# Taylor-made volatility swaps

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**Abstract**

Using little else than the mixing formula and Taylor expansions we show that the volatility swap strike is approximately the implied volatility corresponding to the strike where the vanna and vomma of a vanilla option is zero. As this result does not require any heavy numerical computations, and is valid for a large class of stochastic volatility models, it can be used for fast and model-free indicative pricing of volatility swaps.

# 1 Introduction

In this short note we prove that the volatility swap strike can be approximated by the implied volatility corresponding to the strike where the vanna and vomma of a vanilla option is zero. Even though the result may already be intuitively familiar to some traders, to the best of our knowledge a proof of the validity of the rule-of-thumb has not yet been given in the literature on derivatives pricing.

The next section briefly reviews the Black-Scholes formula for vanilla options, and the Greeks that are relevant for the paper. It is followed by an overview of the mixing formula, which will be the basis of our approximation. Next, we discuss volatility swaps and recall some exisiting approximations for the volatility swap strike before deriving our result. The final section applies our result to volatility swap pricing and compares it to some existing approximations and exact prices.

# 2 Black-Scholes formula

Recall the Black-Scholes formula [1, 10] for European vanilla calls and puts

$$C^{BS}(S,K,\Sigma,T) = Se^{(b-r)T}N(d_1) - Ke^{-rT}N(d_2) \tag{2.1}$$

$$P^{BS}(S,K,\Sigma,T) = Ke^{-rT}N(-d_2) - Se^{(b-r)T}N(-d_1) \tag{2.2}$$

where

$$d_1 = \frac{\ln(Se^{bT}/K) + \frac{1}{2}\Sigma^2 T}{\Sigma\sqrt{T}} \tag{2.3}$$

$$d_2 = d_1 - \Sigma\sqrt{T} \tag{2.4}$$

Here $S$ is the current spot level, $K$ is the strike of the option, $\Sigma$ is the implied volatility of the option, $T$ the time to maturity, and $b = r - q$ is the cost of carry rate. For the purposes of this paper we will also need Black-Scholes Greeks for vanilla options, in particular the delta ($\Delta^{BS}$), vega ($\nu^{BS}$), vanna ($va^{BS}$), and vomma ($vo^{BS}$):

$$\Delta_C^{BS} = \frac{\partial C^{BS}}{\partial S} = e^{(b-r)T}N(d_1) \tag{2.5}$$

$$\Delta_P^{BS} = \frac{\partial P^{BS}}{\partial S} = e^{(b-r)T}[N(d_1) - 1] \tag{2.6}$$

$$\nu^{BS} = \frac{\partial C^{BS}}{\partial \Sigma} = \frac{\partial P^{BS}}{\partial \Sigma} = Se^{(b-r)T}n(d_1)\sqrt{T} \tag{2.7}$$

$$va^{BS} = \frac{\partial^2 C^{BS}}{\partial S \partial \Sigma} = \frac{\partial^2 P^{BS}}{\partial S \partial \Sigma} = -\frac{e^{(b-r)T}d_2}{\Sigma}n(d_1) \tag{2.8}$$

$$vo^{BS} = \frac{\partial^2 C^{BS}}{\partial \Sigma^2} = \frac{\partial^2 P^{BS}}{\partial \Sigma^2} = \frac{d_1 d_2}{\Sigma} \nu^{BS} \tag{2.9}$$

Although the behaviour of all Black-Scholes Greeks is fascinating [6, 7], we are particularly interested in the fact that vanna and vomma are directly proportional to $d_2$. This entails that both quantities will be zero when $d_2 = 0$ and is cruciall for deriving our result. Let $K_{d2}$ be the strike at which $d_2 = 0$, and $\Sigma_{d2}$ be the corresponding implied volatility. Using the formula for $d_2$ it is evident that

$$K_{d2} = S \exp \left\{ \left( b - \frac{1}{2} \Sigma_{d2}^2 \right) T \right\} \tag{2.10}$$

As the implied volatility is a function of strike, solving (2.10) will be an iterative process.

# 3 The mixing formula

Even though the assumptions underlying the Black-Scholes *model* are not realistic, the Black-Scholes *formula* is still used by market participants as a quoting mechanism for vanilla options. Given the market price $C^{MKT}(S, K, T)$ of an option, the implied volatility of the option is by definition the value of $\Sigma$ such that

$$C^{BS}(S, K, \Sigma, T) = C^{MKT}(S, K, T) \tag{3.1}$$

Quoting an option in terms of its price or in terms of its implied volatility are fully equivalent. Implied volatility is not a constant, but depends on the strike, time to maturity, and current spot level as well as other (hidden) variables. This dependence of implied volatility on several variables results in what is called the volatility surface.

As the Black-Scholes model cannot explain observed market prices, other more advanced models have been put forward with less stringent assumptions than Black-Scholes. A majority of these more realistic models have one thing in common: the abandonment of the constant volatility assumption. The price of this, though, is the introduction of more parameters that need to be calibrated to market prices. Stochastic volatility models is one such class of models.

In the family of stochastic volatility models the dynamics of the underlying spot process and its volatility can typically be written as

$$dS = rSdt + \sigma S dW_S \tag{3.2}$$

$$d\sigma = a(\sigma, t)dt + b(\sigma, t)dW_\sigma \tag{3.3}$$

where the volatility process can be correlated to the spot process:

$$dW_S dW_\sigma = \rho dt \tag{3.4}$$

The functions $a(\sigma, t)$ and $b(\sigma, t)$ may contain other parameters, in addition to $\sigma$ and $\rho$, that need to be calibrated to reproduce market prices of vanilla options. Once these parameters are calibrated, the model can be used to price exotic options.

Let $C(S, K, \sigma, T)$ denote the price of a vanilla option under the stochastic volatility model (3.2)-(3.4). Then, once the stochastic volatility model has been calibrated,

$$C(S, K, \sigma, T) = C^{BS}(S, K, \Sigma, T) = C^{MKT}(S, K, T) \qquad (3.5)$$

In addition to this evident relationship between the three prices, the mixing formula [9, 11, 12] establishes another important relationship between the model price of a stochastic volatility model and the Black-Scholes price:

$$C(S, K, \sigma, T) = E_0 \left[ C^{BS} \left( SM_T(\rho), K, \sigma_{0,T}\sqrt{1-\rho^2}, T \right) \right] \qquad (3.6)$$

where

$$M_T(\rho) = \exp\left\{ -\frac{\rho^2}{2} \int_0^T \sigma^2 dt + \rho \int_0^T \sigma dW_\sigma \right\} \qquad (3.7)$$

and $\sigma_{0,T}$ is as in (4.2). Clearly, we can also write

$$C^{BS}(S, K, \Sigma, T) = E_0 \left[ C^{BS} \left( SM_T(\rho), K, \sigma_{0,T}\sqrt{1-\rho^2}, T \right) \right] \qquad (3.8)$$

Carrying out a formal Taylor expansion of (3.8) in the correlation parameter, we arrive at

$$\begin{aligned} C^{BS}(S, K, \Sigma, T) &\approx E_0 \left[ C^{BS}\left(S, K, \sigma_{0,T}, T\right) \right] \\ &+ \rho S E_0 \left[ \Delta^{BS}\left(S, K, \sigma_{0,T}, T\right) \int_0^T \sigma dW_\sigma \right] + O(\rho^2) \qquad (3.9) \end{aligned}$$

# 4 Volatility swaps

A volatility swap is defined as an instrument with the following payoff at maturity $T$:

$$N\left(\sigma_{0,T} - K_{vol}\right) \qquad (4.1)$$

$N$ is the vega notional of the swap, $\sigma_{0,T}$ is the realized volatility over the interval $[0, T]$,

$$\sigma_{0,T} = \sqrt{\frac{1}{T} \int_0^T \sigma^2 dt} \qquad (4.2)$$

and $K_{vol}$ is the fair volatility strike. It is market standard to choose the volatility swap strike in such a way that the swap has zero value at inception. This is the case when

$$K_{vol} = E_0\left[\sigma_{0,T}\right] \qquad (4.3)$$

In other words the volatility strike calculated today is the risk-neutral expectation of future realized volatility.

Although the volatility swap appears to be the simplest possible derivative on volatility (it is after all just a forward contract on volatility) its cousin the variance swap, which has payoff

$$N\left(\sigma_{0,T}^2 - K_{var}^2\right) \tag{4.4}$$

is in fact much easier to price and replicate. The variance swap strike $K_{var}$ can be shown to be equal to the price of a strip of options weighted by the inverse of their respective strikes squared [5]. The variance strike also gives an upper bound for the volatility swap strike, which is a consequence of Jensen's inequality:

$$K_{vol} = E_0\left[\sqrt{\frac{1}{T}\int_0^T \sigma^2 dt}\right] \leq \sqrt{E_0\left[\frac{1}{T}\int_0^T \sigma^2 dt\right]} = K_{var} \tag{4.5}$$

Other than the model-independent upper bound given above, there is only a limited number of (almost) model-free analytical expressions or approximations available for the volatility swap strike. For instance, if the correlation between the underlying process (such as an index or exchange rate) and its volatility process is zero, then the mixing formula can be used to show that the volatility strike is approximately the ATM (at-the-money) implied volatility:

$$K_{vol} \approx \Sigma_{ATM} \tag{4.6}$$

This formula, although elegant, is too restrictive to be of practical use because of the zero correlation assumption. A good approximation should be independent of the correlation between the volatility and the underlying for the class of processes described by (3.2) - (3.4)

Our approximation, to be discussed shortly, generalizes (4.6) by being immune to correlation (at least to first order) while retaining its simplicity of form. We are aware of only one other correlation immune approximation of the volatiliy swap strike, namely the one due to Carr and Lee [3]. Our approach differs from theirs in that it is less technically involved and perhaps intuitively easier to understand. However, the advantage of the Carr and Lee approach is that it also shows what the initial options portfolio is to hedge the volatility swap; something that our result is silent on.

We close this section with an integral expression for the volatility swap strike based on the Laplace transform of the square root function:

$$E[\sigma_{0,T}] = \frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{1 - E[e^{-z\sigma_{0,T}^2}]}{z^{3/2}} dz \tag{4.7}$$

If $E[e^{-z\sigma_{0,T}^2}]$ is known then the integral can be numerically evaluated to give the exact volatility swap strike. Unfortunately $E[e^{-z\sigma_{0,T}^2}]$ is available in closed form only for certain volatility models. However, for the Heston model [8],

$$d\sigma^2 = \lambda(\theta - \sigma^2)dt + \eta\sigma dW_\sigma \tag{4.8}$$

it has been shown [2][4] that

$$E[e^{-z\sigma_{0,T}^2}] = Ae^{-z\sigma^2 B} \tag{4.9}$$

where

$$A = \left[\frac{2\phi e^{(\phi+\lambda)T/2}}{(\phi+\lambda)(e^{\phi T}-1)+2\phi}\right]^{2\lambda\theta/\eta^2} \tag{4.10}$$

$$B = \frac{2(e^{\phi T}-1)}{(\phi+\lambda)(e^{\phi T}-1)+2\phi} \tag{4.11}$$

with $\phi = \sqrt{\lambda^2 + 2z\eta^2}$. This closed-form solution for the Heston model allow us to calculate the volatility swap strike exactly, and will be the benchmark against which we measure the accuracy of our approximation.

As an aside, for numerical implementation in Matlab (and perhaps in other languages and packages as well), the authors have found that (4.10) and (4.11) may cause instability. This is because Matlab calculates the denominator and numerator separately first, and because the term $\exp(\phi T)$ becomes very large, Matlab will return an error even though the integrand as a whole is finite. The solution to this slight problem is to divide both the denominator and numerator by $\exp(\phi T)$ for stable implementation.

# 5 A new approximation for $K_{vol}$

Equation (3.9) with the specific choice $K = K_{d2}$ and the corresponding implied volatility $\Sigma = \Sigma_{d2}$ will be the starting point to derive our result. So,

$$\begin{aligned} C^{BS}(S, K_{d2}, \Sigma_{d2}, T) &\approx E_0\left[C^{BS}(S, K_{d2}, \sigma_{0,T}, T)\right] \\ &+ \rho S E_0\left[\Delta^{BS}(S, K_{d2}, \sigma_{0,T}, T)\int_0^T \sigma dW_\sigma\right] + O(\rho^2) \end{aligned} \tag{5.1}$$

Next, we Taylor expand $C^{BS}$ and $\Delta^{BS}$ on the right hand side of (5.1) around $\Sigma_{d2}$:

$$\begin{aligned} C^{BS}(S, K_{d2}, \sigma_{0,T}, T) \approx & \; C^{BS}(S, K_{d2}, \Sigma_{d2}, T) \\ & + \nu^{BS}(S, K_{d2}, \Sigma_{d2}, T)(\sigma_{0,T} - \Sigma_{d2}) \\ & + \frac{1}{2} vo^{BS}(S, K_{d2}, \Sigma_{d2}, T)(\sigma_{0,T} - \Sigma_{d2})^2 \end{aligned} \tag{5.2}$$

and similarly

$$\begin{aligned} \Delta^{BS}(S, K_{d2}, \sigma_{0,T}, T) \approx & \; \Delta^{BS}(S, K_{d2}, \Sigma_{d2}, T) \\ & + va^{BS}(S, K_{d2}, \Sigma_{d2}, T)(\sigma_{0,T} - \Sigma_{d2}) \end{aligned} \tag{5.3}$$

Since all quantities that do not depend on $\sigma_{0,T}$ can be taken outside of the expectation in (5.1), and using the fact that

$$va^{BS}(S, K_{d2}, \Sigma_{d2}, T) = vo^{BS}(S, K_{d2}, \Sigma_{d2}, T) = 0 \tag{5.4}$$

equation (5.1) simplifies to

$$\begin{aligned} C^{BS}(S, K_{d2}, \Sigma_{d2}, T) \approx & \; C^{BS}(S, K_{d2}, \Sigma_{d2}, T) \\ & + \nu^{BS}(S, K_{d2}, \Sigma_{d2}, T) E_0[(\sigma_{0,T} - \Sigma_{d2})] \end{aligned} \tag{5.5}$$

But this can only be the case if

$$\boxed{K_{vol} = E_0[\sigma_{0,T}] \approx \Sigma_{d2}} \tag{5.6}$$

As (5.6) was arrived at by approximation on approximation, we expect some deviation from the exact result. However, on the up-side, (5.6) is as straightforward as (4.6) but generalizes it to be immune to correlation to first order and therefore of more practical use. To solve it, all one needs to do is to solve (2.10), and if one does not have a solver at hand the volatility swap strike can be found by plotting the vanna or vomma of all available quoted options and optically finding the point where vanna/vomma is zero.

# 6 Numerical results

To test the accuracy of (5.6) we need to compare it to an exact result. For the Heston model there is an exact expression for the volatility strike available, given by (4.7), (4.10) and (4.11). Using the same Heston parameters as in [3] we therefore generate exact volatility swap strikes and implied volatilities for $T = \{0.5, 1, 3, 5\}$ and correlations $\rho = \{-0.9, -0.5, 0, 0.5, 0.9\}$. For each time slice and correlation value

Table 1: Test results

| T | Exact (Heston) | Correlation | | | | |
|---|---|---|---|---|---|---|
| | | -0.9 | -0.5 | 0 | 0.5 | 0.9 |
| 0.5 | 19.02 | 18.93 | 18.98 | 19.02 | 18.98 | 18.95 |
| | | 18.59 | 18.79 | 19.01 | 19.17 | 19.30 |
| 1 | 18.74 | 18.37 | 18.62 | 18.74 | 18.65 | 18.43 |
| | | 17.85 | 18.32 | 18.73 | 18.94 | 18.98 |
| 3 | 18.88 | 18.23 | 18.66 | 18.87 | 18.75 | 18.34 |
| | | 17.43 | 18.18 | 18.84 | 19.21 | 19.24 |
| 5 | 19.12 | 18.54 | 18.92 | 19.12 | 19.00 | 18.63 |
| | | 17.60 | 18.35 | 19.07 | 19.56 | 19.71 |

we then back out the ATM implied volatility, $\Sigma_{ATM}$, as well as our approximation for the volatility strike $\Sigma_{d2}$ by solving (2.10). We then compare this to the exact volatility swap strikes generated by (4.7). For $T = 0.5$ we can also compare our approximation to the one due to Carr and Lee. Throughout we assume $r = q = 0$ for simplicity.

The results of our test is depicted in Table 1. The first row gives our approximation (5.6) and the second row the ATM approximation (4.6). For $T = 0.5$ our approximation matches Carr and Lee in terms of accuracy. Somewhat surprisingly our values are practically identical to Carr & Lee's for $T = 0.5$ even though the derivation of our result is much simpler. The accuracy of our approximation is maintained for longer tenors as well, although there is a small loss of accuracy for large $T$ as correlation approaches 1. A note of caution is in place here: for longer tenors ($T > 2$) stochastic interest rates will start to impact variance and volatility swap prices. The observed market implied volatilities will have to be adjusted for interest rate volatilities before our approximation can be applied. For shorter dated volatility swaps the impact of stochastic rates is limited.

We can conclude that, with the limitations mentioned above, our approximation performs just as well as Carr & Lee's, but it is much easier to implement and to follow (from a technical point of view). A further avenue of research would be a generalization of our method to enable incorporation of stochastic interest rates, as well as possibly higher order immunization to correlation.

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