Valuing Finite-Lived Options as Perpetual

by

Peter Carr and Dmitri Faguet

Cornell University

Johnson Graduate School of Management

Ithaca NY 14853

(607) 255-6255

E-mail:PETER@JOHNSON.CORNELL.EDU

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Abstract

We show how the value of a finite-lived option can be interpreted as the limit of a sequence of perpetual option values subject to default risk. This interpretation yields new closed form approximations for European and American option values in the Black Scholes model. Numerical results indicate that the approximation is both accurate and computationally efficient.

I Introduction

Closed-form solutions for the value of European-style options have been widely known since the seminal papers of Black-Scholes[3] and Merton[28]. Unfortunately, the vast majority of listed options are American-style. Despite a profusion of research on the subject, a completely explicit analytic solution for the value of a finite-lived American option continues to elude finance theorists.

Nonetheless, much progess has been made in developing approaches which *approximate* the value of such options. These approaches bifurcate into either numerical methods or analytic approximations. Numerical methods for valuing American options comprise lattices[34],[11],[36],[21], finite differences[6][43], and Monte Carlo simulation[41],[16]. Analytic approximations include those based on compound options[14],[8],[17], the quadratic formula[24],[2], exponential exercise boundaries[33],[10], integral formulations[22],[9],[44], or regressions between lower and upper bounds[20],[7]. The purpose of this paper is to develop a new analytic approximation based on an analogy between American options and annuities.

A standard approach for valuing a (finite-lived) annuity is to consider it as the difference between a perpetuity and a deferred perpetuity. Since closed-form solutions for *perpetual* American option values have been known since Samuelson/McKean[39], it is tempting to use these solutions to attempt to value (finite-lived) American options. Unfortunately, the value of an American option is not given by the difference between a perpetual American option and its deferred counterpart.

This paper presents an alternative approach for valuing finite-lived securities such as annuities, which also relies on knowing perpetual values. This alternative approach has the advantage of being applicable to American options. In common with all other approaches, it also has the disadvantage of merely *approximating* the desired continuous-time values. However, the approximation for both annuities and American options is given by a simple closed-form formula, which does not involve integrals of any kind. Furthermore, numerical implementation of the formula for American options indicates convergence to the unknown exact value in a computationally efficient manner.

The inspiration for the alternative approach relies on the observation that the values of both annuities and American options decline over time. In contrast, the values of perpetual annuities and American options experience no time decay. In this paper, we mimmick the time decay experienced by an annuity by supposing that the corresponding perpetuity undergoes a series of downgrades in credit quality. Since the terminal value of a finite-lived annuity is zero, the final downgrade of the corresponding perpetuity becomes bankruptcy with zero recovery. Just as it is easier for rating agencies to deal with a finite number of discrete credit classes (eg. AAA, BBB, etc.), we analogously assume that the finite-lived annuity has a finite number of discrete time periods to maturity (eg. n periods, $n-1$ periods etc.).

When dealing with American options, we similarly show that time decay may be mimmicked by credit downgrades of perpetual option values. The lowest possible credit class corresponds to the terminal value of the option, which is a known function of the underlying stock price[1]. Since we are able to derive an exact valuation formula for perpetual options subject to a finite series of credit downgrades, the formula can be used to approximate the value of finite-lived American options.

Our approach offers several advantages over existing methods for valuing American options. First, when compared with numerical methods, the analytic nature of our approach allows us to simultaneously speed up the computation and to eliminate the error due to truncation of the spatial domain (eg. in finite differences) or to price discretization (eg. in lattices). The analytic nature of our solution also offers insights into the signs and magnitudes of comparative statics, and implies that derivatives may be calculated without introducing additional discretization error. Second, when compared with other analytic approaches, our option valuation formulas are easier to implement because they do not use distribution functions or any other special functions (other than the natural log). This key result is due to the fact that the implied risk-neutral density for the underlying stock price may be integrated in closed form. Third, in contrast to most other approaches, our approach yields an explicit analytic approximation of the critical stock price, so long as dividends are modelled as continuous and constant[2]. Consequently, we are able to derive a new analytic approximation for the behavior of the exercise boundary near expiration.

The structure of this paper is as follows. The next section reviews standard results on the pricing of American puts in the Black-Scholes model. The following section presents our technique for approximating the time decay of a finite-lived security with credit downgrades of a perpetual one. The following section discusses the implementation

[1]We couch the presentation in terms of options on the spot price of a stock. Since we allow for a constant proportional dividend, readers should have no difficulty applying the analysis to options on the spot or futures price of a stock index, currency, or commodity.

[2]In contrast, when dividends are modelled as continuous and proportional to the underlying stock price, the critical stock price is implicitly given by an algebraic equation, which is easily solved numerically.

of our American put formula and compares this implementation with extant approaches in terms of both speed and accuracy. The penultimate section extends the analysis to dividends and American calls. The final section summarizes our results and suggests directions for future research.

II American Put Valuation in the Black-Scholes Model

In this section, we focus on the valuation of American puts in the Black-Scholes model. We defer the corresponding development for American calls until dividends have been introduced. The Black-Scholes model assumes that over the option's life $[0, T]$, the economy is described by frictionless markets, no arbitrage, a constant riskless rate $r > 0$, no dividends from the underlying stock, and that the underlying spot price process $\{S_t, t \in (0, T)\}$ is a geometric Brownian motion with a constant volatility rate $\sigma > 0$. Let $P(t, S; T)$ denote the value of an American put as a function of the current time t, the current stock price S, and the maturity date T. Figure 1 graphs the value of an American put against the current stock price, holding t and T fixed. The *critical stock price* $\underline{S}(t), t \in [0, T]$ is defined as the largest stock price S at which the American put value $P(t, S; T)$ equals its *exercise value* $K - S$, where K is the strike price.

As time evolves, the alive American put value falls, which is a phenomenon widely known as time decay. In contrast, the exercise value $K - S$ does not erode with time. The passage of time therefore raises the critical stock price at which exercise occurs. When graphed against time, the critical stock price is a smoothly increasing function termed the *exercise boundary* (see Figure 2). The *continuation region* $\{(t, S); S > \underline{S}(t), t \in [0, T]\}$ lies above this boundary, while its complement, the *exercise region* $\{(t, S); S \leq \underline{S}(t), t \in [0, T]\}$ lies below. In the exercise region, the put value $P(t, S; T)$ is given by its exercise value:

$$P(t, S; T) = K - S, \quad S \in (0, \underline{S}(t)), t \in (0, T). \tag{1}$$

Thus, the problem is to determine the put value in the continuation region at the initial time $t = 0$.

For quite general stochastic processes, the American put's value at this time is given by the solution to an optimal stopping problem:

$$P(0, S; T) = \sup_{\tau \in [0,T]} E_{0,S}\{e^{-r\tau}[K - S_\tau]^+\}, \tag{2}$$

where τ is a stopping time and the conditional expectation is calculated under the risk-neutral probability measure. For initial stock prices above the optimal exercise boundary, the continuity of the stock price process in the Black-

Scholes model implies that the optimal stopping time is a first passage time to this boundary. Consequently, the alive American put may alternatively be valued as:

$$P(0,S;T) = \sup_{B(t);t\in[0,T]} E_{0,S}\{e^{-r(\tau_B \wedge T)}[K - S_{\tau_B}]^+\}, \tag{3}$$

where τ_B is the first passage time[3] from S to an exercise boundary $B(t), t \in [0,T]$. Since the optimal exercise boundary $\{\underline{S}(t), t \in [0,T]\}$ is an unknown function of time (see Figure 2), neither formulation yields a closed-form solution for the alive American put value.

McKean[25] showed that an application of Itô's lemma to (2) implies that the alive American put value and exercise boundary jointly solve a *free boundary problem*, consisting of the Black-Scholes partial differential equation (p.d.e.):

$$\frac{\sigma^2}{2}S^2 P_{ss}(t,S;T) - r[P(t,S;T) - SP_s(t,S;T)] + P_t(t,S;T) = 0, \quad S \in (\underline{S}(t;T),\infty), t \in (0,T), \tag{4}$$

and the following boundary conditions:

$$P(T,S;T) = (K-S)^+, \quad S \in (\underline{S}(T;T),\infty), \quad \text{and} \quad \underline{S}(T;T) = K.$$

$$\lim_{S\uparrow\infty} P(t,S;T) = 0, \qquad \lim_{S\downarrow\underline{S}(t;T)} P(t,S;T) = K - \underline{S}(t;T), \qquad \lim_{S\downarrow\underline{S}(t;T)} P_s(t,S;T) = -1, \qquad t \in (0,T).$$

The hedging arguments given in the seminal papers by Black-Scholes[3] and Merton[28] permit an important economic interpretation of each term in the differential equation. At any point in the continuation region, we may consider a portfolio consisting of one alive American put with value $P(t,S;T)$, long $|P_s(t,S;T)|$ shares with value $|P_s(t,S;T)|S$, and $P(t,S;T) + |P_s(t,S;T)|S$ dollars borrowed. Absence of arbitrage implies that this delta-neutral zero-investment portfolio must maintain zero value through time. This condition in turn implies that the gamma profit measured by the first term in (4) must be offset by the cost of carrying the position and the time decay, given by the second and third terms in (4) respectively.

While the p.d.e formulation lends insight into the local dynamics of the American put value, this formulation does not lend itself to an exact closed-form solution. However, as the maturity date T approaches infinity, time drops out of both the optimal stopping problem and the free boundary problem, leaving much simpler problems

[3] As usual, the first passage time is considered to be infinite if the boundary is never touched.

to solve. The optimal stopping problem for the perpetual put value $P(S;\infty)$ involves the first passage time τ_B to a constant boundary B:

$$\sup_B P(0,S;\infty) = E_{0,S}\{e^{-r\tau_B}(K-B)\}, \tag{5}$$

while the free boundary problem simplifies to the following *ordinary* differential equation (o.d.e.):

$$\frac{\sigma^2}{2}S^2 P_{ss}(S;\infty) + rSP_s(S;\infty) - rP(S;\infty) = 0 \qquad S \in (\underline{S},\infty), \tag{6}$$

and the boundary conditions $\lim_{S\uparrow\infty} P(S;\infty) = 0$, $\lim_{S\downarrow\underline{S}} P(S;\infty) = K-\underline{S}$, $\lim_{S\downarrow\underline{S}} P_s(S;\infty) = -1$. From Samuelson[39], both problems yield the following exact solution for the alive perpetual put value $P(S;\infty)$ and critical stock price $\underline{S}$:

$$P(S;\infty) = (K-\underline{S})\left(\frac{S}{\underline{S}}\right)^{-\frac{2r}{\sigma^2}}, S > \underline{S}, \text{ where } \underline{S} = \frac{r}{r+\frac{\sigma^2}{2}}K. \tag{7}$$

Just as time drops out when considering perpetual claims, the stock price drops out when considering claims whose payoffs are independent of the stock price. For example, the time t value of a T-maturity annuity with fixed continuous payments of ϕ dollars per year is governed by the following o.d.e.:

$$-rA(t;T) + A_t(t;T) + \phi = 0, \qquad t \in [0,T), \text{ subject to } A(T;T) = 0. \tag{8}$$

As was true for American options, a perpetuity is valued by setting the time derivative to zero:

$$-rA(t;\infty) + \phi = 0. \tag{9}$$

The solution $A(t;\infty) = \frac{\phi}{r}$ can be used to generate the well-known solution for the finite-lived annuity:

$$A(t;T) = \frac{\phi}{r}[1-e^{-r(T-t)}]. \tag{10}$$

Note that the perpetuity value is independent of time, whereas the finite-lived annuity experiences time decay.

Equation (10) clearly indicates that the value of a finite-lived annuity is given by the value of a perpetuity $\frac{\phi}{r}$, less the value of a deferred perpetuity $e^{-r(T-t)}\frac{\phi}{r}$. Unfortunately, using the corresponding approximation for the American put will undervalue substantially because this approach implicitly forces the exercise boundary $\underline{S}_t$ to the sub-optimal perpetual boundary $\underline{S}$ given in (7). In the next section, we develop an approximation procedure for valuing finite-lived annuities which also applies to American put options.

III Approximating Time Decay as Credit Downgrades

The analysis in the previous section assumed that all claims are not subject to default risk. However, if we assume as in Merton[29], Madan and Unal[27], or Duffie and Singleton[13] that default is triggered by a Poisson process, then the governing differential equations change. In particular, suppose that the first jump of a standard Poisson process with constant intensity λ forces the risky perpetuity value $A^{(1)}$ to a given recovery value $A^{(0)}$. Then the generalization of (9) to risky perpetuities is:

$$-rA^{(1)} + \lambda[A^{(0)} - A^{(1)}] + \phi = 0, \tag{11}$$

where the dependence on time has been suppressed as a reminder that we are dealing with perpetuities. The new term is given by the product of the instantaneous probability of a jump λ and the change in value which the jump triggers. Since we will assume that the perpetuity has no recovery value ($A^{(0)} = 0$), the solution is simply:

$$A^{(1)} = \frac{\phi}{r + \lambda}, \tag{12}$$

indicating that λ may be interpreted as a credit spread.

More generally, we may assume the existence of $n+1$ credit classes and that each jump of a standard Poisson process triggers a credit downgrade of the perpetuity from class m to class $m-1$, for $m = 1, \ldots, n$. The resulting difference equation governing the value of a risky perpetuity in credit class m is:

$$-rA^{(m)} + \lambda[A^{(m-1)} - A^{(m)}] + \phi = 0, \qquad m = 1, \ldots, n, \text{ subject to } A^{(0)} = 0. \tag{13}$$

Comparing (13) for a risky perpetuity with (8) for a riskless annuity indicates a mechanism by which credit downgrades may be used to mimmick time decay. In particular, suppose that we divide the riskless annuity's life $(0, T)$ into n equal steps, each of length $\triangle \equiv \frac{T}{n}$. Further suppose that we defer the effect of time passing on the annuity value $A(t;T)$ in (8) until the end of each time step. Let $\hat{A}^{(m)}$ denote the resulting approximation for the annuity value $A(T - m\triangle; T)$ with m periods to maturity. Then integrating (8) across time and dividing by $\triangle$ implies that $\hat{A}^{(m)}$ solves a difference equation of the same form as (13):

$$-r\hat{A}^{(m)} + \frac{1}{\triangle}[\hat{A}^{(m)} - \hat{A}^{(m-1)}] + \phi = 0, \qquad m = 1, \ldots n, \text{ subject to } \hat{A}^{(0)} = 0. \tag{14}$$

Re-arranging gives the sensible result:

$$\hat{A}^{(m)} = \frac{\phi\triangle + \hat{A}^{(m-1)}}{1 + r\triangle} = R(\phi\triangle + \hat{A}^{(m-1)}), \qquad m = 1, \ldots, n, \tag{15}$$

where $R \equiv \frac{1}{1+r\triangle}$ is the single period discount factor. Iterating on m implies that the solution to (14) with n steps to maturity is:

$$\hat{A}^{(n)} = \phi\triangle[R + R^2 + \ldots + R^n] = \frac{\phi}{r}R(1 - R^n). \tag{16}$$

Comparing with (10) indicates that this discrete-time approximation converges to the correct continuous-time solution as the number of steps n becomes large. Furthermore, if we equate the intensity λ in (15) to the reciprocal of the time step length $\triangle$ in (14), then the resulting difference equations for $A^{(m)}$ and $\hat{A}^{(m)}$ are formally identical. As a result, (16) also gives the exact value of a perpetuity in the n-th credit class, where credit downgrades occur according to a Poisson process with intensity $\lambda = \frac{1}{\triangle}$.

Proceeding analogously for American options, let $P^{(m)}(S)$ and $\underline{S}_m$ respectively denote our approximations for American put value $P(T - m\triangle, S; T)$ and the critical stock price $\underline{S}(T - m\triangle; T)$ with m steps to maturity, $m = 0, 1, \ldots, n$, with $P^{(0)}(S) \equiv (K - S)^+$ and $\underline{S}_0 \equiv K$. If we suppose that the put's value, delta, gamma, and critical stock price do not change except at the end of each calendar time step, then integrating the p.d.e. (4) across calendar time and dividing by $\triangle$ implies that:

$$\frac{\sigma^2}{2}S^2 P_{ss}^{(m)}(S) - r[P^{(m)}(S) - SP_s^{(m)}(S)] + \frac{1}{\triangle}[P^{(m-1)}(S) - P^{(m)}(S)] = 0, \text{ for } S \in (\underline{S}_m, \infty), \tag{17}$$

subject to the boundary conditions:

$$\lim_{S\uparrow\infty} P^{(m)}(S) = 0, \lim_{S\downarrow\underline{S}_m} P^{(m)}(S) = K - \underline{S}_m, \lim_{S\downarrow\underline{S}_m} P_s^{(m)}(S) = -1, \text{ for } m = 1, \ldots, n, \text{ with } P^{(0)}(S) \equiv (K - S)^+. \tag{18}$$

We may interpret $P^{(m)}(S)$ as the value of a perpetual put in the m-th credit class. The o.d.e. indicates that the put enters the next lowest credit class at the next jump time of a Poisson process with intensity $\lambda = \frac{1}{\triangle}$.

Comparing (17) with (4) indicates that the net effect of our approximation procedure has been to replace the time derivative $P_t(t, S; T) \equiv \frac{\partial P}{\partial t}(t, S; T)$ in (4) with the finite difference $\frac{\triangle P^{(m)}(S)}{\triangle} \equiv \frac{P^{(m-1)}(S) - P^{(m)}(S)}{\triangle}$ in (17). However, the spatial derivatives are not replaced with their finite differences, in contrast to standard finite difference schemes or the binomial model[4]. The notion of discretizing time while leaving space continuous is

[4]The binomial model uses a forward finite difference for the maturity derivative leading to an explicit scheme. Our use of a backward difference indicates that our procedure may be considered as the limiting case of a fully implicit scheme, where the size of each space step is infinitessimally small.

known in the numerical methods literature as the *method of horizontal lines* or *Rothe's method* (see Rothe[38] and Rektorys[35]). Its application to free boundary problems has been promulgated in Meyer[30],[31] and in Meyer & van der Hoek[32], who use it to numerically value American options. Goldenberg and Schmidt[15] test this numerical scheme against other approaches and find that it is highly accurate, although slightly slower[5] than some other approaches.

Our approach is similar to the work of Meyer & van der Hoek and Goldenberg & Schmidt, except that we treat the method of lines as a way to generate *explicit formulas* for the approximate value of an American option and its critical stock price. In contrast to the numerical method of lines, our analytic method of lines is immune to error introduced by bounding and discretizing the spatial domain. The accuracy of our formulas may be anticipated *a priori* by noting that as the maturity date T approaches infinity holding the number of steps n fixed, then our modified problem, (17), approaches that of the perpetual put, (6). As a result, the solution at any time step to our modified problem will converge to the correct perpetual solution, (7). Conversely, as n gets arbitrarily large with T held fixed, then the finite difference $\frac{\triangle P^{(n)}(S)}{\triangle}$ in (17) converges to the time derivative $P_t(0, S; T)$ in (4). As a result, we conjecture[6]that the solution $(P^{(n)}(S), \underline{S}_n)$ to our modified problem converges to the unknown solution $(P(0, S; T), \underline{S}(0; T))$ of the exact problem (2) or (4).

Re-arranging (17) leads to the following sequence[7] of (inhomogeneous second order Euler) o.d.e.'s:

$$\frac{\sigma^2}{2} S^2 P_{ss}^{(m)}(S) + rSP_s^{(m)}(S) - (r + \lambda)P^{(m)}(S) = -\lambda P^{(m-1)}(S), \qquad S \in (\underline{S}_m, \infty), m = 1, \ldots, n, \tag{19}$$

where recall $\lambda = \frac{1}{\triangle} = \frac{n}{T}$ is both the intensity of the default process for the perpetual put and the number of time steps per year for the finite-lived put.

Setting $m = 1$ in (18) and (19), the inhomogeneous term is $-\lambda P^{(m-1)}(S) = -\lambda P^{(0)}(S) = -\lambda(K - S)^+$ and the boundary value problem defining $P^{(1)}(S)$ can be solved analytically. See Figure 3 for a graph of this first step solution against the stock price. Setting $m = 2$ in (18) and (19), $-\lambda P^{(1)}(S)$ becomes the inhomogeneous term, and $P^{(2)}(S)$ may be solved for analytically. See Figure 4 for a graph of this second step solution and Figure 5 for a graph of the solution over the first two steps. Continuing in this fashion, the entire sequence of o.d.e.'s in (19)

[5] However, given the speed of modern computers, they argue that its inherent accuracy makes it the method of choice among those tested.

[6] While numerical implementation of our solution will prove to be consistent with this conjectured convergence, a formal proof of convergence remains an open question.

[7] We define $P^{(m-1)}(S) \equiv K - S$ for $S \in (\underline{S}_m, \underline{S}_{m-1})$.

can be solved explicitly.

As previously mentioned, the final solution $P^{(n)}(S)$ is also the value of a perpetual put in the n-th credit class. Using the Feynman-Kac Theorem (see Duffie[12]), the value of this perpetual claim may be expressed as:

$$P^{(n)}(S) = E_{0,S} \int_0^{\tau_n} e^{-(r+\lambda)u} \lambda P^{(n-1)}(S_u) du + E_{0,S} e^{-(r+\lambda)\tau_n}(K - \underline{S}_n), \tag{20}$$

where τ_n is the first passage time to $\underline{S}_n$. The first term is the value arising if the first credit downgrade precedes exercise, while the second term captures the value of exercising prior to any downgrade. Comparing (20) with Samuelson's solution (5) indicates that this second term is structurally identical[8] to the value of a riskless perpetual put. Thus, the default risk (or maturity of the finite-lived claim) is captured by the first term in (20).

The analysis leading to the problem specifications (19) or (20) for the value of an American put applies to European puts as well. The sequence of boundary value problems which governs the approximate value $p^{(m)}(S)$ of a European put is:

$$\frac{\sigma^2}{2} S^2 p_{ss}^{(m)}(S) + rSp_s^{(m)}(S) - (r+\lambda)p^{(m)}(S) = -\lambda p^{(m-1)}(S), \qquad S > 0, m = 1, \ldots, n, \tag{21}$$

$$\text{subject to } \lim_{S \uparrow \infty} p^{(m)}(S) = 0, \qquad \lim_{S \downarrow 0} p^{(m)}(S) = KR^m, \qquad \text{for } m = 1, \ldots, n, \text{ with } p^{(0)}(S) \equiv (K - S)^+.$$

The corresponding sequence of continuous-time perpetual claim valuation problems is:

$$p^{(m)}(S) = E_{0,S} \int_0^{\infty} e^{-(r+\lambda)u} \lambda p^{(m-1)}(S_u) du, \qquad \text{for } m = 1, \ldots, n, \text{ with } p^{(0)}(S) \equiv (K - S)^+. \tag{22}$$

The corresponding solution to either (21) or (22) for the European put value when $m = n$ is:

$$p_0^{(n)}(S) = \left(\frac{S}{K}\right)^{\gamma - \epsilon} \sum_{k=0}^{n-1} \frac{\left(2\epsilon \ln\left(\frac{S}{K}\right)\right)^k}{k!} \sum_{l=0}^{n-k-1} \binom{n-1+l}{n-1} [KR^n q^n p^{l+k} - K\hat{q}^n \hat{p}^{l+k}], \qquad S > K, \tag{23}$$

where:

$$R \equiv \frac{1}{1 + r\triangle}, \qquad \gamma = \frac{1}{2} - \frac{r}{\sigma^2}, \qquad \epsilon \equiv \sqrt{\gamma^2 + \frac{2}{R\sigma^2 \triangle}}, \tag{24}$$

and where:

$$p \equiv \frac{\epsilon - \gamma}{2\epsilon} \in (0,1), q \equiv 1 - p = \frac{\epsilon + \gamma}{2\epsilon}, \hat{p} \equiv \frac{\epsilon - \gamma + 1}{2\epsilon} \in (0,1), \text{ and } \hat{q} \equiv 1 - \hat{p} = \frac{\epsilon + \gamma - 1}{2\epsilon} \tag{25}$$

[8] However, the discount rate and critical stock price differ.

may be thought of as pseudo-probabilities. Note that the solution is only valid for $S > K$, i.e. for a European put which is currently out-of-the-money. The solution for the initial value of an in-the-money European put is given by put-call-parity:

$$p_1^{(n)}(S) = KR^n - S + c_1^{(n)}(S), \qquad S \leq K, \tag{26}$$

where $c_1^{(n)}(S)$ is the initial value of an out-of-the-money European call, given by:

$$c_1^{(n)}(S) = \left(\frac{S}{K}\right)^{\gamma+\epsilon} \sum_{k=0}^{n-1} \frac{\left(2\epsilon \ln\left(\frac{K}{S}\right)\right)^k}{k!} \sum_{l=0}^{n-k-1} \binom{n-1+l}{n-1} [K\hat{p}^n \hat{q}^{k+l} - KR^n p^n q^{k+l}], \qquad S \leq K. \tag{27}$$

For completeness, we record the initial value of an in-the-money European call as:

$$c_0^{(n)}(S) = S - KR^n + p_0^{(n)}(S), \qquad S > K, \tag{28}$$

where $p_0^{(n)}(S)$ is given by (23). These European option formulas are slightly more complex than the Black-Scholes formulas in that a different formula must be used depending on whether the stock price is above or below the strike price. However, the formulas are also simpler in that no special functions such as normal distribution functions require evaluation.

One can also use either method to solve for the value of an *American* put. However, one must first determine the sequence of critical stock prices $\underline{S}_1, \ldots, \underline{S}_n$, where monotonicity of the free boundary implies $\underline{S}_1 > \underline{S}_2 > \ldots > \underline{S}_n$ (see Figure 8). Like the European put formula, the American put formula will depend on whether the stock price is above or below the strike. In addition, for stock prices below the strike, the American put formula for $P^{(n)}(S)$ will depend on which interval $(\underline{S}_i, \underline{S}_{i-1})$ contains the current spot price S.

Assuming that the critical stock prices are known, the solution to either (19) or (20) for the value of an American put with n periods to maturity is:

$$P^{(n)}(S) = \begin{cases} p_0^{(n)}(S) + b_1^{(n)}(S) & \text{if } S > \underline{S}_0 \equiv K \\ KR^{n-i+1} - S + b_i^{(n)}(S) + A_i^{(n)}(S;1) & \text{if } S \in (\underline{S}_i, \underline{S}_{i-1}], i = 1, \ldots, n \\ K - S & \text{if } S \leq \underline{S}_n. \end{cases} \tag{29}$$

where for $i = 1, \ldots, n$:

$$\begin{aligned} b_i^{(n)}(S) &\equiv \sum_{j=1}^{n-i+1} \left(\frac{S}{\underline{S}_{n-j+1}}\right)^{\gamma-\epsilon} \sum_{k=0}^{j-1} \frac{\left(2\epsilon \ln\left(\frac{S}{\underline{S}_{n-j+1}}\right)\right)^k}{k!} \sum_{l=0}^{j-k-1} \binom{j-1+l}{j-1} q^j p^{k+l} R^j K r \triangle \\ A_i^{(n)}(S;h) &\equiv \sum_{j=h}^{n-i+1} \left(\frac{S}{\underline{S}_{n-j+1}}\right)^{\gamma+\epsilon} \sum_{k=0}^{j-1} \frac{\left(2\epsilon \ln\left(\frac{\underline{S}_{n-j+1}}{S}\right)\right)^k}{k!} \sum_{l=0}^{j-k-1} \binom{j-1+l}{j-1} p^j q^{k+l} R^j K r \triangle. \end{aligned}$$

The formula in the first line of (29) for the out-of-the-money value of an American put reflects the well-known decomposition into the value of the corresponding European put and the early exercise premium (see [22], [18], [9], [44], and [19]). The formula in the second line is a new decomposition of the American put value into the value if forced to sell at a given date[9] prior to expiration, and the premia which arise because exercise can occur before or after this date. Appendix 1 provides an economic justification for this decomposition. The final line of (29) indicates that the put should be exercised immediately if the stock price S is at or below the critical stock price $\underline{S}_n$.

A significant advantage of our new decomposition is that it yields an *explicit* formula for the sequence of critical stock prices $\underline{S}_1, \ldots, \underline{S}_n$. Continuity at the strike price for each time step $m = 1, \ldots, n$ implies $c_1^{(m)}(K) = A_1^{(m)}(K;1)$, which in turn implies the following explicit recursive solution for each critical stock price $\underline{S}_m$:

$$\underline{S}_m = K \left(\frac{pRKr\triangle}{c_1^{(m)}(K) - A_1^{(m)}(K;2)} \right)^{\frac{1}{\gamma+\epsilon}}, \qquad m = 1, \ldots, n, \tag{30}$$

where from (27), the at-the-money European call value with m periods to maturity simplifies to:

$$c_1^{(m)}(K) = \sum_{l=0}^{m-1} \binom{m-1+l}{m-1} [K\hat{p}^m \hat{q}^l - KR^m p^m q^l], \qquad m = 1, \ldots, n. \tag{31}$$

To initiate the sequence, we simply set $m = 1$:

$$\underline{S}_1 = K \left(\frac{pRr\triangle}{\hat{p} - Rp} \right)^{\frac{1}{\gamma+\epsilon}}. \tag{32}$$

Letting the time step size shrink yields the following simple approximation for the expiration behavior of the exercise boundary[10]:

$$\lim_{\triangle \downarrow 0} \underline{S}_1 \approx K \left(\frac{r\triangle}{\varrho} \right)^{\varrho}, \text{ where } \varrho \equiv \sqrt{\frac{\sigma^2 \triangle}{2}}.$$

IV Implementation

Our solution (29) for the American put value $P^{(n)}(S;T)$ is a triple sum. Clearly, we need the number of steps n to be small in order to achieve computational efficiency. This section describes an ingenious technique called Richardson extrapolation, which can be used to provide accurate answers in at most 3 time steps. Richardson extrapolation

[9] The given date used is the unique time t solving $\underline{\hat{S}}(t;T) = S$, where $\underline{\hat{S}}(t;T)$ is our step function approximation (see Figure 8) to the exercise boundary.

[10] See Barles et. al.[1], Van Moerbeke[42], and Wilmott et. al.[43] for alternative approximations.

has been used previously to accelerate valuation schemes for American[11] options. Geske and Johnson[14] first used Richardson extrapolation in a financial context to speed up and simplify their compound option valuation model. Breen[5] applied this idea to accelerate the binomial model of Cox, Ross, and Rubinstein[11]. Yu[44] and Subrahmanyam and Yu[40] use the approach to accelerate a modification of the integral representation of McKean[25]. Finally, Broadie and Detemple[7] use it to accelerate a hybrid of the binomial and Black-Scholes models.

Richardson extrapolation works off the same principle as option valuation itself. The key insight of the Black-Scholes analysis is that the positive correlation between a call option and its underlying stock allows the formation of a riskless portfolio involving opposing positions. If trading can only occur discretely, then the portfolio will have risk, but which is of an order of magnitude below that of either component security. In the same vein, the positive correlation between the errors in two successive put value approximations (eg. $P^{(1)}(S;T)$ and $P^{(2)}(S;T)$) allows the formation of a weighted average which has lower error than either approximation. In particular, when the error of each approximation is of order $O(\triangle)$ as assumed here, the two point Richardson extrapolation, $P^{1:2}(S;T)$, is obtained from incrementing the more accurate two step value with the improvement over the less accurate single step value:

$$P^{1:2}(S;T) = P^{(2)}(S;T) + [P^{(2)}(S;T) - P^{(1)}(S;T)] = -P^{(1)}(S;T) + 2P^{(2)}(S;T).$$

It can be shown that the order of the error of the improved approximation $P^{1:2}(S;T)$ is $O(\triangle^2)$.

When trading occurs in discrete time, the variance of a portfolio may be reduced by adding more options. In a similar manner, the error in a Richardson extrapolation may be reduced by adding more approximations. In a three point Richardson extrapolation, the single and double step values are combined with the three step value to obtain the following approximation $P^{1:3}(S;T)$ for the limiting value:

$$P^{1:3}(S;T) \equiv \frac{1}{2}P^{(1)}(S;T) - 4P^{(2)}(S;T) + \frac{9}{2}P^{(3)}(S;T). \tag{33}$$

The resulting error is of $O(\triangle^3)$. Figures 6 and 7 illustrate the idea behind a 3 step extrapolation. From Marchuk

[11] Boyle, Evnine, and Gibbs[4] also use the approach to value multivariate options.

and Shaidurov[23], p. 24, an N point Richardson extrapolation is the following weighted[12] average of N put values:

$$P^{1:N}(S;T) \equiv \sum_{n=1}^{N} \frac{(-1)^{N-n} n^N}{n!(N-n)!} P^{(n)}(S;T). \tag{34}$$

The critical stock price can be obtained by imposing either of the smooth pasting conditions in (18) or[13] by Richardson extrapolation:

$$\underline{S}^{1:N}(T) \equiv \sum_{n=1}^{N} \frac{(-1)^{N-n} n^N}{n!(N-n)!} \underline{S}_n(T).$$

In our implementation of the three point Richardson extrapolation (33), we found it useful to modify the weights slightly. To understand the nature of this modification, it is instructive to examine a typical test case: $S = 100, K = 100, T = 1, r = .1, \delta = 0$, and $\sigma = .3$. The true value based on the binomial method with 2000 time steps appears to be 8.3378. Table 1 shows that for this test case, extrapolated put values obtained from our approach converge rapidly to this true value, with penny accuracy obtained in only 5 points. In contrast, the unextrapolated values converge very slowly from below. The undervaluation observed in this test case was observed in other cases as well.

The source of this undervaluation may be understood from two perspectives. Treating the American put value as the solution to an optimal stopping problem (2), we are freezing the exercise boundary over each time step (see Figure 8), and therefore optimizing the put value over a restricted set. Treating the American put as a solution to a free boundary problem (4), we are approximating the average value of the put's time derivative over each maturity step with its value at the end of each maturity step. Since the put value is an increasing concave function of maturity (see Figure 7), our first order scheme undervalues. Viewed from either perspective, the approximation error is largest for short maturity options. Fortunately, the early exercise premium is also smallest for such options.

We may nevertheless mitigate the undervaluation by adjusting the Richardson weights upwards in a manner that depends on the time to maturity. After some numerical experimentation, we settled on the following "fine-tuning" of the three point Richardson extrapolation:

$$P^{1:3m}(S,T) \equiv \frac{1}{2} P^{(1)}(S;T) - 4[1 - .0002(5-T)^+] P^{(2)}(S;T) + \frac{9}{2} P^{(3)}(S;T). \tag{35}$$

[12] The weights always sum to unity and alternate in sign. In general, higher order approximations involve weights with greater absolute value. As a result, implementing higher order extrapolations on a computer requires double precision to control roundoff error.

[13] We prefer the former method when accuracy is important and the latter method when speed matters.

Thus, no adjustment is made if the time to maturity exceeds 5 years. For shorter maturities, the middle weight is modified so that the three weights sum to *slightly* more than one. The maximum adjustment to the middle weight is .004, which occurs when $T = 0$. When applied to our test case, the modified value is 8.3332, giving penny accuracy in only 3 time steps. The extrapolated put value given by (35) is the central result of this paper.

Broadie and Detemple[7] conduct extensive numerical simulations of a wide array of methods for valuing American options. Their results indicate that our approach dominates most methods in terms of speed and accuracy. Indeed, their results indicate that our above three point extrapolation is on the "efficient frontier", intermediate in terms of speed and accuracy between the quadratic formula and their capped option formulae. We believe that our three point extrapolation given by (35) represents a satisfactory tradeoff between speed and accuracy. Furthermore, our approach displays more flexibility than other efficient approaches, in that speed or accuracy can be emphasized whenever one consideration is paramount, simply by varying the number of Richardson points used. In particular, arbitrary accuracy can be achieved in contrast to other efficient methods.

V Extension to Positive Dividends and American Calls

Merton[28] generalized the Black-Scholes analysis to continuously-paid dividends which are either constant or proportional to the price of the underlying. He did not permit a dividend rate which is linear in the spot price, presumably due to the difficulty in generating analytic solutions under this assumption. While we are also unable to deal with a linear dividend rate, this section values American options explicitly when the dividend payout rate has both a fixed and a proportional component. We also show that our approximation to the put's critical stock price is still given by an explicit formula when dividends are constant, but must be determined numerically when there is a proportional component to the dividend flow. Finally, we develop corresponding results for American call options.

To obtain a truly fixed component ϕ of the dividend flow, we follow Roll[37] in assuming that this component has been escrowed out of the stock price. In other words, the time t stock price S_t decomposes into:

$$S_t = \frac{\phi}{r}[1 - e^{-r(T-t)}] + s_t, \qquad t \in [0, T], \tag{36}$$

where the first term is the present value at t of the constant flow ϕ until T, and the residual s_t is the *stripped price*, reflecting the stripping off of the fixed component of the dividend flow from the stock price. We assume that the risk-neutralized process for the stripped price $\{s_t, t \in [0,T]\}$ is the following geometric Brownian motion:

$$s_t = s \exp\left[\left(r - \delta - \frac{\sigma^2}{2}\right)t + \sigma W_t\right], \qquad t \in [0,T], \tag{37}$$

where $\{W_t, t \in [0,T]\}$ is a standard Brownian motion, and from (36), the initial value is:

$$s = S - \frac{\phi}{r}[1 - e^{-rT}]. \tag{38}$$

Thus, the dollar dividend rate d_t has both a fixed and a proportional component:

$$d_t = \phi + \delta s_t, \qquad t \in [0,T]. \tag{39}$$

The parameter ϕ captures the stickiness of dividends in the short run, while δ captures the tendency for dividends to increase with stock prices in the long run. If $\delta = 0$, then ϕ is the constant dividend rate, while if $\phi = 0$, then δ is the constant dividend *yield*, since $s_t = S_t$ from (36).

V-A Positive Dividends and American Puts

We generalize the previous analysis by letting $P(t,s;T)$ denote the value of an American put as a function of the current time t, the current *stripped* price s, and the maturity date T. We also define the *critical stripped price* $\underline{s}(t)$ as the unique stripped price s at which the alive American put value $P(t,s;T)$ just equals its exercise value $K - s - \frac{\phi}{r}[1 - e^{-r(T-t)}]$, for $t \in [0,T]$. From (36), the critical stock price $\underline{S}(t)$ is now defined by:

$$\underline{S}(t) \equiv \frac{\phi}{r}[1 - e^{-r(T-t)}] + \underline{s}(t), \qquad t \in [0,T]. \tag{40}$$

To obtain discrete-time analogs, note that (16) implies that the discrete-time value of a finite-lived annuity paying ϕ_n each period for n periods is:

$$\hat{A}^{(n)} = \frac{\phi_n}{r} R(1 - R^n),$$

where $R \equiv \frac{1}{1+r\triangle}$ is the single period discount factor. We define the periodic cash flow ϕ_n so that the discrete-time value matches its continuous counterpart:

$$\phi_n \equiv \phi \frac{1 - e^{-rT}}{R(1 - R^n)}.$$

We define $P^{(m)}(s)$ as our approximation[14] for the American put value when m periods remain, $m = 1, \ldots, n$. Our approximation for the critical stripped price, $\underline{s}_m$, is the unique s satisfying $P^{(m)}(s) = K - s - \phi_n R \frac{1-R^m}{1-R}, m = 1, \ldots, n$.

We may again approximate European option values by solving for the mimmicking perpetual values:

$$p^{(n)}(s) = \begin{cases} \left(\frac{s}{K}\right)^{\gamma-\epsilon} \sum_{k=0}^{n-1} \frac{\left(2\epsilon \ln\left(\frac{s}{K}\right)\right)^k}{k!} \sum_{l=0}^{n-k-1} \binom{n-1+l}{n-1} [KR^n q^n p^{k+l} - KD^n \hat{q}^n \hat{p}^{k+l}] & \text{if } s > K \\ KR^n - sD^n + c^{(n)}(s) & \text{if } s \leq K \end{cases} \tag{41}$$

$$c^{(n)}(s) = \begin{cases} sD^n - KR^n + p^{(n)}(s) & \text{if } s > K \\ \left(\frac{s}{K}\right)^{\gamma+\epsilon} \sum_{k=0}^{n-1} \frac{\left(2\epsilon \ln\left(\frac{K}{s}\right)\right)^k}{k!} \sum_{l=0}^{n-k-1} \binom{n-1+l}{n-1} [KD^n \hat{p}^n \hat{q}^{k+l} - KR^n p^n q^{k+l}] & \text{if } s \leq K, \end{cases} \tag{42}$$

where now:

$$\gamma = \frac{1}{2} - \frac{r-\delta}{\sigma^2}, \qquad D \equiv \frac{1}{1+\delta\triangle}, \tag{43}$$

while $R, \epsilon, p, q, \hat{p}$, and $\hat{q}$ are again given by (24) and (25).

For $\delta = 0$ and $\phi \geq rK$, American puts are not rationally exercised early. Consequently, the American put value $P^{(n)}(s)$ is given by (41) in this case. For $\delta > 0$ or $\phi < rK$, the discrete-time decomposition of the American put is:

$$P^{(n)}(s) = \begin{cases} p_0^{(n)}(s) + b_1^{(n)}(s) & \text{if } s > \underline{s}_0 \equiv K \\ v_i^{(n)}(s) + b_i^{(n)}(s) + A_i^{(n)}(s;1) & \text{if } s \in (\underline{s}_i, \underline{s}_{i-1}], i = 1, \ldots, n \\ K - S & \text{if } s \leq \underline{s}_n, \end{cases} \tag{44}$$

where for $i = 1, \ldots, n$:

$$v_i^{(n)}(s) = KR^{n-i+1} - sD^{n-i+1} - \phi_n R \frac{R^{n-i+1} - R^n}{1-R}, \tag{45}$$

approximates the value of a short forward position maturing in $n - i + 1$ periods, while:

$$b_i^{(n)}(s) = \sum_{j=1}^{n-i+1} \left(\frac{s}{\underline{s}_{n-j+1}}\right)^{\gamma-\epsilon} \sum_{k=0}^{j-1} \frac{\left(2\epsilon \ln\left(\frac{s}{\underline{s}_{n-j+1}}\right)\right)^k}{k!} \sum_{l=0}^{j-k-1} \binom{j-1+l}{j-1} [q^j p^{k+l} R^j (Kr - \phi_n) - \hat{q}^j \hat{p}^{k+l} D^j \underline{s}_{n-j+1} \delta]\triangle$$

$$\mathrm{A}_i^{(n)}(s;h) = \sum_{j=h}^{n-i+1} \left(\frac{s}{\underline{s}_{n-j+1}}\right)^{\gamma+\epsilon} \sum_{k=0}^{j-1} \frac{\left(2\epsilon \ln\left(\frac{\underline{s}_{n-j+1}}{s}\right)\right)^k}{k!} \sum_{l=0}^{j-k-1} \binom{j-1+l}{j-1} [p^j q^{k+l} R^j (Kr - \phi_n) - \hat{p}^j \hat{q}^{k+l} D^j \underline{s}_{n-j+1} \delta]\triangle.$$

Continuity in s at the strike price for each time step $m = 1, \ldots, n$ again implies $c_1^{(m)}(K) = A_1^{(m)}(K;1)$, which in turn implies that each critical stripped price $\underline{s}_m$ implicitly solves:

$$c_1^{(m)}(K) - A_1^{(m)}(K;2) = \left(\frac{K}{\underline{s}_m}\right)^{\gamma+\epsilon} [pR(Kr - \phi_n) - \hat{p}D\underline{s}_m\delta]\triangle, \qquad m = 1, \ldots, n, \tag{46}$$

[14] If we realistically assume that exercise of the put at the end of a calendar time period occurs immediately *after* that period's dividend is paid, then S is the *ex*-dividend stock price and s is the ex-dividend price after stripping off the fixed component of the dividend flow.

where from (42), the at-the-money call value on the left hand side (LHS) of (46) simplifies to:

$$c_1^{(m)}(K) = \sum_{l=0}^{m-1} \binom{m-1+l}{m-1} [KD^m \hat{p}^m \hat{q}^l - KR^m p^m q^l] \qquad m = 1, \ldots, n. \tag{47}$$

It is straightforward to solve (46) numerically for each critical stripped price $\underline{s}_m$, since $\underline{s}_m$ does not appear on the LHS. Setting $\delta = 0$ in (46) implies the following explicit solution for the critical stripped prices when the dividend rate is constant at ϕ:

$$\underline{s}_m = K \left(\frac{pR(Kr - \phi_n)\triangle}{c_1^{(m)}(K) - A_1^{(m)}(K;2)} \right)^{\frac{1}{\gamma+\epsilon}}, \qquad m = 1, \ldots, n, \tag{48}$$

where the call value $c_1^{(m)}(K)$ is now given by (31). This solution is a good initial guess when numerically solving (46). From (40), the critical stock price $\underline{S}_n$ is determined by:

$$\underline{S}_n = \frac{\phi}{r}(1 - e^{-rT}) + \underline{s}_n, \tag{49}$$

where $\underline{s}_n$ is given by setting $m = n$ in (48) when $\delta = 0$ and solves (46) with $m = n$ otherwise.

V-B Positive Dividends and American Calls

Let $C(t, s; T)$ denote the value of an American call as a function of the current time t, the current *stripped* price s, and the maturity date T. We also define the call's *critical stripped price* $\bar{s}(t)$ as the unique stripped price s at which the alive American call value $C(t, s; T)$ just equals its exercise value $s + \frac{\phi}{r}[1 - e^{-r(T-t)}] - K$, for $t \in [0, T]$. From (36), the call's critical stock price $\bar{S}(t)$ is defined by:

$$\bar{S}(t) \equiv \frac{\phi}{r}[1 - e^{-r(T-t)}] + \bar{s}(t), \qquad t \in [0, T]. \tag{50}$$

In discrete time, $C^{(m)}(s)$ denotes our approximation[15] for the American call value when m periods remain, $m = 1, \ldots, n$. Our approximation for the critical stripped price, $\bar{s}_m$, is the unique s satisfying $C^{(m)}(s) = s + \phi_n \triangle R \frac{1-R^m}{1-R} - K, m = 1, \ldots, n$. For $\delta = 0$ and $\phi \leq rK$, American calls are not rationally exercised early. Consequently, the American call value $C^{(n)}(s)$ is given by (42) in this case. For $\delta > 0$ or $\phi > rK$, the discrete-time decomposition of the American call is:

$$C^{(n)}(s) = \begin{cases} S - K & \text{if } s \geq \bar{s}_n \\ -v_i^{(n)}(s) + \alpha_i^{(n)}(s) + B_i^{(n)}(s;1) & \text{if } s \in [\bar{s}_{i-1}, \bar{s}_i), i = 1, \ldots, n \\ c_0^{(n)}(s) + \alpha_1^{(n)}(s) & \text{if } s < \bar{s}_0 \equiv K \end{cases} \tag{51}$$

[15] If we realistically assume that exercise of the call at the end of a calendar time period occurs immediately *before* that period's dividend is paid, then S is the *cum*-dividend stock price and s is the cum-dividend price after stripping off the fixed component of the dividend flow.

where for $i = 1, \ldots, n$, $-v_i^{(n)}(s)$ is the initial value of a long[16] forward position maturing in $n - i + 1$ periods,

$$\alpha_i^{(n)}(s) = \sum_{j=1}^{n-i+1} \left(\frac{s}{\bar{s}_{n-j+1}}\right)^{\gamma+\epsilon} \sum_{k=0}^{j-1} \frac{\left(2\epsilon \ln\left(\frac{\bar{s}_{n-j+1}}{s}\right)\right)^k}{k!} \sum_{l=0}^{j-k-1} \binom{j-1+l}{j-1} [\hat{p}^j \hat{q}^{k+l} D^j \bar{s}_{n-j+1}\delta - p^j q^{k+l} R^j (Kr - \phi_n)]\triangle$$

$$B_i^{(n)}(s; h) = \sum_{j=h}^{n-i+1} \left(\frac{s}{\bar{s}_{n-j+1}}\right)^{\gamma-\epsilon} \sum_{k=0}^{j-1} \frac{\left(2\epsilon \ln\left(\frac{s}{\bar{s}_{n-j+1}}\right)\right)^k}{k!} \sum_{l=0}^{j-k-1} \binom{j-1+l}{j-1} [\hat{q}^j \hat{p}^{k+l} D^j \bar{s}_{n-j+1}\delta - q^j p^{k+l} R^j (Kr - \phi_n)]\triangle.$$

Appendix 2 also provides an economic justification for the decompositions in (51).

Continuity in s at the strike price for each time step $m = 1, \ldots, n$ implies that the critical stripped price $\bar{s}_m$ implicitly solves:

$$p_0^{(m)}(K) - B_1^{(m)}(K; 2) = \left(\frac{K}{\bar{s}_m}\right)^{\gamma-\epsilon} [\hat{q} D \bar{s}_m \delta - qR(Kr - \phi_n)]\triangle, \qquad m = 1, \ldots, n, \tag{52}$$

where from (41), the at-the-money put value on the LHS of (52) simplifies to:

$$p_0^{(m)}(K) = \sum_{l=0}^{m-1} \binom{m-1+l}{m-1} [KR^m q^m p^l - KD^m \hat{q}^m \hat{p}^l] \qquad m = 1, \ldots, n. \tag{53}$$

It is straightforward to solve (52) numerically for the critical stripped price $\bar{s}_m$, since it does not appear on the LHS. If $\phi_n = rK$, then (52) yields the following explicit solution for the critical stripped price:

$$\underline{s}_m = K \left(\frac{K\hat{q}D\delta\triangle}{p_0^{(m)}(K) - B_1^{(m)}(K; 2)}\right)^{\frac{1}{\gamma-\epsilon-1}}, \qquad m = 1, \ldots, n. \tag{54}$$

This solution is a good initial guess when numerically solving (52). From (**??**), the call's critical stock price $\bar{S}_n$ is determined by:

$$\bar{S}_n = \frac{\phi}{r}(1 - e^{-rT}) + \bar{s}_n, \tag{55}$$

where $\bar{s}_n$ is given by setting $m = n$ in (54) when $\phi = rK$ and solves (52) with $m = n$ otherwise. Appendix 2 collects all the formulas needed to implement European and American puts and calls when the underlying has a continuous payout with a fixed component ϕ and a proportional component δ.

VI Summary and Future Research

We implemented a new approach to valuing American options, which is fast, accurate, and flexible. The approach uses the known solution for perpetual options to value finite-lived American options. The insight allows one to

[16] See (45) for $v_i^{(n)}(s)$.

replace a free boundary problem involving a p.d.e. with a recurrent sequence of simpler free boundary problems involving o.d.e's. Alternatively, a finite-horizon optimal stopping problem involving a first passage time to a time-dependent boundary is replaced with a simpler sequence of infinite-horizon stopping problems each involving a passage time to a constant boundary. Under either perspective, each problem in the sequence may be solved analytically, resulting in an explicit solution for the put value, with the critical stock prices determined recursively. Richardson extrapolation is used to dramatically enhance convergence, with a modified three point scheme performing particularly well in numerical tests.

Since American options are among the most difficult derivative securities to value, it appears plausible that our approach can be applied to valuing other options, such as Asian, barrier, compound, installment, and lookback options. While analytic solutions or approximations exist for these securities, it is also plausible that our approach can be used to handle realities such as multiple barriers, discrete sampling, term structure effects, and volatility smiles. Our approach may also be applied to other areas of finance where optimal stopping problems or p.d.e.'s arise, such as signalling models or optimal consumption and portfolio theory. In the interests of brevity, these directions are left for future research.

References

[1] Barles, G., J. Burdeau, M. Romano, and N. Samsœn, 1993, Estimation de la frontière libre des options américaines au voisinage de l'échéance, *C. R. Acad. Sci. Paris*, t. 316, Série I, p. 171–174.

[2] Barone-Adesi G. and R. Whaley, 1987, Efficient Analytic Approximation of American Option Values, *The Journal of Finance*, **42**, 301–320.

[3] Black, F. and M. Scholes, 1973, The Pricing of Options and Corporate Liabilities, *The Journal of Political Economy*, **81**, 637–659.

[4] Boyle, P., J. Evnine and S. Gibbs, 1989, Numerical Evaluation of Multivariate Contingent Claims, *Review of Financial Studies*, **2**, 2, 241-250.

[5] Breen, R., 1991, The Accelerated Binomial Option Pricing Model, *Journal of Financial and Quantitative Analysis*, **26**, 2, 153–164.

[6] Brennan, M. and E. Schwartz, 1977, The Valuation of American Put Options, *Journal of Finance*, **32**, 449–462.

[7] Broadie, M. and J. Detemple, 1995, American Option Valuation: New Bounds, Approximations, and a Comparison of Existing Methods, *Review of Financial Studies*, forthcoming.

[8] Bunch, D. and H. Johnson, 1992, A Simple and Numerically Efficient Valuation Method for American Puts Using a Modified Geske-Johnson Approach, *Journal of Finance*, **47**, 2, 809–816.

[9] Carr, P., R. Jarrow, and R. Myneni, 1992, Alternative Characterizations of American Put Options, *Mathematical Finance*, 87–106.

[10] Chesney, M., 1989, Pricing American Currency Options: An Analytical Approach, HEC working paper.

[11] Cox, J., S. Ross, and M. Rubinstein, 1979, Option Pricing: A Simplified Approach, *Journal of Financial Economics*, **7**, 229–263.

[12] Duffie, D., 1988, An Extension of the Black-Scholes Model of Security Valuation, *Journal of Economic Theory* **46**, 1, 194–204.

[13] Duffie, D.,and K. Singleton, 1995, Modeling Term Structures of Defaultable Bonds, Stanford University working paper.

[14] Geske, R. and H. Johnson, 1984, The American Put Option Valued Analytically, *Journal of Finance*, **39**, 1511–1524.

[15] Goldenberg D. and R. Schmidt, 1995, Estimating the Early Exercise Boundary and Pricing American Options, Rensselaer Polytechnic Institute working paper.

[16] Grant, D., G. Vora, and D. Weeks, 1993, Path-Dependent Options: Extending the Monte Carlo Simulation Approach, University of New Mexico working paper.

[17] Ho T., R. Stapleton, and M. Subrahmanyam, 1994, A Simple Technique for the Valuation and Hedging of American Options, *The Journal of Derivatives*, Fall 1994, 52–66.

[18] Jacka, S.D., 1991, Optimal Stopping and the American Put, *Journal of Mathematical Finance*, **1**, 2, 1–14.

[19] Jamshidian, F., 1989, Formulas for American Options, Merrill Lynch Capital Markets working paper.

[20] Johnson, H., 1983, An Analytic Approximation for the American Put Price, *Journal of Financial and Quantitative Analysis*, **18**, 1, 141–148.

[21] Kamrad, B. and P. Ritchken, 1991, Multinomial Approximating Models for Options with k State Variables, Case Western Reserve University working paper.

[22] Kim, I. J., 1990, The Analytic Valuation of American Puts, *Review of Financial Studies*, **3**, 4, 547–572.

[23] Marchuk, G. and V. Shaidurov, 1983, Difference Methods and Their Extrapolations, Springer Verlag, NY.

[24] MacMillan, L., 1986, Analytic Approximation for the American Put Option, *Advances in Futures and Options Research,* Volume 1, 119-139.

[25] McKean, Jr., H. P., 1965, Appendix: A Free Boundary Problem for the Heat Equation Arising from a Problem in Mathematical Economics, *Industrial Management Review*, **6**, 32–39.

[26] Mason, S., 1977, The Numerical Analysis of Certain Free Boundary

[27] Madan, D., and H. Unal, 1993, Pricing the Risks of Default, University of Maryland working paper.

[28] Merton, R. C., 1973, Theory of Rational Option Pricing, *Bell Journal of Economics and Management Science*, **4**, 141–183.

[29] Merton, R. C., 1976, Option Pricing When Underlying Security Returns are Discontinuous, *Journal of Financial Economics*, **3**, 125–144.

[30] Meyer, G. H., 1970, On a Free Boundary Problem for Linear Ordinary Differential Equations and the One-Phase Stephan Problem, *Numerical Mathematics*, **16**, 248–67.

[31] Meyer, G. H., 1979, One-Dimensional Free Boundary Problems *SIAM Review*, **19**, 17–34.

[32] Meyer, G. H. and J. van der Hoek, 1994, The Evaluation of American Options with the Method of Lines, Georgia Institute of Technology working paper.

[33] Omberg, E., 1987, The Valuation of American Puts with Exponential Exercise Policies, *Advances in Futures and Options Research,* Volume 2, 117–142.

[34] Parkinson, M., 1977, Option Pricing: The American Put, *Journal of Business*, **50**, 21–36.

[35] Rektorys, K., 1982, The Method of Discretization in Time and Partial Differential Equations, D. Reidel Publishing Co., Dordrecht-Boston, MA.

[36] Rendleman R., and B. Bartter, 1979, Two-State Option Pricing, *Journal of Finance*, **34**, 1093-1110.

[37] Roll, R., 1977, An Analytic Valuation Formula for Unprotected American Call Options on Stocks with Known Dividends, *Journal of Financial Economics*, **5** 251–58.

[38] Rothe, E., 1930, Zweidimensionale parabolische Randwertaufgaben als Grenzfall eindimensinaler Randwertaufgaben, *Math. Ann.* 102.

[39] Samuelson, P. A., 1965, Rational Theory of Warrant Pricing, *Industrial Management Review*, **6**, 13–31.

[40] Subrahmanyam, M., and G. Yu, 1993, Pricing and Hedging American Options: A Unified Method and its Efficient Implementation, New York University working paper.

[41] Tilley, J., 1992, Valuing American Options in a Path Simulation Model, Morgan Stanley working paper.

[42] Van Moerbeke, P., 1976, On Optimal Stopping and Free Boundary Problems, *Archive for Rational Mechanics and Analysis*, **60**, 101–148.

[43] Wilmott, P., J. Dewynne, and S. Howison, 1993, Option Pricing: Mathematical Models and Computation, Oxford Financial Press, Oxford, U.K.

[44] Yu G. 1992, Valuation of American Options: A Simplified Approach and Applications, New York University working paper.

Appendix 1

This appendix presents a new decomposition of the value of an American put into the value if forced to sell the underlying at a given date $T_x \in [0, T]$, and the premia which arise because exercise can occur before or after this date. This decomposition may be understood from the following continuous-time trading strategy, which converts the initial purchase of an American put into the final payoff of a European *call*. Suppose that at $t = 0$ an investor buys an alive American put maturing at T. In the period $t \in (0, T_x)$, the holder of the put exercises it each time the stock price crosses the exercise boundary from above and repurchases the put each time the stock price crosses from below. In order that the transitions self-finance, the investor keeps K dollars in an interest earning bank account and is short one share in the exercise region for $t \in [0, T_x]$. Thus, when the stock price is in this region during this period, the investor withdraws interest continuously so as to keep his bank balance flat at K and covers the dividends from the short share position. Consequently, if the stock price is still in the exercise region at T_x, the investor can liquidate the position for a net payoff $K - S_{T_x}$, matching that of a short forward position expiring at T_x. However, if the stock price is in the continuation region at T_x, obtaining this payoff requires that the investor purchase one share at T_x and borrow the delivery price K. For stock prices in the continuation region during the period $t \in (T_x, T)$, the investor withdraws any dividend payments and continuously pays interest on these borrowings so as to keep his liability constant at K. As a result, his overall position in this region during this period consists of the alive American put,long one share, and less the constant liability of K. During this period, the investor liquidates this position by exercising the put each time the stock price crosses the exercise boundary from above and re-enters the position each time the stock price crosses from below. The optimality of the exercise boundary again ensures that these transitions are self-financing. At expiration, the position is worthless if the stock price finishes below the strike and pays the difference between the stock price and the strike price otherwise.

Equating the initial cost of the strategy to the present value of the cash flows resulting from it gives the following fundamental decomposition of the continuous-time value of an alive American put:

$$P(0, s; T) = v(0, s; T_x) + b(0, s; 0, T_x) + a(0, s; T_x, T), \qquad s > \underline{s}(0). \tag{56}$$

The value if forced to sell at T_x is the value of a short forward position:

$$v(0, s; T_x) = e^{-rT_x} E_{0,s}(K - S_{T_x}) = Ke^{-rT_x} - se^{-\delta T_x} - \frac{\phi}{r}[e^{-rT_x} - e^{-rT}]. \tag{57}$$

Letting $1(A)$ denote the indicator function of the event A, and $N(\cdot)$ denote the standard normal distribution function, the premium for allowing the sale to occur before T_x is given by the value of the interest *less dividends* (i.e. net interest) received below the boundary before T_x:

$$\begin{aligned} b(0,s;0,T_x) &= E_{0,s}\int_0^{T_x} e^{-rt}[rK-\phi-\delta s_t]1(s_t<\underline{s}(t))dt \\ &= \int_0^{T_x}[(rK-\phi)e^{-rt}N(-d_2(\underline{s}(t),t))-\delta s e^{-\delta t}N(-d_1(\underline{s}(t),t))]dt, \end{aligned}$$

where:

$$d_2(\underline{s}(t),t)\equiv\frac{\ln(s/\underline{s}(t))+(r-\delta-\sigma^2/2)t}{\sigma\sqrt{t}}, \qquad d_1(\underline{s}(t),t)\equiv d_2(\underline{s}(t),t)+\sigma\sqrt{t}.$$

Finally, the premium for allowing the sale to occur after T_x or never is the value of a European call less the *net* interest received above the boundary after T_x:

$$\begin{aligned} a(0,s;T_x,T) &= E_{0,s}e^{-rT}(S_T-K)^+ - E_{0,s}\int_{T_x}^{T} e^{-rt}[rK-\phi-\delta s_t]1(s_t\geq\underline{s}(t))dt \\ &= se^{-\delta T}N(d_1(K,T))-Ke^{-rT}N(d_2(K,T))-\int_{T_x}^{T}[(rK-\phi)e^{-rt}N(d_2(\underline{s}(t),t))-\delta s e^{-\delta t}N(d_1(\underline{s}(t),t))]dt. \end{aligned}$$

Note that the simpler decomposition of the alive American put value into the corresponding European put value and the early exercise premium is obtained by setting $T_x = T$. In contrast, the discrete-time analog (44) of (56) is generated by setting T_x to the time T_s which solves $\underline{\hat{S}}(T_s) = S$, where $\underline{\hat{S}}$ is the step function approximation of the exercise boundary (see Figure 8) and S is the current stock price, assumed to be in the interval $(\underline{\hat{S}}(0), \underline{\hat{S}}(T))$. The final term $A_i^{(n)}(s;1)$ in our discrete-time decomposition (44) for the American put value reflects a simplification obtained by imposing smoothness[17] at the critical stock price at every time step.

Using a strategy mirroring the one underlying the decomposition of an American put, the alive American call can be shown to decompose into three terms:

$$C(0,s;T) = -v(0,s;T_x)+\alpha(0,s;0,T_x)+\beta(0,s;T_x,T), \qquad s<\underline{s}(0). \tag{58}$$

The first term is the value if forced to buy at T_x and is given by the value of a *long*[18] forward position maturing at T_x. The premium for allowing the purchase to occur prior to T_x is now the value of the dividends less interest

[17] It can be shown that at any time step, our solution for the American put is twice differentiable for all stock prices strictly above the critical price. Furthermore, at the critical stock price, our solution is continuous and differentiable. It is not twice differentiable at the critical price, as is true of the unknown solution to the original free boundary problem (4).

[18] See (57) for $v(0,s;T_x)$.

(i.e. net dividends) received *above* the boundary before T_x:

$$\begin{aligned}\alpha(0,s;0,T_x) &= E_{0,s}\int_0^{T_x} e^{-rt}[\phi+\delta s_t - rK]1(s_t > \bar{s}(t))dt \\ &= \int_0^{T_x}[\delta s e^{-\delta t}N(d_1(\bar{s}(t),t)) - (rK-\phi)e^{-rt}N(d_2(\bar{s}(t),t))]dt.\end{aligned}$$

Finally, the premium for allowing the purchase to occur after T_x or never is now the value of the net *dividends* received below the boundary after T_x:

$$\begin{aligned}&\beta(0,s;T_x,T) \\ &= E_{0,s}e^{-rT}(K-S_T)^+ - E_{0,s}\int_{T_x}^{T} e^{-rt}[\phi+\delta s_t - rK]1(s_t < \bar{s}(t))dt \\ &= Ke^{-rT}N(-d_2(K,T)) - se^{-\delta T}N(-d_1(K,T)) - \int_{T_x}^{T}[\delta s e^{-\delta t}N(-d_1(\bar{s}(t),t)) - (rK-\phi)e^{-rt}N(-d_2(\bar{s}(t),t))]dt.\end{aligned}$$

As in the put case, the discrete-time analog sets T_x to the time which equates a step function approximation of the exercise boundary to the initial stock price. The final term $B_i^{(n)}(s;1)$ in (51) reflects a simplification obtained by imposing smoothness at the critical stock price at every time step.

Appendix 2

This appendix collects all the formulas needed to implement European and American puts and calls when the underlying has a continuous payout with a fixed component ϕ and a proportional component δ. Letting $s = S - \frac{\phi}{r}[1 - e^{-rT}]$, the N-point Richardson extrapolation of the European put formula is:

$$p^{1:N}(s;T) \equiv \sum_{n=1}^{N} \frac{(-1)^{N-n} n^N}{n!(N-n)!} p^{(n)}(s;T),$$

where:

$$p^{(n)}(s;T) = \begin{cases} \left(\frac{s}{K}\right)^{\gamma-\epsilon} \sum_{k=0}^{n-1} \frac{\left(2\epsilon \ln\left(\frac{s}{K}\right)\right)^k}{k!} \sum_{l=0}^{n-k-1} \binom{n-1+l}{n-1} [KR^n q^n p^{k+l} - KD^n \hat{q}^n \hat{p}^{k+l}] & \text{if } s > K \\ KR^n - sD^n + c^{(n)}(s;T) & \text{if } s \leq K, \end{cases}$$

with $c^{(n)}(s;T), s \leq K$ given on the next page, and where:

$$\gamma \equiv \frac{1}{2} - \frac{r-\delta}{\sigma^2}, \triangle \equiv \frac{T}{n}, R \equiv \frac{1}{1+r\triangle}, D \equiv \frac{1}{1+\delta\triangle}, \epsilon \equiv \sqrt{\gamma^2 + \frac{2}{R\sigma^2\triangle}}, p \equiv \frac{\epsilon - \gamma}{2\epsilon}, q \equiv 1-p, \hat{p} \equiv \frac{\epsilon - \gamma + 1}{2\epsilon}, \text{ and } \hat{q} \equiv 1 - \hat{p}.$$

The N-point Richardson extrapolation of the American put formula is:

$$P^{1:N}(s;T) \equiv \sum_{n=1}^{N} \frac{(-1)^{N-n} n^N}{n!(N-n)!} P^{(n)}(s;T).$$

where:

$$P^{(n)}(s;T) = \begin{cases} p^{(n)}(s;T) + b_1^{(n)}(s) & \text{if } s > \underline{s}_0 \equiv K \\ v_i^{(n)}(s) + b_i^{(n)}(s) + A_i^{(n)}(s;1) & \text{if } s \in (\underline{s}_i, \underline{s}_{i-1}], i = 1, \ldots, n \\ K - S & \text{if } s \leq \underline{s}_n, \end{cases}$$

where for $i = 1, \ldots, n$, $v_i^{(n)}(s) = KR^{n-i+1} - sD^{n-i+1} - \phi_n R \frac{R^{n-i+1} - R^n}{1-R}$, $\phi_n \equiv \frac{\phi}{r} \frac{1-e^{-rT}}{R(1-R^n)}$,

$$b_i^{(n)}(s) = \sum_{j=1}^{n-i+1} \left(\frac{s}{\underline{s}_{n-j+1}}\right)^{\gamma-\epsilon} \sum_{k=0}^{j-1} \frac{\left(2\epsilon \ln\left(\frac{s}{\underline{s}_{n-j+1}}\right)\right)^k}{k!} \sum_{l=0}^{j-k-1} \binom{j-1+l}{j-1} [q^j p^{k+l} R^j (Kr - \phi_n) - \hat{q}^j \hat{p}^{k+l} D^j \underline{s}_{n-j+1} \delta] \triangle,$$

$$A_i^{(n)}(s;h) = \sum_{j=h}^{n-i+1} \left(\frac{s}{\underline{s}_{n-j+1}}\right)^{\gamma+\epsilon} \sum_{k=0}^{j-1} \frac{\left(2\epsilon \ln\left(\frac{\underline{s}_{n-j+1}}{s}\right)\right)^k}{k!} \sum_{l=0}^{j-k-1} \binom{j-1+l}{j-1} [p^j q^{k+l} R^j (Kr - \phi_n) - \hat{p}^j \hat{q}^{k+l} D^j \underline{s}_{n-j+1} \delta] \triangle.$$

If $\delta = 0$, the critical stripped prices are given by $\underline{s}_m = K \left(\frac{pR(Kr-\phi_n)\triangle}{c_1^{(m)}(K) - A_1^{(m)}(K;2)}\right)^{\frac{1}{\gamma+\epsilon}}$, $m = 1, \ldots, n$, where $c_1^{(m)}(K) = \sum_{l=0}^{m-1} \binom{m-1+l}{m-1} [KD^m \hat{p}^m \hat{q}^l - KR^m p^m q^l]$. If $\delta > 0$, the critical stripped prices solve:

$$c_1^{(m)}(K) - A_1^{(m)}(K;2) = \left(\frac{K}{\underline{s}_m}\right)^{\gamma+\epsilon} [pR(Kr - \phi_n) - \hat{p}D\underline{s}_m \delta] \triangle, \qquad m = 1, \ldots, n.$$

The N-point Richardson extrapolation of the put's critical stock price is $\underline{S}^{1:N}(T) \equiv \frac{\phi}{r}[1 - e^{-rT}] + \sum_{n=1}^{N} \frac{(-1)^{N-n} n^N}{n!(N-n)!} \underline{s}_n$.

Similarly, letting $s = S - \frac{\phi}{r}[1 - e^{-rT}]$, the N-point Richardson extrapolation of the European call formula is:

$$c^{1:N}(s;T) \equiv \sum_{n=1}^{N} \frac{(-1)^{N-n} n^N}{n!(N-n)!} c^{(n)}(s;T),$$

where:

$$c^{(n)}(s;T) = \begin{cases} sD^n - KR^n + p^{(n)}(s;T) & \text{if } s > K \\ \left(\frac{s}{K}\right)^{\gamma+\epsilon} \sum_{k=0}^{n-1} \frac{\left(2\epsilon \ln\left(\frac{K}{s}\right)\right)^k}{k!} \sum_{l=0}^{n-k-1} \binom{n-1+l}{n-1} [KD^n \hat{p}^n \hat{q}^{k+l} - KR^n p^n q^{k+l}] & \text{if } s \leq K, \end{cases}$$

with $p^{(n)}(s;T), s > K$ given on the previous page, and where again:

$$\gamma \equiv \frac{1}{2} - \frac{r-\delta}{\sigma^2}, \triangle \equiv \frac{T}{n}, R \equiv \frac{1}{1+r\triangle}, D \equiv \frac{1}{1+\delta\triangle}, \epsilon \equiv \sqrt{\gamma^2 + \frac{2}{R\sigma^2\triangle}}, p \equiv \frac{\epsilon-\gamma}{2\epsilon}, q \equiv 1-p, \hat{p} \equiv \frac{\epsilon-\gamma+1}{2\epsilon}, \text{ and } \hat{q} \equiv 1-\hat{p}.$$

The N-point Richardson extrapolation of the American call formula is:

$$C^{(n)}(s;T) = \begin{cases} S - K & \text{if } s \geq \bar{s}_n \\ -v_i^{(n)}(s) + \alpha_i^{(n)}(s) + B_i^{(n)}(s;1) & \text{if } s \in [\bar{s}_{i-1}, \bar{s}_i), i = 1, \ldots, n \\ c^{(n)}(s) + \alpha_1^{(n)}(s) & \text{if } s < \bar{s}_0 \equiv K \end{cases}$$

where for $i = 1, \ldots, n$,

$$-v_i^{(n)}(s) = sD^{n-i+1} + \phi_n R \frac{R^{n-i+1} - R^n}{1-R} - KR^{n-i+1}, \phi_n \equiv \frac{\phi}{r} \frac{1-e^{-rT}}{R(1-R^n)},$$

$$\alpha_i^{(n)}(s) = \sum_{j=1}^{n-i+1} \left(\frac{s}{\bar{s}_{n-j+1}}\right)^{\gamma+\epsilon} \sum_{k=0}^{j-1} \frac{\left(2\epsilon \ln\left(\frac{\bar{s}_{n-j+1}}{s}\right)\right)^k}{k!} \sum_{l=0}^{j-k-1} \binom{j-1+l}{j-1} [\hat{p}^j \hat{q}^{k+l} D^j \bar{s}_{n-j+1}\delta - p^j q^{k+l} R^j (Kr - \phi_n)]\triangle,$$

$$B_i^{(n)}(s;h) = \sum_{j=h}^{n-i+1} \left(\frac{s}{\bar{s}_{n-j+1}}\right)^{\gamma-\epsilon} \sum_{k=0}^{j-1} \frac{\left(2\epsilon \ln\left(\frac{s}{\bar{s}_{n-j+1}}\right)\right)^k}{k!} \sum_{l=0}^{j-k-1} \binom{j-1+l}{j-1} [\hat{q}^j \hat{p}^{k+l} D^j \bar{s}_{n-j+1}\delta - q^j p^{k+l} R^j (Kr - \phi_n)]\triangle.$$

If $\phi_n = rK$, the critical stripped price $\bar{s}_m$ is given by:

$$\underline{s}_m = K \left(\frac{K\hat{q}D\delta\triangle}{p_0^{(m)}(K) - B_1^{(m)}(K;2)}\right)^{\frac{1}{\gamma-\epsilon-1}}, \qquad m = 1, \ldots, n,$$

where $p_0^{(m)}(K) = \sum_{l=0}^{m-1} \binom{m-1+l}{m-1} [KR^m q^m p^l - KD^m \hat{q}^m \hat{p}^l]$.

Otherwise, the critical stripped price $\bar{s}_m$ implicitly solves:

$$p_0^{(m)}(K) - B_1^{(m)}(K;2) = \left(\frac{K}{\bar{s}_m}\right)^{\gamma-\epsilon} [\hat{q}D\bar{s}_m\delta - qR(Kr - \phi_n)]\triangle, \qquad m = 1, \ldots, n.$$

The N-point Richardson extrapolation of the call's critical stock price is $\bar{S}^{1:N}(T) \equiv \frac{\phi}{r}[1 - e^{-rT}] + \sum_{n=1}^{N} \frac{(-1)^{N-n} n^N}{n!(N-n)!} \bar{s}_n$.

Table 1: Convergence without and with Richardson Extrapolation

S = 100, K = 100, T = 1, r = 0.1, δ = 0, σ = 0.3

Number of Steps n or Points N	Unextrapolated Put Value $P^{(n)}$	Extrapolated Put Value $P^{1:N}$
1	7.0405	7.0405
2	7.6175	8.1946
3	7.8353	8.3089
4	7.9505	8.3257
5	8.0220	8.3311
6	8.0709	8.3333
7	8.1065	8.3345
8	8.1335	8.3353
9	8.1548	8.3358
10	8.1720	8.3362
11	8.1862	8.3365
12	8.1981	8.3367
13	8.2082	8.3369
14	8.2169	8.3370
15	8.2246	8.3371

Figure 2: Exercise Boundary vs. Calendar time.

Figure 4: Approximate Solution Using Two Time Steps.

Figure 6: Three Point Richardson Extrapolation

Figure 8: Piecewise Constant Approximation to Exercise Boundary