$ since it depends on S_T, given that $\mathbb{E}[\sigma_t^2|S_t] = \sigma_{\mathrm{loc}}^2(t,S_t)$ and that S_T and S_t are positively correlated.

Remarks

- For all reasonable data, $\sigma^2_{\mathrm{loc}}(t, S_t)$ and $\sigma^2_{\mathrm{loc}}(t', S_{t'})$ at two different dates $t, t' \in [T, T+\tau)$ will likely be positively correlated.
- If we use the approximation $\mathbb{E}[\sigma_t^2|S_t] \approx \mathbb{E}[\sigma_t^2|S_{T_i}]$ for $t \in [T_i, T_i+\tau)$, we get

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{\tau}\int_{T_i}^{T_i+\tau} \sigma_t^2\,dt \middle| S_{T_i}\right] &= \frac{1}{\tau}\int_{T_i}^{T_i+\tau} \mathbb{E}\left[\sigma_t^2 \middle| S_{T_i}\right]\,dt \\
&\approx \frac{1}{\tau}\int_{T_i}^{T_i+\tau} \mathbb{E}\left[\sigma_t^2 \middle| S_t\right]\,dt \\
&= \frac{1}{\tau}\int_{T_i}^{T_i+\tau} \sigma^2_{\mathrm{loc}}(t, S_t)\,dt
\end{aligned}$$

which implies that

$$\frac{1}{\tau}\int_{T_i}^{T_i+\tau} \sigma^2_{\mathrm{loc}}(t, S_t)\,dt \lesssim_c \frac{1}{\tau}\int_{T_i}^{T_i+\tau} \sigma_t^2\,dt.$$

Open questions

- Instantaneous VIX: In the case where $\sigma^2_{\text{loc}}(t, S_t) \leq_c \sigma^2_t$, how can we recursively build martingale transports that capture the observed joint dynamics of SPX returns and VIX, e.g., VIX moving continuously and VIX increasing when SPX returns are negative? **Conditional martingale (optimal) transport**
- Real case (30-day VIX): Under what conditions does the timewise convex ordering of $\sigma^2_{\text{loc}}(t, S_t)$ and σ^2_t imply that the distributions of $\mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau} \sigma^2_{\text{loc}}(t, S_t)\, dt \middle| \mathcal{F}_T\right]$ and $\mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau} \sigma^2_t\, dt \middle| \mathcal{F}_T\right]$ are in convex order? Can we build a diffusive model that calibrates jointly to the SPX and VIX option prices? If not, can we build an arbitrage strategy ("if no-jump" arbitrage)? **Stability of convex order under sum and projection**

Why jumps can help

-
- For a diffusive model to calibrate jointly to SPX and VIX options, the distribution of $\mathbb{E}\left[\frac{1}{\tau}\int_T^{T+\tau}\sigma_t^2\,dt\middle|\mathcal{F}_T\right]$ should be as narrow as possible, but without killing the SPX skew. Ergodic/stationary (σ_t) are not solutions, as they produce flat SPX skew.
- Jump-Lévy processes are precisely examples of processes that can generate deterministic realized variance together with a smile on the underlying.
- This explains why jumps have proved useful in this problem.

A few selected references

- Beiglböck, M., Juillet, N.: *On a problem of optimal transport under marginal martingale constraints*, Ann. Probab. 44(1):42–106, 2016.
- Baldeaux, J, Badran, A.: *Consistent Modelling of VIX and Equity Derivatives Using a 3/2 plus Jumps Model*, Appl. Math. Finance 21(4):299–312, 2014.
- Carr, P., Madan, D.: *Joint modeling of VIX and SPX options at a single and common maturity with risk management applications*, IIE Transactions 46(11):1125–1131, 2014.
- Carr, P., Nadtochiy, S.: *Local variance gamma and explicit calibration to option prices*, Math. Finance 27(1):151–193, 2017.
- The CBOE volatility index-VIX, www.cboe.com/micro/vix/vixwhite.pdf (accessed on August 18, 2017).
- Cont, R., Kokholm, T.: *A consistent pricing model for index options and volatility derivatives*, Math. Finance 23(2):248–274, 2013.
- De Marco, S., Henry-Labordere, P.: *Linking vanillas and VIX options: A constrained martingale optimal transport problem*, SSRN, 2015.
- Dupire, B.: *Pricing with a smile*, Risk, January, 1994.

A few selected references

- Fouque, J.-P., Saporito, Y.: *Heston Stochastic Vol-of-Vol Model for Joint Calibration of VIX and S&P 500 Options*, arXiv preprint, 2017. Available at arxiv.org/abs/1706.00873.
- Gatheral, J.: *Consistent modeling of SPX and VIX options*, presentation at Bachelier Congress, 2008.
- Gatheral, J.: *Joint modeling of SPX and VIX*, presentation at Peking University, 2013.
- Gyöngy, I.: *Mimicking the One-Dimensional Marginal Distributions of Processes Having an It Differential*, Probability Theory and Related Fields, 71, 501-516, 1986.
- Kokholm, T., Stisen, M.: *Joint pricing of VIX and SPX options with stochastic volatility and jump models*, The Journal of Risk Finance 16(1):27–48, 2015.
- Michon, A.: *Skorokhod embedding problem and subreplication of variance options*, Research internship at Bloomberg LP, 2016.
- Pacati, C., Pompa, P., Renò, R.: *Smiling Twice: The Heston++ Model*, SSRN preprint, 2015. Available at ssrn.com/abstract=2697179.

A few selected references

- Papanicolaou, A., Sircar, R.: *A regime-switching Heston model for VIX and S&P 500 implied volatilities*, Quantitative Finance 14(10):1811–1827, 2014.
- Henry-Labordère, P.: *Model-free Hedging: A Martingale Optimal Transport Viewpoint*, Chapman & Hall/CRC Financial Mathematics Series, 2017.
- Sepp, A.: *Achieving consistent modeling of VIX and equity derivatives*, Imperial College Mathematical Finance Seminar, November 2, 2011.
- Strassen, V.: *The existence of probability measures with given marginals*, Ann. Math. Statist., 36:423–439, 1965.

Happy birthday Jim!