Vanna Volga and Smile-consistent Implied Volatility Surface of Equity Index Option

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Abstract

Vanna-Volga method, known as the traders rule of thumb, is commonly used in FX option market to manage implied volatility surface and hedge against the movement of underlying asset price. However, this method has not attracted much attention in other derivative markets. This essay investigates Vanna-Volga method and two approximation of Vanna-Volga implied volatility in equity option market. By pricing European call option written on S&P 500 index, the numerical results evidence the efficiency of Vanna-Volga method and its two approximation for building smile-consistent implied volatility of equity index option.

Keywords: arbitrage free, equity index option, implied volatility, Vanna, Vega, Volga.

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1 Introduction

By virtue of huge liquidity and price transparency, the FX market is in a strong position when comes to taking a precise approach to study the intricacies of options [2]. Vanna-Volga method is commonly adopted to price the first generation of exotic option in FX option market, but it has not attracted much attention in other derivative markets. This paper investigates the efficiency of Vanna-Volga method and two approximation of Vanna-Volga implied volatility in equity option market.

For option pricing, the nontrivial issue is how to build a smooth, consistent and arbitrage-free implied volatility surface that encapsulates the information of the distribution of underlying asset price for a given maturity. Building implied volatility surface requires the full continuum of option price across expiry and strike. However, only a discrete set of option prices are observable in the market. Therefore, for a given expiry, the entries of liquidly traded options on volatility surface can be computed directly by their market prices, whereas the rest of volatility surface must resort to interpolation and extrapolation. Consequently, we need an efficient tool for arbitrage-free interpolation and extrapolation of volatility surface in both strike and maturity dimension.

Vanna-Volga method is an efficient approach when it is used to interpolate smile-consistent implied volatility of currency option for a given maturity. This method is easy to be implemented and only three market quotes of liquidly traded instruments are required. Vanna-Volga method depends on the construction of a locally replicating portfolio which is vega-neutral in Black-Scholes flat-smile world. It can yield implied volatility for any options' delta, particularly for those outside the basic range set by the Δ_{25} put and call quotes. It allows one to compute different vega risk in a consistent manner and hedge exotic option. However, Vanna-Volga method may produce negative price for extremely large Risk Reversal values in currency market and for extreme strikes in equity market. Therefore, Vanna-Volga procedure must be handled with care when the wings are valued.

Vanna-Volga method was first introduced in literature by Lipton and McGhee (2002). They analyse various pricing inconsistencies that arise from the non-rigorous nature of the technique, and adjust the Black-Scholes value of double-no-touch options by incorporating the hedging cost. Vanna-Volga method is applied on one-touch option in currency market in Wystup (2003). Castagna and Mercurio (2007) detail the Vanna-Volga procedure and provide the mathematical justification on vanilla option. They suggest that Vanna-Volga method can be extended efficiently to other markets. Fisher (2007) suggests a number of corrections of Vanna-Volga to handle the pricing inconsistencies. Shkolnikov (2009) presents a more rigorous and theoretical justification and extends Vanna-Volga method to include interest-rate risk. Bossens, Rayee, Skantzos and Deelstra (2010) describe two variations and propose a simple calibration method for pricing a wide range of exotic options.

By taking the advice in Castagna and Mercurio (2007), this paper investigates the efficiency of Vanna-Volga method when it is used for pricing equity index option. To the best of my knowledge, this is the first paper studying the performance of Vanna-Volga method on pricing equity option

[2]In FX market, the typical books have a small number of underlying securities and a massively complex positions, which often broken into only two or three big option books with a huge number of strikes. The books of the equity stock market are usually consist of a number of small but not complex positions in a wide range of underlying securities. This fact gives rise to that the equity market has grown more in the direction of correlation-based products.

written on S&P 500 index. Apart from Vanna-Volga pricing formula, two approximation of Vanna-Volga implied volatility are also studied in this paper. By pricing the same European call option, the numerical experiments evidence the efficiency of these three approaches. They can perfectly match the market price within the interval of three quotes. For ITM option, the bias between market price and the 1st approximation of Vanna-Volga implied volatility is the smallest, even for the extremely short maturities. The study results show the high efficiency of Vanna-Volga method and its two approximation for generating the smile-consistent implied volatility in equity option market.

The rest of the paper is organized as follows. Vanna-Volga method and two approximation of Vanna-Volga implied volatility are introduced in Section 2. Section 3 details the application of Vanna-Volga method in equity option market. The numerical experiments are presented in Section 4. Finally, Section 5 concludes.

2 Vanna-Volga Method

Vanna-Volga method is based on the construction of locally risk free replicating portfolio whose hedging costs are added to the Black-Scholes option price to produce smile-consistent prices. It yields a good approximation of volatility smile, especially within the range delimited by the two extreme strikes. Vanna-Volga method has several advantages. First, it is an efficient tool for interpolating and extrapolating volatility for a given maturity while reproducing exactly the market quoted volatilities. Second, it can be employed in any market where at least three volatility quotes are available for a given maturity. Third, this method can derive implied volatilities for any options delta, particularly for those outside the basic range set by the Δ_{25} put and call quotes. Fourth, this non-parametric method produces a consistent and complete smile with just three prices for each maturity. Fifth, it is supported by a clear financial rationale based on a hedging argument. Finally, this method allows for the automatic calibration to the three input volatilities derived from market prices and acts as an explicit function of them.

2.1 Vegga, Vanna and Volga

The measurement of the sensitivity of option value with respect to the change of either the state variable or the model parameter is known as the *Greek*. The *Greeks* Vega, Vanna and Volga are related to the sensitivities of option value with respect to the change of volatility.

2.1.1 Vega

If the financial derivative has a convex structure, it has a Vega; if the financial derivative has a linear structure such as a forward, then it does not have Vega. The highly positive or negative Vega implies that the portfolio is very sensitive to the small changes of volatility. If the value of Vega is close to zero, it suggests that the volatility has little impact on the value of the portfolio. Vega tells the change in value of the portfolio with respect to a discrete move in volatility for a given percentage level, such as the change of option price with respect to a one percent point change of the volatility.

Vega follows a bell shape. The ATM Vega is the peak and it decreases more and more for deep ITM and OTM options. The ATM Vega is stable to volatility but it is convex for deep ITM and OTM Vegas. Generally, the Vega of most derivatives decreases with time. For some exotic options, Vega increases with time under certain conditions, such as lookback and reverse knock out options.

Vega is important for design and maintenance of an effective hedging, and the hedging shouls be adjusted as Vega moves. It is easy to understand Vega of plain vanilla option. However, when it comes to exotic option, it becomes crucial to monitor the change of Vega with respect to other parameters, such as the spot and implied volatility. Vega is derived by

$$\text{Vega} = S_0\sqrt{T-t}N'(d_1)e^{-q(T-t)} \tag{1}$$

In case of no dividend, i.e. $q = 0$, equation (1) is equivalent to:

$$\text{Vega} = S_0\sqrt{T-t}N'(d_1) \tag{2}$$

where

$$\begin{aligned} N'(d_1) &= e^{(-d_1^2/2)}\frac{1}{2\pi} \\ d_1 &= \frac{ln(S/K) + r - q + \sigma^2/2(T-t)}{\sigma\sqrt{T-t}} \end{aligned} \tag{3}$$

Vega in terms of Gamma is given by:

$$\begin{aligned} \text{Vega} &= \text{Gamma} \cdot S_t^2(T-t)\sigma \\ &= \frac{\partial\Delta}{\partial S}S_t^2(T-t) \\ &= \frac{N'(d_1)}{S_t\sigma\sqrt{T-t}}S_t^2(T-t) \end{aligned} \tag{4}$$

The Black-Scholes model cannot take care of the sensitivity of Vega due to the Vega-neutral position is subject to changes of spot and volatility. Therefore, we need to know the sensitivity of Vega to the changes in spot and implied volatility. The measurement of its sensitivity to these two parameters are represented by Vanna and Volga, respectively.

2.1.2 Vanna

The option's Vanna, which is the second order cross Greek, represents the risk to the skew increasing. It is used to monitor the *Vega exposure* or *cross Gamma risk on Delta* with respect to the change of the spot. Vanna can be defined in three different ways:

- $\frac{\partial V}{\partial S}$: the change of Vega V with respect to the change in underlying price S
- $\frac{\partial \Delta}{\partial \sigma}$: the sensitivity of Delta Δ with respect to the change in volatility σ
- $\frac{\partial^2 P}{\partial \sigma \partial S}$: the sensitivity of option value P with respect to a joint movement in

volatility σ and the underlying price S

In Black-Scholes model, Vanna of simple option with closed form is derived by:

$$\text{Vanna} = e^{-qt}\sqrt{T-t}N'(d_1)(d2/\sigma) \tag{5}$$

In case of no dividend, i.e. $q = 0$, equation (5) is equivalent to:

$$\text{Vanna} = \sqrt{T-t}N'(d_1)(1-d_1) \tag{6}$$

The algebraic expression of Vanna in terms of Vega reads:

$$\begin{aligned}\text{Vanna} &= \text{Vega} \cdot \frac{d_2}{S\sigma} \\ &= S\sqrt{T-t}N'(d_1)\frac{d_2}{S\sigma} \\ &= \sqrt{T-t}N'(d_1)(\frac{d_2}{\sigma})\end{aligned} \tag{7}$$

The call and put options with the same strike K have same Vanna.

2.1.3 Volga

The option's Volga or volatility Gamma represents the sensitivity of Vega with respect to the change in volatility and shows the risk to the smile becoming more pronounced. It measures the convexity of option price with respect to volatility. The relationship between convexity and duration is same as the relationship between Gamma and Delta. The option with high volga can benifit from volatility of volatility. Volga can be defined in two different ways:

- $\frac{\partial V}{\partial \sigma}$: the change in Vega V with respect to a change in volatility σ
- $\frac{\partial^2 P}{\partial \sigma^2}$: the second derivative of option value P with respect to changes in volatility σ

$$\text{Volga} = e^{-qt}\sqrt{T-t}N'(d_1)(\frac{d_1 d_2}{\sigma}) \tag{8}$$

where $d_2 = d_1 - \sigma\sqrt{T-t}$. Volga in terms of Vega is expressed as:

$$\begin{aligned}\text{Volga} &= \text{Vega} \cdot \frac{d_1 d_2}{S\sigma} \\ &= S\sqrt{T-t}N'(d_1)\frac{d_1 d_2}{S\sigma} \\ &= \sqrt{T-t}N'(d_1)\frac{d_1 d_2}{\sigma}\end{aligned} \tag{9}$$

2.2 The Vanna-Volga Option Pricing Formula

The Vanna-Volga option price $C^{\text{VV}}(K)$ is obtained by adding to the Black-Scholes theoretical price $C^{\text{BS}}(K)$ the cost difference of the hedging portfolio induced by the market implied volatilities with respect to the constant volatility σ:

$$C^{\mathrm{VV}}(K) = C^{\mathrm{BS}}(K) + \sum_{i=1}^{3} x_i(K)\Big(C^{\mathrm{M}}(K_i) - C^{\mathrm{BS}}(K_i)\Big) \tag{10}$$

where $C^{\mathrm{M}}(K)$ denotes the observed market call option price for strike K.

The Vanna-Volga option pricing formula (10) was proposed without assumption of the distribution of the underlying asset price. The first step is to build a hedging portfolio of three options $C(K_i, T)$ with same maturity T but different strikes $K_i, \{i = 1, 2, 3\}$, so that the portfolio can hedge the price variations of the call $C(K, T)$ with maturity T and strike K , up to the second order in the underlying and the volatility. Denothing by Δ_t and x_i the units of underlying asset and options with strike K_i held at time t, respectively, under diffusion dynamics both for S_t and σ_t, by *Itô*'s lemma we have:

$$\begin{aligned} dC^{BS}(t,K) - \Delta_t dS_t - \Delta_t \delta S_t dt - \sum_{i=1}^{3} x_i dC_i^{BS}(t) =& [\frac{\partial C^{BS}(t,K)}{\partial t} - \sum_{i=1}^{3} x_i \frac{\partial C^{BS}(t)}{\partial t} - \Delta_t \delta S_t]dt \\ &+ [\frac{\partial C^{BS}(t,K)}{\partial S} - \Delta_t - \sum_{i=1}^{3} x_i \frac{\partial C_i^{BS}(t)}{\partial S}]dS_t \\ &+ [\frac{\partial C^{BS}(t,K)}{\partial \sigma} - \sum_{i=1}^{3} x_i \frac{\partial C_i^{BS}(t)}{\partial \sigma}]d\sigma_t \\ &+ \frac{1}{2}[\frac{\partial^2 C^{BS}(t,K)}{\partial S^2} - \sum_{i=1}^{3} x_i \frac{\partial^2 C_i^{BS}(t)}{\partial S^2}](dS_t)^2 \\ &+ \frac{1}{2}[\frac{\partial^2 C^{BS}(t,K)}{\partial \sigma^2} - \sum_{i=1}^{3} x_i \frac{\partial^2 C_i^{BS}(t)}{\partial \sigma^2}]d\sigma_t d\sigma_t \\ &+ [\frac{\partial^2 C^{BS}(t,K)}{\partial S \partial \sigma} - \sum_{i=1}^{3} x_i \frac{\partial^2 C_i^{BS}(t)}{\partial S \partial \sigma}]dS_t d\sigma_t \end{aligned} \tag{11}$$

We zero out the coefficients of $dS_t, d\sigma_t, d\sigma_t d\sigma_t$, and $dS_t d\sigma_t$, so that no stochastic terms are involved in its differential. Accordingly, the replicating portfolio is locally risk-free at time t (given that Gamma and other higher order risks can be ignored) and has a return at risk free rate. Applying the Black-Scholes partial differential equation, we get:

$$dC^{BS}(t,K) - \Delta_t dS_t - \Delta_t \delta S_t dt - \sum_{i=1}^{3} x_i dC_i^{BS}(t) = r\Big(C^{BS}(t,K) - \Delta_t S_t - \sum_{i=1}^{3} x_i C_i^{BS}(t)\Big)dt \tag{12}$$

Equation (12) shows that, when volatility is stochastic and option are priced by the Black-Scholes formula, one still have a locally perfect hedge.

It is assumed that the position is Delta-hedged, and the replicating portfolio in Black-Scholes flat-smile world is both Vega-neutral and Gamma-neutral. Under these assumptions, the weights $x_i\,\{i = 1, 2, 3\}$ can be solved by imposing that the replicating portfolio and call option have the same Vega (i.e. $\partial C^{BS}/\partial\sigma$), Vanna (i.e. $\partial^2 C^{BS}/\partial\sigma\partial S$), and Volga (i.e. $\partial^2 C^{BS}/\partial\sigma^2$) :

$$\begin{aligned}\frac{\partial C^{BS}}{\partial\sigma}(K) &= \sum_{i=1}^{3} x_i(K)\frac{\partial C^{BS}}{\partial\sigma}(K_i)\\ \frac{\partial^2 C^{BS}}{\partial\sigma^2}(K) &= \sum_{i=1}^{3} x_i(K)\frac{\partial^2 C^{BS}}{\partial\sigma^2}(K_i)\\ \frac{\partial^2 C^{BS}}{\partial\sigma\partial S_0}(K) &= \sum_{i=1}^{3} x_i(K)\frac{\partial^2 C^{BS}}{\partial\sigma\partial S_0}(K_i)\end{aligned} \tag{13}$$

By solving equations of the system (13) with

$$\frac{\partial C^{BS}}{\partial\sigma}(K) = S_0\ \mathrm{e}^{-qT}\sqrt{T}\ \phi\big(d_1(K)\big) \tag{14}$$

$$\frac{\partial^2 C^{BS}}{\partial\sigma^2}(K) = \frac{\nu(K)}{\sigma}d_1(K)d_2(K) \tag{15}$$

$$\frac{\partial^2 C^{BS}}{\partial\sigma\partial S_0}(K) = -\frac{\nu(K)}{S_0\ \sigma\sqrt{T}}d_2(K) \tag{16}$$

$$d_1(K) = \frac{\ln(S_0/K) + (r - q + 1/2\ \sigma^2)\ T}{\sigma\sqrt{T}} \tag{17}$$

$$d_2(K) = d_1(K) - \sigma\sqrt{T} \tag{18}$$

we derive the unique solution of the weights:

$$\begin{aligned}x_1(K) &= \frac{\nu(K)}{\nu(K_1)}\frac{\ln(K_2/K)\ln(K_3/K)}{\ln(K_2/K_1)\ln(K_3/K_1)}\\ x_2(K) &= \frac{\nu(K)}{\nu(K_2)}\frac{\ln(K/K_1)\ln(K_3/K)}{\ln(K_2/K_1)\ln(K_3/K_2)}\\ x_3(K) &= \frac{\nu(K)}{\nu(K_3)}\frac{\ln(K/K_1)\ln(K/K_2)}{\ln(K_3/K_1)\ln(K_3/K_2)}\end{aligned} \tag{19}$$

where $\nu(K) = \frac{\partial C^{BS}}{\partial\sigma}(K)$, and $\phi(\cdot)$ is the normal density function. If $K = K_j$, then $x_i(K) = 1$ for $i = j$ and zero otherwise. The Vanna-Volga option price $C^{\mathrm{VV}}(K)$ is twice differentiable and satisfies the following no-arbitrage conditions:

1. $\lim_{K\to 0} C^{\mathrm{VV}}(K) = S_0 e^{-\delta T}$ and $\lim_{K\to+\infty} C^{\mathrm{VV}}(K) = 0$
2. $\lim_{K\to 0}(\partial C/\partial K)(K) = -e^{-rT}$ and $\lim_{K\to+\infty} K(\partial C/\partial K)(K) = 0$

These properties follow from the fact that, for each i, both $x_i(K)$ and $\partial x_i(K)/\partial K$ go to zero for $K \to 0$ or $K \to +\infty$. To avoid arbitrage, the Vanna-Volga option price $C^{\mathrm{VV}}(K)$ should be a convex function of the strike K, i.e. $\partial^2 C(K)/\partial K^2 > 0$ for each $K > 0$. This property holds for typical market parameters. Therefore, the equation (10) leads to the arbitrage-free option price.

2.3 The Risk-neutral Density

For each strike, the option price identifies its consistent and the unique risk-neutral density. The study of Breeden and Litzenberger (1978) shows that the second derivative of call option price with respect to strike price yields the risk-neutral density $p(K,T)$:

$$
\begin{aligned}
p(K,T) &= \mathrm{e}^{rT}\ \frac{\partial^2 C}{\partial K^2} && (20)\\
&= \mathrm{e}^{rT}\Big(\frac{\partial^2 C^{BS}}{\partial K^2}(K) + \sum_{i\in 1,2,3}\frac{\partial^2 x_i}{\partial K^2}(K)\Big(C^M(K_i) - C^{BS}(K_i)\Big)\Big) && (21)\\
&= \mathrm{e}^{rT}\frac{\partial^2 C^{BS}}{\partial K^2}(K) + \mathrm{e}^{rT}\sum_{i\in 1,2,3}\frac{\partial^2 x_i}{\partial K^2}(K)\Big(C^M(K_i) - C^{BS}(K_i)\Big) && (22)
\end{aligned}
$$

with

$$
\begin{aligned}
\frac{\partial^2 x_1}{\partial K^2}(K) =& \frac{\nu(K)}{K^2\sigma^2 T\nu(K_1)\ln(K_2/K_1)\ln(K_3/K_1)}\\
&\times\Big(\Big(d_1(K)^2 - \sigma\sqrt{T}d_1(K) - 1\Big)\ln\frac{K_2}{K}\ln\frac{K_3}{K}\\
&- 2\sigma\sqrt{T}d_1(K)\ln\frac{K_2K_3}{K^2} + \sigma^2 T\Big(\ln\frac{K_2K_3}{K^2} + 2\Big)\Big) && (23)\\
\frac{\partial^2 x_3}{\partial K^2}(K) =& \frac{\nu(K)}{K^2\sigma^2 T\nu(K_3)\ln(K_3/K_1)\ln(K_3/K_2)}\\
&\times\Big(\Big(d_1(K)^2 - \sigma\sqrt{T}d_1(K) - 1\Big)\ln\frac{K_2}{K}\ln\frac{K_1}{K}\\
&- 2\sigma\sqrt{T}d_1(K)\ln\frac{K_1K_2}{K^2} + \sigma^2 T\Big(\ln\frac{K_1K_2}{K^2} + 2\Big)\Big) && (24)
\end{aligned}
$$

The first term in the right hand side of formula (22) is Black-Scholes log-normal density with drift $r - q$ and volatility $\sigma = \sigma_2$.

2.4 The 1st and the 2nd Approximation of Vanna-Volga Implied Volatility

The option pricing formula (10) combined with the system of equations (19) lead to a straightforward approximation for implied volatility. By expanding both members of equation (10) at first order in $\sigma = \sigma_2$, the Vanna-Volga call option price is approximated as:

$$
C^{VV}(K) \approx C^{BS}(K) + \sum_{i=1}^{3} x_i(K)\nu(K_i)(\sigma_i - \sigma) \qquad (25)
$$

Since the unique solution of weights x_i and the fact that $\sum_{i=1}^{3} x_i(K)\nu(K_i) = \nu(K)$, equation (25) leads to

$$
C^{VV}(K) \approx C^{BS}(K) + \nu(K)\Big(\sum_{i=1}^{3}\mathcal{X}_i(K)\sigma_i - \sigma\Big) \qquad (26)
$$

where

$$\begin{aligned}\mathcal{X}_1(K) &= \frac{\ln(K_2/K)\,\ln(K_3/K)}{\ln(K_2/K_1)\,\ln(K_3/K_1)}\\ \mathcal{X}_2(K) &= \frac{\ln(K/K_1)\,\ln(K_3/K)}{\ln(K_2/K_1)\,\ln(K_3/K_2)}\\ \mathcal{X}_3(K) &= \frac{\ln(K/K_1)\,\ln(K/K_2)}{\ln(K_3/K_1)\,\ln(K_3/K_2)}\end{aligned} \tag{27}$$

Comparing equation (26) with the first-order Taylor expansion

$$C^{VV}(K) \approx C^{BS}(K) + \nu(K)\Big(\varrho(K) - \sigma\Big) \tag{28}$$

we derive the **first approximation** of implied volatility:

$$\varrho(K) \approx \varrho_1(K) := \mathcal{X}_1(K)\sigma_1 + \mathcal{X}_2(K)\sigma_2 + \mathcal{X}_3(K)\sigma_3 \tag{29}$$

The equation (29) shows that the implied volatility $\varrho(K)$ can be approximated by a simple linear combination of volatilities σ_1, σ_2 and σ_3, and their weights $\mathcal{X}_1$, $\mathcal{X}_2$ and $\mathcal{X}_3$ sum to one. Apparently, this approximation is a quadratic function of log-strike. It suggests that we can resort to a simple parabolic interpolation when log coordinates are used. Nevertheless, due to this approximation is a quadratic function of log-strike, the arbitrage free condition derived by Lee (2004) for the asymptotics of implied volatility are violated. The second approximation proposed by Castagna and Mercurio (2007 b), which is asymptotically constant at extreme strikes, aims to overcome this drawback. By expanding both members of equation (10) at second order in $\sigma = \sigma_2$, we get

$$C(K) \approx C^{BS}(K) + \sum_{i=1}^{3} x_i(K)\left[\nu(K_i)(\sigma_i - \sigma) + \frac{1}{2}\frac{\partial^2 C^{BS}}{\partial^2\sigma}(K_i)(\sigma_i - \sigma)^2\right] \tag{30}$$

The second-order Taylor expansion yields

$$C(K) - C^{BS}(K) \approx \nu(K)\Big(\varrho(K) - \sigma\Big) + \frac{1}{2}\frac{\partial^2 C^{BS}}{\partial^2\sigma}(K)\Big(\varrho(K) - \sigma\Big)^2 \tag{31}$$

Comparing equation (30) with equation (31), we get

$$\begin{aligned}&\nu(K)\Big(\varrho(K) - \sigma\Big) + \frac{1}{2}\frac{\partial^2 C^{BS}}{\partial^2\sigma}(K)\Big(\varrho(K) - \sigma\Big)^2\\ &\approx \sum_{i=1}^{3} x_i(K)\left[\nu(K_i)(\sigma_i - \sigma) + \frac{1}{2}\frac{\partial^2 C^{BS}}{\partial^2\sigma}(K_i)(\sigma_i - \sigma)^2\right]\end{aligned} \tag{32}$$

The **second approximation** of implied volatility is obtained by solving equation (32) :

$$\varrho(K) \approx \varrho_2(K) := \sigma_2 + \frac{-\sigma_2 + \sqrt{\sigma_2^2 + d_1(K)d_2(K)\big[2\sigma_2 D_1(K) + D_2(K)\big]}}{d_1(K)d_2(K)} \tag{33}$$

where

$$
\begin{aligned}
D_1(K) =& \frac{\ln(K_2/K)\ \ln(K_3/K)}{\ln(K_2/K_1)\ \ln(K_3/K_1)}\ \sigma_1 + \frac{\ln(K/K_1)\ \ln(K_3/K)}{\ln(K_2/K_1)\ \ln(K_3/K_2)}\ \sigma_2 \\
&+ \frac{\ln(K/K_1)\ln(K/K_2)}{\ln(K_3/K_1)\ \ln(K_3/K_2)}\ \sigma_3 - \sigma_2 \\
D_2(K) =& \frac{\ln(K_2/K)\ \ln(K_3/K)}{\ln(K_2/K_1)\ \ln(K_3/K_1)}\ d_1(K_1)\ d_2(K_1)(\sigma_1 - \sigma_2)^2 \\
&+ \frac{\ln(K/K_1)\ \ln(K/K_2)}{\ln(K_3/K_1)\ \ln(K_3/K_2)}\ d_1(K_3)\ d_2(K_3)(\sigma_3 - \sigma_2)^2
\end{aligned}
\tag{34}
$$

Castagna and Mercurio (2007 b) argued that the second approximation is not only accurate within the interval $[K_1, K_3]$, but also in the wings, even for extreme values of put Deltas. However, although the radiant is positive in most practical applications, the volatility $\varrho(K)$ may not be defined due to the presence of a square-root term.

In order to exhibit the goodness of the 1st and 2nd approximation of implied volatility, a graphical example is provided by Figure 1. It compares the volatility smiles of currency option generated by Vanna-Volga option pricing formula, 1st approximation and 2nd approximation. The plots are generated with the following data: $S_0 = 1.195$, $T = \{14D, 1M, 6M, 9M, 1Y, 2.5Y, 5Y, 10Y, 15Y\}$, $K_1 = 1.18$, $K_2 = 1.22$, $K_3 = 1.265$, $\sigma_1 = 9.43\%$, $\sigma_2 = 9.05\%$, $\sigma_3 = 8.93\%$. The discount factor for domestic and foreign markets are $r_d = 0.9902752$ and $r_f = 0.9945049$, respectively. For each T, the discount factor is rescaled by $-\log(r_d)/T$ and $-\log(r_f)/T$. The plots discover that both the 1st and 2nd approximation can match Vanna-Volga implied volatility perfectly at the ATM region. For expiration less than 2.5 years, the implied volatility generated by the 2nd approximation is much closer to Vanna-Volga implied volatility, even for the wings. The 1st approximation overestimates the volatility on both wings for all expirations. However, as $T \to \infty$, the 2nd approximation tends to produce the same implied volatility as the 1st approximation.

3 Applying Vanna-Volga Method in Equity Option Market

For a given expiration, although a volatility smile has as many degrees of freedom as considered strikes, it is reasonable to assume that there are only three degrees of freedom: *level*, *steepness* and *convexity*. Practically, most of shape variations can be explained by a parallel shift of the smile, by a tilt to the right or to the left, or by a relative change of wings with respect to the center strike. In FX option market, the application of Vanna-Volga method requires three quotes for a given expiration:

- **ATM volatility associated with delta-neutral straddle** [3]: it is the indicator of the level of volatility smile
- **Risk Reversal for 25 delta call and put** [4]: it is the measure of the steepness of the smile

[3] Delta-neutral straddle denotes $\Delta_C + \Delta_P = 0$, with Δ_C and Δ_P represents delta of call and put option, respectively.

[4] 25 delta means the level of delta is 25%; 25 delta call is a call option whose delta is 25%; 25 delta put is a put option whose delta is −25%.

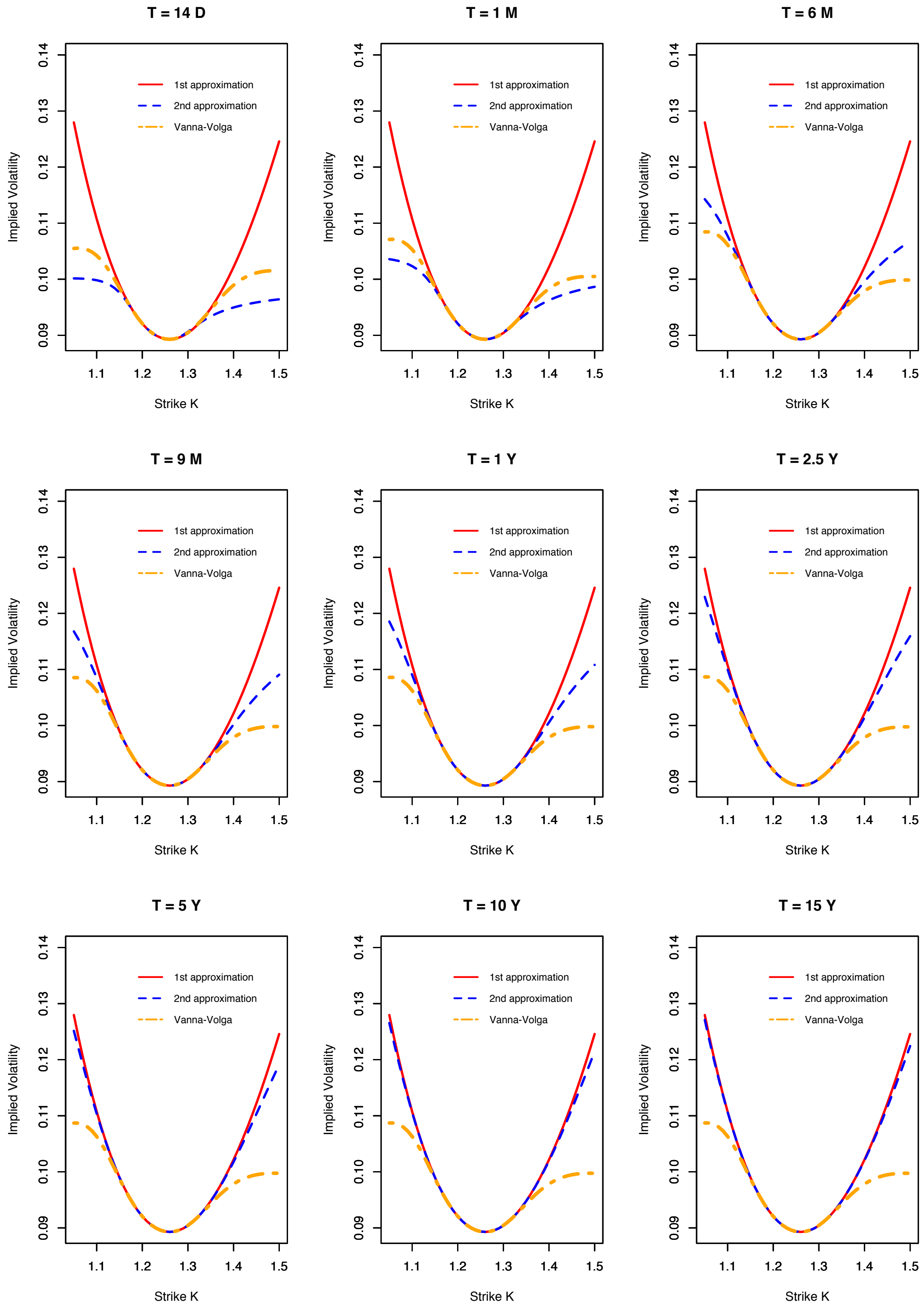

Figure 1: Vanna Volga Implied Volatility, 1st and 2nd Approximation of Implied Volatility

- **Vega Weighted Butterfly with 25 delta wings**: it is the measure of convexity

These two delta level are introduced due to they are almost midway between the center of the smile and the extreme wings (zero delta put and zero delta call) and due to they are the strikes associated with high Volga, hence containing a good deal of information on the underlying asset's fourth moment, and thus on the curvature of the smile. Wystup (2008) argued that it is not clear up front which target delta to use for Risk Reversal and Butterfly. In his study, the delta level is determined merely based on the basis of its liquidity.

In FX option market, the implied volatility smile is built by sticky delta rule. The convention of equity option market is different from that of the FX option market. When Vanna-Volga method is applied in equity market, we need a volatility matrix that is presented in the same compact form as in the FX option market (i.e., the volatility matrix provides us the ATM implied volatility, the 25Δ Risk Reversal, and the 25Δ Butterfly for each expiration).

The procedure for building such a volatility matrix is outlined in the **Algorithm** below:

1. For each expiration, find out the strike $\hat{K}_2$ which is the nearest to the forward price and its corresponding implied volatility $\hat{\sigma}_2$. The implied volatility $\hat{\sigma}_2$ will be used for iteration in step 3 and 4.

2. Detect two strikes $\hat{K}_1$ and $\hat{K}_3$ which yield the absolute value of delta of put and call as near as possible to 25%, satisfying $\hat{K}_1 < \hat{K}_2 < \hat{K}_3$. And back out their corresponding volatilities $\hat{\sigma}_1$ and $\hat{\sigma}_3$. The implied volatilities $\hat{\sigma}_1$ and $\hat{\sigma}_3$ will be used for iteration in step 4.

3. Compute the ATM strike K_2 and its corresponding volatility σ_2 by an iterative procedure. The ATM strike is referred to zero-delta straddle. For each given expiry, it is chosen so that a put and a call have the same delta but with different sign. Accordingly, denoting by σ_{ATM} and K_{ATM} the ATM volatility and ATM strike, respectively, we get

$$e^{-qT}\Phi\left(\frac{ln(S_0/K_{ATM})(r-q+\frac{1}{2}\sigma_{ATM}^2)T}{\sigma_{ATM}\sqrt{T}}\right) = e^{-qT}\Phi\left(-\frac{ln(S_0/K_{ATM})(r-q+\frac{1}{2}\sigma_{ATM}^2)T}{\sigma_{ATM}\sqrt{T}}\right) \tag{35}$$

   Remember that $K_2 = K_{ATM}$ and $\sigma_2 = \sigma_{ATM}$. From equation (35), we know ATM strike can be computed by

$$K_{ATM} = S_0 e^{(r-q+\frac{1}{2}\sigma^2)T} \tag{36}$$

   As long as we obtain the ATM strike, we can compute option price using equation (10). Then, back out the ATM implied volatility using Black-Scholes pricing formula. Explicitly, the iterative procedure is:

   i. Set the constant volatility σ to be $\hat{\sigma}_2$,

   ii. $K^i_{ATM} = S_0 e^{(r-q+\frac{1}{2}\sigma^2_{i-1})T}$ (i denotes ith iteration; for $i = 1, \sigma_{i-1} = \sigma$),

   iii. $C(K^i_{ATM}) = C^{BS}(K^i_{ATM}) + \sum_{j=1}^{3} x_j(K^i_{ATM})[C^{MKT}(K_j) - C^{BS}(K_j)]$, with C^{BS} derived by plugging in the Black-Scholes equation the constant volatility σ,

   iv. $\sigma_i = (C^{BS})^{-1}\Big(C(K^i_{ATM})\Big)$ (this formula implies that plugging σ_i into the Black-Scholes formula will yield $C(K^i_{ATM})$),

v. Iterate from point (ii) until $K^i_{ATM} - K^{i-1}_{ATM} < \epsilon$, ϵ suitably small,

vi. Obtain the ATM strike K_{ATM} with its implied volatility σ_{ATM}.

4. Compute implied volatilities and strikes of 25 delta call and put by iteration. In order to derive the strike, we first need to calculate the implied volatility of 25 delta call and put in terms of Risk Reversal σ_{RR} and Vega Weighted Butterfly σ_{VWB} . The formulae of σ_{RR} and σ_{VWB}

$$\sigma_{RR} = \sigma_{25\Delta C} - \sigma_{25\Delta P} \tag{37}$$

$$\sigma_{VWB} = \frac{\sigma_{25\Delta C} + \sigma_{25\Delta P}}{2} - \sigma_{ATM} \tag{38}$$

lead to

$$\sigma_{25\Delta C} = \sigma_{ATM} + \sigma_{VWB} + \frac{1}{2}\sigma_{RR} \tag{39}$$

$$\sigma_{25\Delta P} = \sigma_{ATM} - \sigma_{VWB} + \frac{1}{2}\sigma_{RR} \tag{40}$$

The delta of call and put are computed by

$$\begin{aligned} e^{-rT}\Phi(\frac{ln(S_0/K_{25\Delta C}) + (r + \frac{1}{2}\sigma^2_{25\Delta P})T}{\sigma_{25\Delta P}\sqrt{T}}) = 0.25 \\ -e^{-rT}\Phi(-\frac{ln(S_0/K_{25\Delta C}) + (r + \frac{1}{2}\sigma^2_{25\Delta P})T}{\sigma_{25\Delta P}\sqrt{T}}) = -0.25 \end{aligned} \tag{41}$$

the straightforward algebra yields

$$K_{25\Delta C} = S_0 e^{\alpha\sigma_{25\Delta C}\sqrt{T} + (r + \frac{1}{2}\sigma^2_{25\Delta C})T} \tag{42}$$

$$K_{25\Delta P} = S_0 e^{-\alpha\sigma_{25\Delta P}\sqrt{T} + (r + \frac{1}{2}\sigma^2_{25\Delta P})T} \tag{43}$$

$$\alpha = -\Phi^{-1}(\mathbb{H}\ e^{rT}) \tag{44}$$

where $\mathbb{H}$ denotes the absolute value of delta level, and Φ^{-1} is the inverse normal distribution function. We assume that α is positive for typical market parameters and maturities up to two years. The strikes must satisfy $K_{25\Delta P} < K_{ATM} < K_{25\Delta C}$.

The iterative procedure for 25 delta call is:

i. $K^i_{25\Delta C} = S_0 e^{\alpha\sigma_{i-1,25\Delta C}\sqrt{T} + (r + \frac{1}{2}\sigma^2_{i-1,25\Delta C})T}$,
(*i* denotes *i*th iteration; for $i = 1$, $\sigma_{i-1,25\Delta C} = \hat{\sigma}_3$),

ii. $C(K^i_{25\Delta C}) = C^{BS}(K^i_{25\Delta C}) + \sum_{j=1}^{3} x_j(K^i_{25\Delta C})[C^{MKT}(K_j) - C^{BS}(K_j)]$, with C^{BS} derived by plugging in the Black-Scholes equation the constant volatility σ,

iii. $\sigma_{i,25\Delta C} = (C^{BS})^{-1}C(K^i_{25\Delta C})$, (this formula implies that plugging $\sigma_{i,25\Delta C}$ into the Black-Scholes formula will yield $C(K^i_{25\Delta C})$),

iv. iterate from step (ii) until $K^i_{25\Delta C} - K^{i-1}_{25\Delta C} < \epsilon$, ϵ suitably small,

v. Now, obtain strikes $K_{25\Delta C}$ and the corresponding implied volatilities $\sigma_{25\Delta C}$.

The procedure i to v should be repeated for each expiration.

The iterative procedure for 25 delta put is:

i. $K^i_{25\Delta P} = S_0 e^{-\alpha\sigma_{i-1,25\Delta P}\sqrt{T}+(r+\frac{1}{2}\sigma^2_{i-1,25\Delta P})T}$,
(i denotes ith iteration; for $i = 1$, $\sigma_{i-1,25\Delta P} = \hat{\sigma}_1$),

ii. $C(K^i_{25\Delta P}) = C^{BS}(K^i_{25\Delta P}) + \sum_{j=1}^{3} x_j(K^i_{25\Delta P})[C^{MKT}(K_j) - C^{BS}(K_j)]$, with C^{BS} derived by plugging in the Black-Scholes equation the constant volatility σ,

iii. $\sigma_{i,25\Delta P} = (C^{BS})^{-1}C(K^i_{25\Delta P})$, (this formula implies that plugging $\sigma_{i,25\Delta P}$ into the Black-Scholes formula will yield $C(K^i_{25\Delta P})$),

iv. iterate from step (ii) until $K^i_{25\Delta P} - K^{i-1}_{25\Delta P} < \epsilon$,$\epsilon$ suitably small,

v. Now, obtain strikes $K_{25\Delta P}$ and the corresponding implied volatilities $\sigma_{25\Delta P}$.

The procedure i to v should be repeated for each expiration.

5. So far, we have built the volatility smile for the traded expirations by implementing above procedures. Now, we can interpolate / extrapolate a volatility surface in terms of fixed time-to-maturity periods.

4 Numerical Experiments

My study investigates the Vanna-Volga method and its two approximation by pricing call option written on S&P 500 index, April 22, 2016. The data is provided by OptionMetrics. The forward price F is derived via Put-Call parity. The pseudo code in Appendix D outlines the computation of the forward price using optimization method. After deriving the forward price F, the dividend q is computed by

$$q = r - \frac{\log(F/S_0)}{T} \tag{45}$$

where r denotes risk-free rate, and the underlying price S_0 of April 22, 2016 is 2091.58. My results of forward price and the related dividend for different expirations are presented in Table 1.

After obtaining the forward price and dividend, I implement the first two steps of the iterative procedure (outlined in **Algorithm** in Section 3) for each expiration to detect $\hat{K}_i$ and $\hat{\sigma}_i$, $i = 1, 2, 3$. The results are listed in Table 2. For each expiration, the ATM strike $\hat{K}_2$ is close to the forward price presented in Table 1. The values of $\Delta(P)$ and $\Delta(C)$ in the 5th and 10th column show that $\hat{K}_1$ and $\hat{K}_3$ are strikes of the put and call whose delta level approximates 25. The strikes of each expiration satisfy that $\hat{K}_1 < \hat{K}_2 < \hat{K}_3$.

Table 1: Forward Price and Dividend

Expiry	Days	Forward	Dividend
2016-04-29	7	2091.561	0.0216
2016-05-13	21	2089.225	0.0239
2016-07-29	98	2083.421	0.0209
2016-09-30	161	2076.923	0.0229
2016-12-30	252	2070.583	0.0222
2017-03-17	329	2064.752	0.0223
2017-12-15	602	2047.915	0.0218
2018-12-21	973	2032.327	0.0213

Table 2: Detected Strikes and Implied Volatilities before Iteration

Expiry	Days	Put			ATM		Call		
		$\hat{K}_1$	$\hat{\sigma}_1$	$\Delta(P)$	$\hat{K}_2$	$\hat{\sigma}_2$	$\hat{K}_3$	$\hat{\sigma}_3$	$\Delta(C)$
2016-04-29	7	2069	0.1312	-0.2524	2092	0.1156	2114	0.1047	0.2491
2016-05-13	21	2053	0.1283	-0.2523	2089	0.1117	2128	0.0956	0.2502
2016-07-29	98	1990	0.1701	-0.2507	2083	0.1402	2193	0.1048	0.2502
2016-09-30	161	1952	0.1820	-0.2492	2077	0.1509	2232	0.1124	0.2495
2016-12-30	252	1913	0.1924	-0.2503	2071	0.1601	2250	0.1242	0.2840
2017-03-17	329	1884	0.1994	-0.2497	2065	0.1662	2250	0.1333	0.3144
2017-12-15	602	1825	0.2106	-0.2604	2048	0.1806	2250	0.1533	0.3723
2018-12-21	973	1825	0.2116	-0.2912	2032	0.1910	2250	0.1700	0.4084

With $\hat{K}_i$ and $\hat{\sigma}_i$, $i = 1, 2, 3$, I compute K_i and σ_i, $i = 1, 2, 3$, by implementing step 3 and 4 of **Algorithm** detailed in Section 3. After iteration, I obtain the ATM implied volatility, and implied volatilities of 25 delta call and put. With these information, Risk Reversal and Vega Weighted Butterfly are computed using equations (37) and (38), respectively. The volatility matrix expressed in compact form as in FX option market is shown in Table 3.

Next, for different delta levels, I interpolate and extrapolate the volatility surface between expiries ranging from 4 days to 10 years. The absolute value of investigated delta levels are $\Delta =$ 0.01, 0.05, 0.10, 0.15, 0.20, 0.25, 0.30, 0.50. The graphical result of implied volatility surface is exhibited by Figure 2. The absolute value of put delta and call delta are on the left and right hand side of the x-axis, respectively. The ATM zero-delta strike is the center of the x-axis. For saving the space, numerical results for only $\Delta = 0.25$ are presented in Table 4. The features of volatility surface

Table 3: Implied Volatility Surface in Compact Form Obtained by Iteration

Expiry	σ_{ATM}	$RR_{25\Delta}$	$VBF_{25\Delta}$
2016-04-29	0.1161	-0.0251	0.0022
2016-05-13	0.1113	-0.0300	0.0017
2016-07-29	0.1383	-0.0553	0.0044
2016-09-30	0.1485	-0.0574	0.0044
2016-12-30	0.1567	-0.0628	0.0036
2017-03-17	0.1600	-0.0657	0.0050
2017-12-15	0.1737	-0.0564	0.0077
2018-12-21	0.1820	-0.0405	0.0082

are rather easily recognisable when the surface is expressed in terms of ATM straddle, Risk Reversal and Vega Weighted Butterfly. Risk Reversal decrease as expiry increase, whereas Vega Weighted Butterfly increase as expiry increase. The volatility surface exhibits non-flat instantaneous profile and strike and term structure.

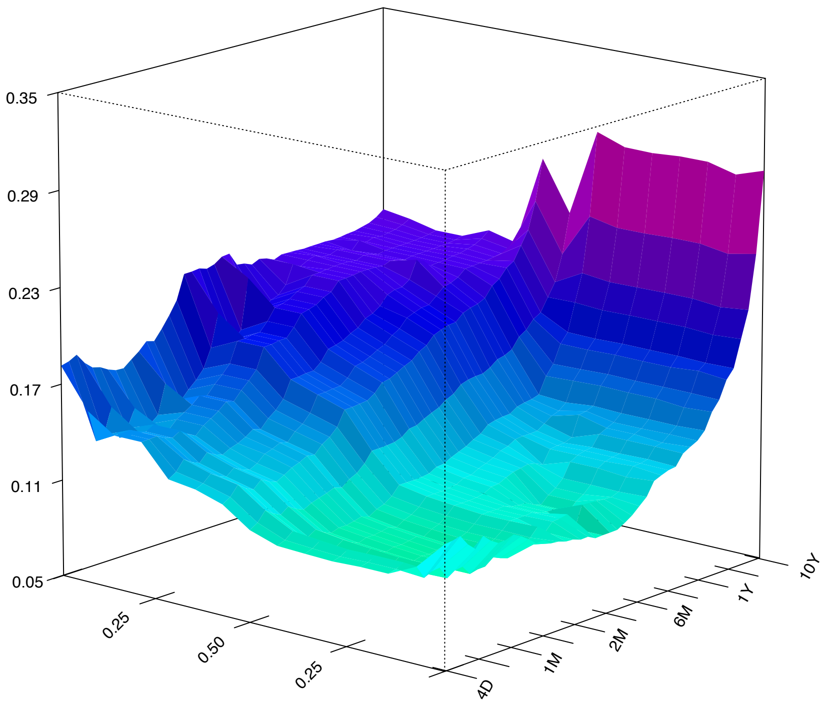

Figure 2: Implied Volatility Surface

With information of ATM volatility, Risk Reversal and Vega Weighted Butterfly, for each expiry, the implied volatilities of 25 delta call and put are computed using equation (39) and (40), respectively. The results listed in the 5th and 6th columns in Table 4 show that implied volatilities of call and put increase as $T \to \infty$. Once obtaining the information of Table 4, the strikes for 25 delta put, ATM and 25 delt call (i.e. K_1, K_2, K_3) can be computed using equation (43), (36) and (42), respectively. My results of strikes of each expiry are presented in Table 5. For each expiry, strikes satisfy $K_1 < K_2 < K_3$.

So far, I have obtained strike K_i and implied volatility σ_i, $i = 1, 2, 3$, for each expiry. Now, the Vanna-Volga pricing formula, and its 1st and 2nd approximation can be used to price the equity option with K_i and σ_i, $i = 1, 2, 3$, for each expiry. In order to evaluate the goodness of these three approaches, my paper compares the model price with sparse market price provided by OptionMetrics. The results are graphically illustrated by plots in Appendix A. The orange cross sign in each plot represents the real market implied volatility. Implied volatility smile generated by Vanna-Volga pricing formula, its 1st and 2nd approximation are denoted by purple, red and blue dashed line, respectively. The plots discover some interesting results. First, for each expiry, implied volatility produced by these three approaches are extremely accurate inside the interval $[K_1, K_3]$. Second, for each expiry, implied volatility generated by the 1st approximation is much closer to the real implied volatility of ITM option. Third, the 2nd approximation can perfectly approximate Vanna-Volga implied volatility of ITM, ATM and OTM options. The bias between Vanna-Volga implied volatility and its 2nd approximation decreases as expiry T becomes larger. Fourth, Vanna-Volga implied volatility and its 2nd approximation approach the real implied volatility of ITM option as expiry T becomes larger. The plots in Appendix C compare the risk-neutral density generated using formula (22) and Black-Scholes formula.

The more explicit comparison of market price and model price are presented in Table 6. For saving the space, only numerical results for $T = (5D, 28D, 69D, 84D)$ are listed in the table. The real market implied volatility is σ_{market}. Implied volatility computed by Vanna-Volga, 1st and 2nd approximation are σ_{VV}, σ_{1st} and σ_{2nd}, respectively. The last three columns are Bias 1 = $\sigma_{market} - \sigma_{VV}$, Bias 2 = $\sigma_{market} - \sigma_{1st}$, and Bias 3 = $\sigma_{market} - \sigma_{2nd}$. The results show that, for each expiry, Bias 2 is always the smallest for each strike. Bias 3 is very close to Bias 1 in all cases. For ATM option, the biases of three approaches are always the smallest. As expiry increases, Bias 1 and Bias 3 for ITM option decrease. Appendix B provides the plots to compare the bias from three approaches.

5 Conclusion

This paper applies Vanna-Volga method and two approximation of Vanna-Volga implied volatility on pricing equity option written on S&P500 index. My findings are as follow. First, for pricing ITM option, the 1st approximation performs better than Vanna-Volga method and the 2nd approximation, particularly for short maturity. Second, as $T \to \infty$, the results of Vanna-Volga method and the 2nd approximation approach the result of the 1st approximation. Third, Vanna-Volga method and its two approximation generate accurate implied volatilities inside the interval $[K_1, K_3]$.

In a nutshell, the advantages of Vanna-Volga method and its two approximation are evident. Only three quotes are required for using these approaches. They yield the reliable volatility smile for equity option, even for the extremely short maturity. Since calibration is unnecessary, therefore we can avoid the issues related to instability and global minimum searching. Vanna-Volga method depends on the shape of volatility surface. It performs well when volatility surface is standard, i.e. symmetric smile and typical skew.

Table 4: Risk Reversal, Butterfly, and Implied Volatility for 25 Delta Level

Expiry	$\sigma(ATM)$	Risk Reversal	Butterfly	$\sigma(C)$	$\sigma(P)$	Delta Level
2016-04-27	0.1101	-0.0239	0.0043	0.1024	0.1263	0.25
2016-05-02	0.1126	-0.0265	0.0020	0.1013	0.1279	0.25
2016-05-09	0.1108	-0.0289	0.0020	0.0983	0.1272	0.25
2016-05-20	0.1129	-0.0345	0.0022	0.0979	0.1323	0.25
2016-05-31	0.1145	-0.0370	0.0020	0.0981	0.1351	0.25
2016-06-03	0.1183	-0.0393	0.0023	0.1010	0.1403	0.25
2016-06-24	0.1315	-0.0478	0.0034	0.1110	0.1588	0.25
2016-06-30	0.1318	-0.0485	0.0035	0.1111	0.1596	0.25
2016-07-01	0.1323	-0.0484	0.0034	0.1115	0.1599	0.25
2016-07-15	0.1340	-0.0513	0.0039	0.1122	0.1635	0.25
2016-08-19	0.1409	-0.0549	0.0041	0.1176	0.1725	0.25
2016-08-31	0.1432	-0.0559	0.0041	0.1193	0.1752	0.25
2016-09-16	0.1451	-0.0560	0.0043	0.1213	0.1773	0.25
2016-10-26	0.1507	-0.0588	0.0042	0.1256	0.1843	0.25
2016-12-16	0.1551	-0.0614	0.0039	0.1283	0.1897	0.25
2017-01-20	0.1574	-0.0645	0.0036	0.1288	0.1933	0.25
2017-01-25	0.1576	-0.0646	0.0038	0.1291	0.1937	0.25
2017-03-17	0.1600	-0.0657	0.0050	0.1322	0.1978	0.25
2017-04-25	0.1648	-0.0661	0.0055	0.1373	0.2033	0.25
2017-06-16	0.1671	-0.0646	0.0062	0.1410	0.2056	0.25
2018-04-25	0.1766	-0.0508	0.0079	0.1591	0.2099	0.25
2019-04-25	0.1849	-0.0352	0.0083	0.1756	0.2108	0.25
2020-04-27	0.1931	-0.0195	0.0087	0.1921	0.2116	0.25
2021-04-26	0.2013	-0.0039	0.0092	0.2085	0.2125	0.25
2023-04-25	0.2178	0.0272	0.0100	0.2414	0.2141	0.25
2026-04-26	0.2425	0.0741	0.0113	0.2908	0.2167	0.25

Table 5: Strikes for ATM, 25 Delta Call and 25 Delta Put

Expiry	$K(\Delta_{25P})$	$K(ATM)$	$K(\Delta_{25C})$
2016-04-27	2070.602	2091.295	2108.242
2016-05-02	2061.515	2091.029	2114.731
2016-05-09	2052.497	2090.635	2120.602
2016-05-20	2038.914	2089.461	2127.675
2016-05-31	2027.914	2088.460	2133.559
2016-06-03	2023.603	2088.725	2136.921
2016-06-24	1998.821	2087.905	2152.623
2016-06-30	1994.786	2088.276	2156.052
2016-07-01	1993.860	2088.172	2156.703
2016-07-15	1982.560	2087.453	2162.807
2016-08-19	1958.375	2088.429	2182.260
2016-08-31	1948.937	2086.649	2186.230
2016-09-16	1939.691	2086.556	2193.758
2016-10-26	1916.857	2086.677	2211.296
2016-12-16	1892.389	2086.452	2229.160
2017-01-20	1878.359	2088.097	2240.861
2017-01-25	1876.979	2088.913	2243.510
2017-03-17	1857.177	2089.097	2260.853
2017-04-25	1841.675	2091.050	2279.680
2017-06-16	1825.763	2091.912	2298.990
2018-04-25	1761.587	2103.406	2415.444
2019-04-25	1723.682	2131.626	2564.199
2020-04-27	1699.121	2164.642	2731.858
2021-04-26	1697.437	2223.071	2951.761
2023-04-25	1712.569	2365.841	3515.555
2026-04-26	1814.389	2746.937	5035.416

Table 6: Comparison of Implied Volatility

Strike	**Expiry**	σ_{market}	σ_{VV}	σ_{1st}	σ_{2nd}	**Bias 1**	**Bias 2**	**Bias 3**
1900	2016-04-27	0.4357	0.1224	0.5804	0.1534	0.3133	-0.1447	0.2823
1920	2016-04-27	0.3994	0.1246	0.4950	0.1536	0.2748	-0.0956	0.2458
1940	2016-04-27	0.3531	0.1275	0.4188	0.1537	0.2256	-0.0657	0.1994
1960	2016-04-27	0.3306	0.1311	0.3514	0.1537	0.1994	-0.0208	0.1768
1980	2016-04-27	0.2573	0.1357	0.2925	0.1536	0.1216	-0.0352	0.1037
2000	2016-04-27	0.2399	0.1411	0.2419	0.1529	0.0988	-0.0020	0.0870
2020	2016-04-27	0.1977	0.1459	0.1994	0.1510	0.0518	-0.0017	0.0467
2040	2016-04-27	0.1622	0.1459	0.1647	0.1463	0.0163	-0.0025	0.0160
2060	2016-04-27	0.1383	0.1355	0.1377	0.1352	0.0028	0.0006	0.0030
2080	2016-04-27	0.1197	0.1182	0.1180	0.1182	0.0015	0.0017	0.0015
2100	2016-04-27	0.1051	0.1055	0.1055	0.1055	-0.0004	-0.0004	-0.0004
2120	2016-04-27	0.0990	0.1000	0.1000	0.1000	-0.0010	-0.0010	-0.0010
2140	2016-04-27	0.1117	0.1022	0.1013	0.1023	0.0095	0.0104	0.0094
1900	2016-05-20	0.1854	0.1420	0.1834	0.1476	0.0434	0.0020	0.0377
1920	2016-05-20	0.1750	0.1440	0.1763	0.1478	0.0311	-0.0013	0.0272
1940	2016-05-20	0.1723	0.1455	0.1691	0.1476	0.0268	0.0032	0.0247
1960	2016-05-20	0.1657	0.1461	0.1618	0.1469	0.0196	0.0039	0.0189
1980	2016-05-20	0.1574	0.1454	0.1545	0.1453	0.0120	0.0029	0.0121
2000	2016-05-20	0.1498	0.1429	0.1470	0.1425	0.0069	0.0027	0.0072
2020	2016-05-20	0.1420	0.1383	0.1395	0.1381	0.0036	0.0024	0.0038
2040	2016-05-20	0.1339	0.1320	0.1319	0.1320	0.0020	0.0020	0.0019
2060	2016-05-20	0.1255	0.1245	0.1243	0.1245	0.0010	0.0012	0.0010
2080	2016-05-20	0.1164	0.1166	0.1166	0.1166	-0.0002	-0.0002	-0.0002
2100	2016-05-20	0.1079	0.1088	0.1088	0.1088	-0.0009	-0.0009	-0.0009
2120	2016-05-20	0.0994	0.1010	0.1009	0.1010	-0.0016	-0.0015	-0.0016

Note: Bias 1 = $\sigma_{market} - \sigma_{VV}$, Bias 2 = $\sigma_{market} - \sigma_{1st}$, Bias 3 = $\sigma_{market} - \sigma_{2nd}$.

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Table 6 – continued from previous page

Strike	**Expiry**	σ_{market}	σ_{VV}	σ_{1st}	σ_{2nd}	**Bias 1**	**Bias 2**	**Bias 3**
2140	2016-05-20	0.0924	0.0924	0.0930	0.0925	0.0000	-0.0006	-0.0002
1900	2016-06-30	0.1946	0.1753	0.1866	0.1749	0.0194	0.0080	0.0197
1920	2016-06-30	0.1883	0.1737	0.1810	0.1731	0.0146	0.0073	0.0152
1940	2016-06-30	0.1822	0.1711	0.1754	0.1706	0.0111	0.0069	0.0116
1960	2016-06-30	0.1758	0.1676	0.1696	0.1673	0.0081	0.0061	0.0085
1980	2016-06-30	0.1693	0.1633	0.1639	0.1632	0.0060	0.0054	0.0061
2000	2016-06-30	0.1626	0.1582	0.1581	0.1582	0.0044	0.0045	0.0044
2020	2016-06-30	0.1556	0.1526	0.1522	0.1527	0.0030	0.0034	0.0029
2040	2016-06-30	0.1489	0.1466	0.1463	0.1467	0.0023	0.0026	0.0022
2060	2016-06-30	0.1419	0.1405	0.1403	0.1405	0.0014	0.0015	0.0013
2080	2016-06-30	0.1346	0.1343	0.1343	0.1344	0.0003	0.0003	0.0002
2100	2016-06-30	0.1275	0.1283	0.1283	0.1283	-0.0008	-0.0008	-0.0008
2120	2016-06-30	0.1203	0.1223	0.1222	0.1222	-0.0019	-0.0018	-0.0019
2140	2016-06-30	0.1134	0.1162	0.1160	0.1162	-0.0028	-0.0026	-0.0027
1900	2016-07-15	0.1945	0.1781	0.1859	0.1775	0.0164	0.0085	0.0170
1920	2016-07-15	0.1882	0.1758	0.1806	0.1752	0.0124	0.0076	0.0130
1940	2016-07-15	0.1820	0.1726	0.1752	0.1722	0.0094	0.0069	0.0098
1960	2016-07-15	0.1759	0.1687	0.1697	0.1685	0.0072	0.0062	0.0074
1980	2016-07-15	0.1695	0.1641	0.1642	0.1641	0.0054	0.0053	0.0054
2000	2016-07-15	0.1631	0.1590	0.1587	0.1591	0.0041	0.0045	0.0040
2020	2016-07-15	0.1568	0.1535	0.1531	0.1536	0.0033	0.0037	0.0032
2040	2016-07-15	0.1503	0.1477	0.1474	0.1478	0.0025	0.0028	0.0024
2060	2016-07-15	0.1435	0.1419	0.1418	0.1420	0.0016	0.0017	0.0015
2080	2016-07-15	0.1364	0.1361	0.1361	0.1361	0.0003	0.0003	0.0003
2100	2016-07-15	0.1298	0.1304	0.1304	0.1304	-0.0006	-0.0006	-0.0006
2120	2016-07-15	0.1226	0.1247	0.1246	0.1247	-0.0022	-0.0021	-0.0021

Note: Bias 1 = $\sigma_{market} - \sigma_{VV}$, Bias 2 = $\sigma_{market} - \sigma_{1st}$, Bias 3 = $\sigma_{market} - \sigma_{2nd}$.

Continued on next page

Table 6 – continued from previous page

Strike	**Expiry**	σ_{market}	σ_{VV}	σ_{1st}	σ_{2nd}	**Bias 1**	**Bias 2**	**Bias 3**
2140	2016-07-15	0.1160	0.1190	0.1188	0.1189	-0.0031	-0.0029	-0.0030
1900	2016-08-19	0.1928	0.1831	0.1863	0.1826	0.0097	0.0065	0.0102
1920	2016-08-19	0.1877	0.1800	0.1816	0.1796	0.0077	0.0061	0.0081
1940	2016-08-19	0.1824	0.1763	0.1769	0.1761	0.0062	0.0056	0.0063
1960	2016-08-19	0.1769	0.1722	0.1721	0.1722	0.0047	0.0048	0.0047
1980	2016-08-19	0.1715	0.1677	0.1673	0.1678	0.0038	0.0042	0.0037
2000	2016-08-19	0.1660	0.1629	0.1625	0.1631	0.0031	0.0035	0.0029
2020	2016-08-19	0.1603	0.1580	0.1577	0.1581	0.0023	0.0026	0.0022
2040	2016-08-19	0.1541	0.1530	0.1528	0.1531	0.0011	0.0013	0.0010
2060	2016-08-19	0.1484	0.1480	0.1479	0.1481	0.0004	0.0005	0.0004
2080	2016-08-19	0.1429	0.1430	0.1430	0.1430	-0.0001	-0.0001	-0.0001
2100	2016-08-19	0.1367	0.1381	0.1381	0.1381	-0.0014	-0.0013	-0.0013
2120	2016-08-19	0.1307	0.1332	0.1331	0.1332	-0.0025	-0.0024	-0.0024
2140	2016-08-19	0.1247	0.1283	0.1282	0.1283	-0.0036	-0.0035	-0.0036

Note: Bias 1 = $\sigma_{market} - \sigma_{VV}$, Bias 2 = $\sigma_{market} - \sigma_{1st}$, Bias 3 = $\sigma_{market} - \sigma_{2nd}$.

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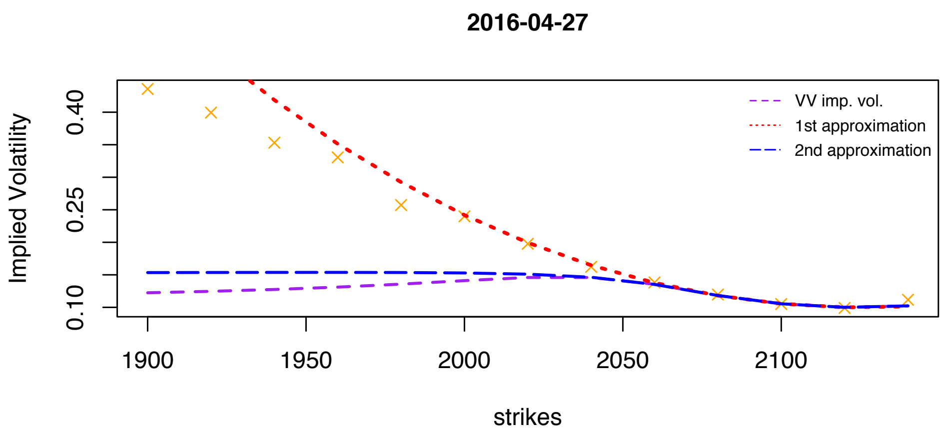
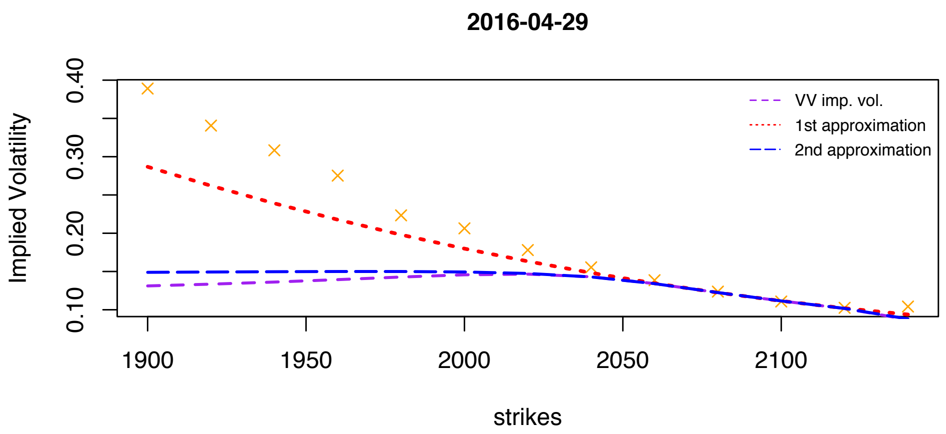
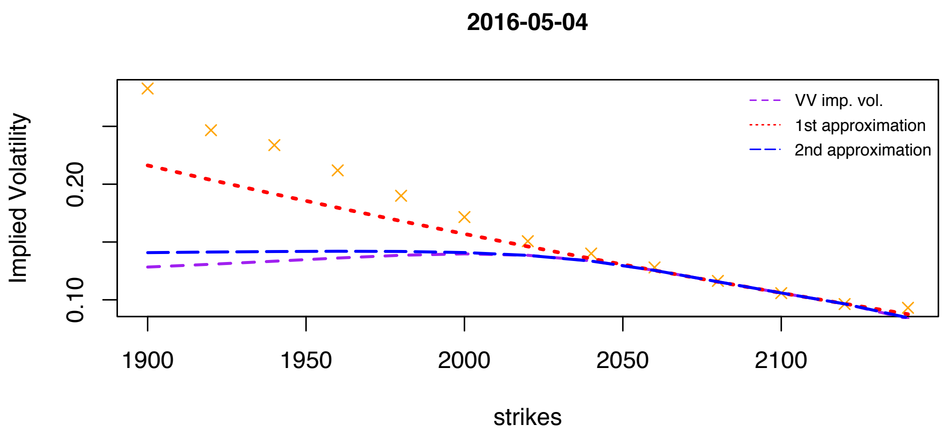
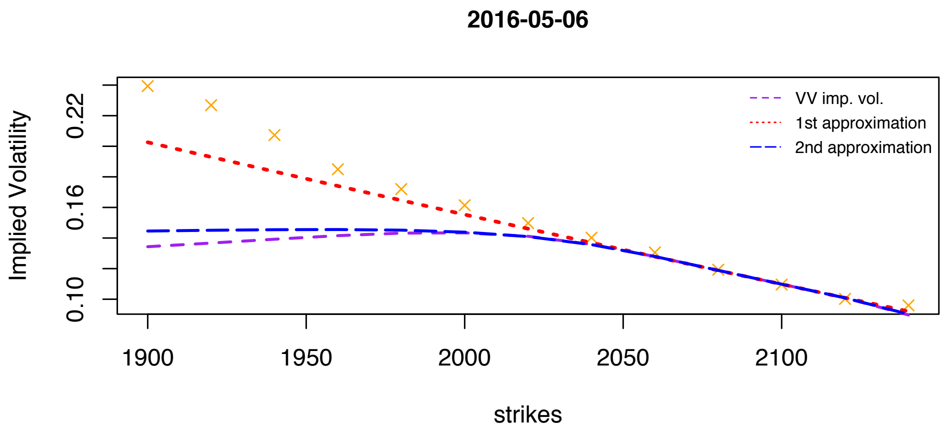
A Comparison of Volatility Smile

Note: the orange cross in each plot denotes observed market implied volatility.





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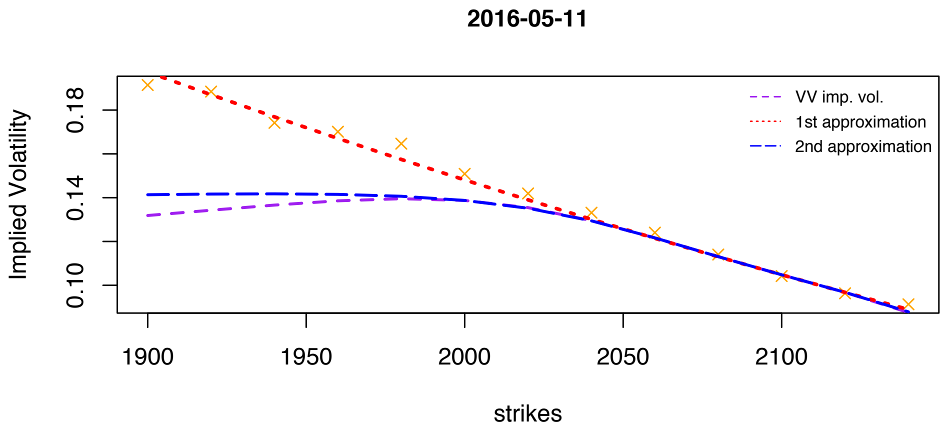
Implied Volatility
0.10
0.16
0.22
1900
1950
2000
2050
2100
strikes
VV imp. vol.
1st approximation
2nd approximation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Implied Volatility
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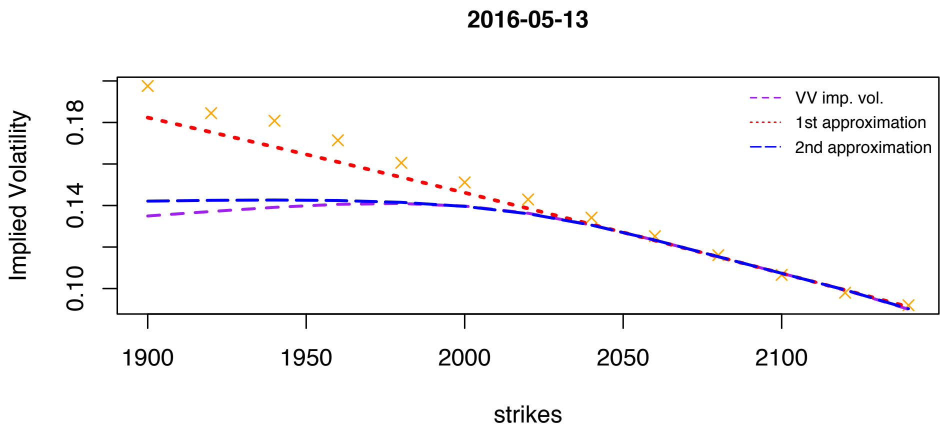
0.14
0.18
1900
1950
2000
2050
2100
strikes
VV imp. vol.
1st approximation
2nd approximation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Implied Volatility
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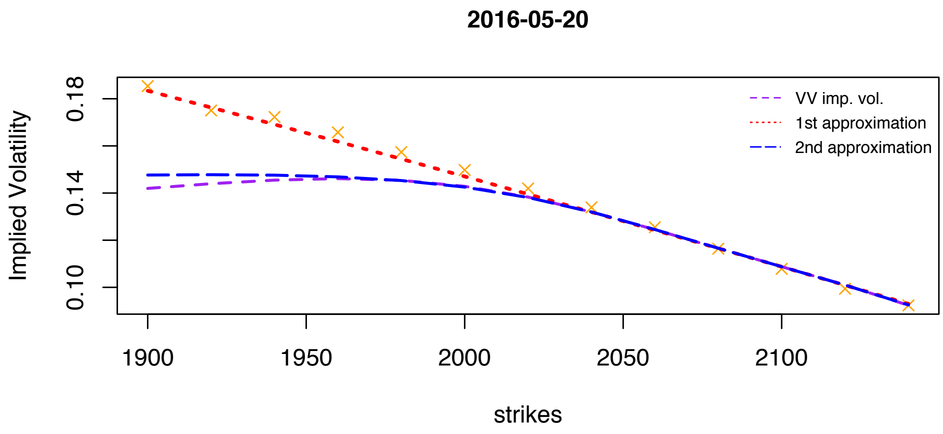
0.14
0.18
1900
1950
2000
2050
2100
strikes
VV imp. vol.
1st approximation
2nd approximation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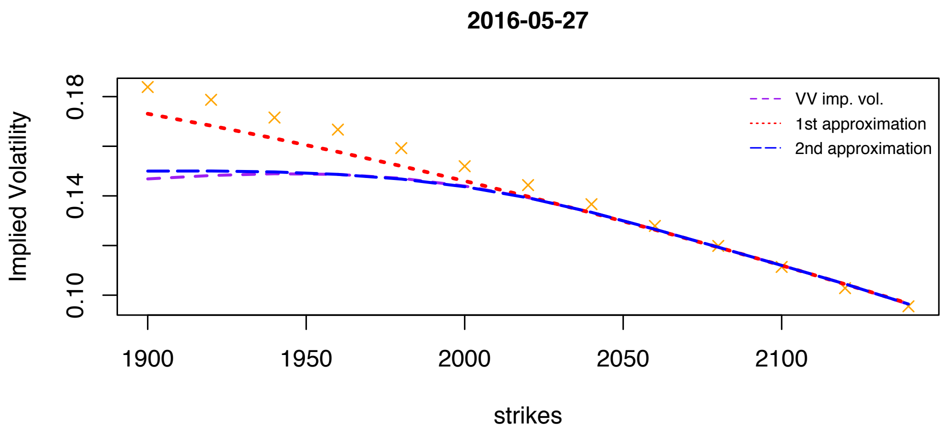
Implied Volatility
0.10
0.14
0.18
1900
1950
2000
2050
2100
strikes
VV imp. vol.
1st approximation
2nd approximation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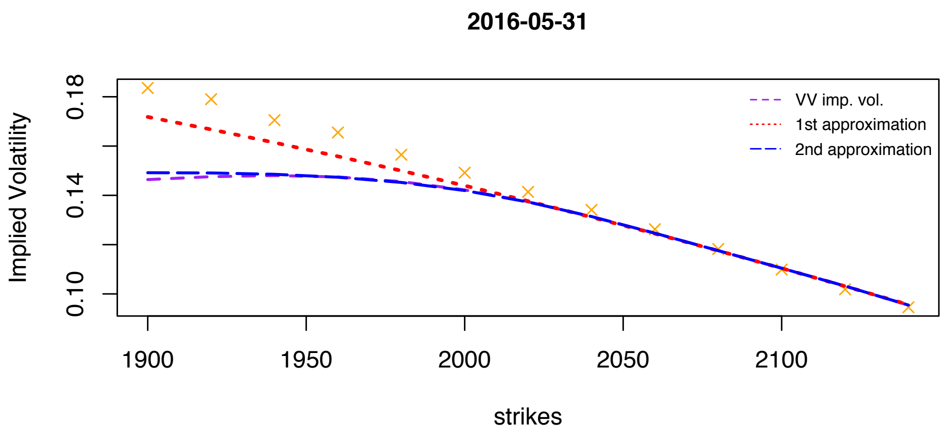
0.14
0.18
1900
1950
2000
2050
2100
strikes
VV imp. vol.
1st approximation
2nd approximation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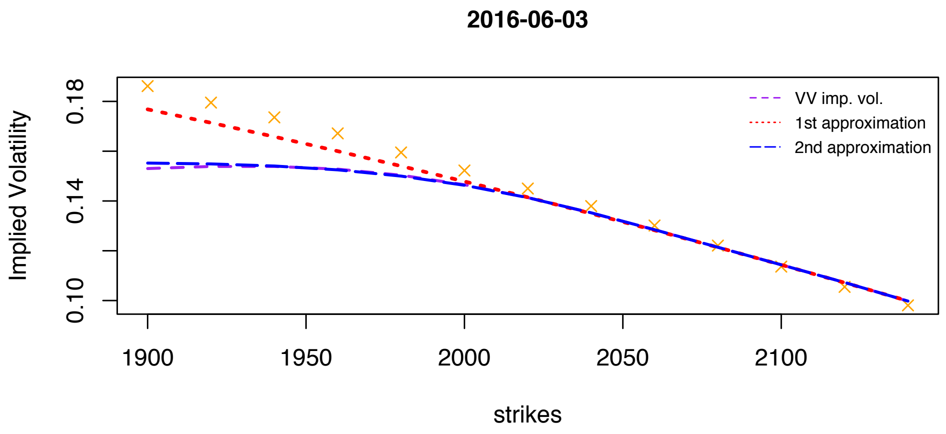
0.14
0.18
1900
1950
2000
2050
2100
strikes
VV imp. vol.
1st approximation
2nd approximation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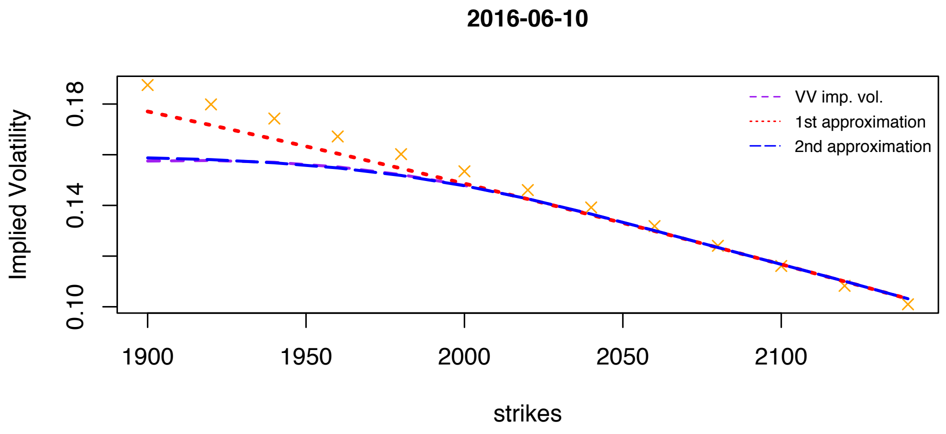
Implied Volatility
0.10
0.14
0.18
1900
1950
2000
2050
2100
strikes
VV imp. vol.
1st approximation
2nd approximation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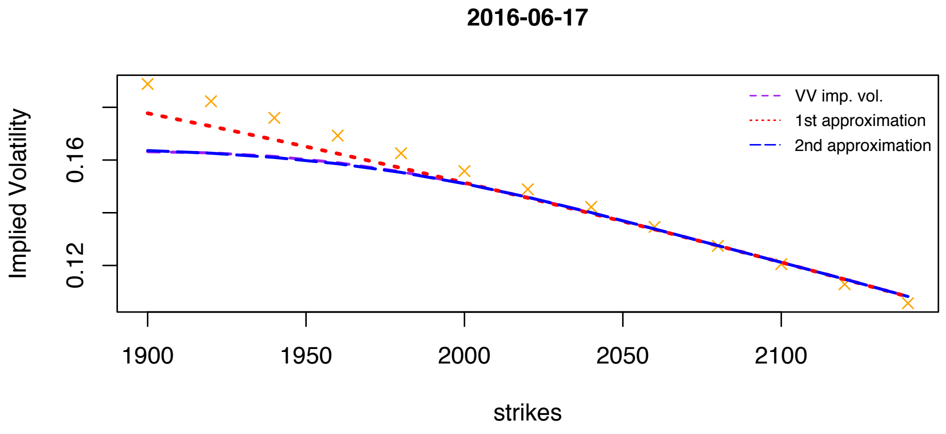
0.14
0.18
1900
1950
2000
2050
2100
strikes
VV imp. vol.
1st approximation
2nd approximation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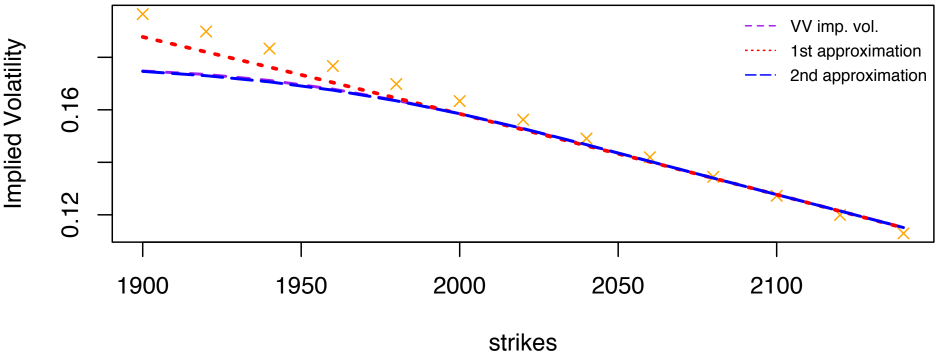
0.16
1900
1950
2000
2050
2100
strikes
VV imp. vol.
1st approximation
2nd approximation

2016-06-24



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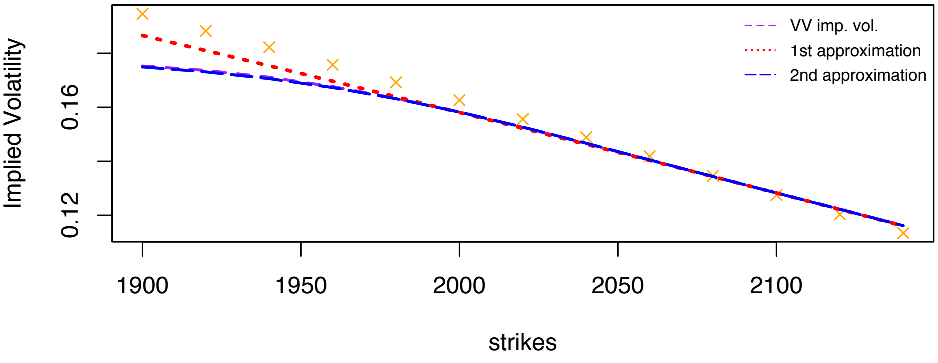

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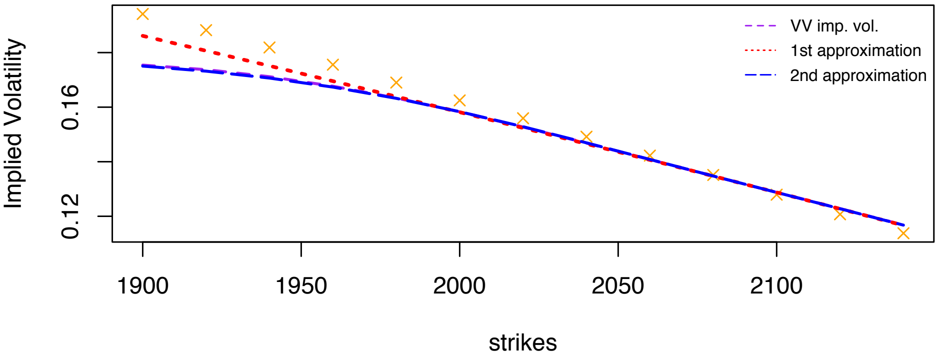

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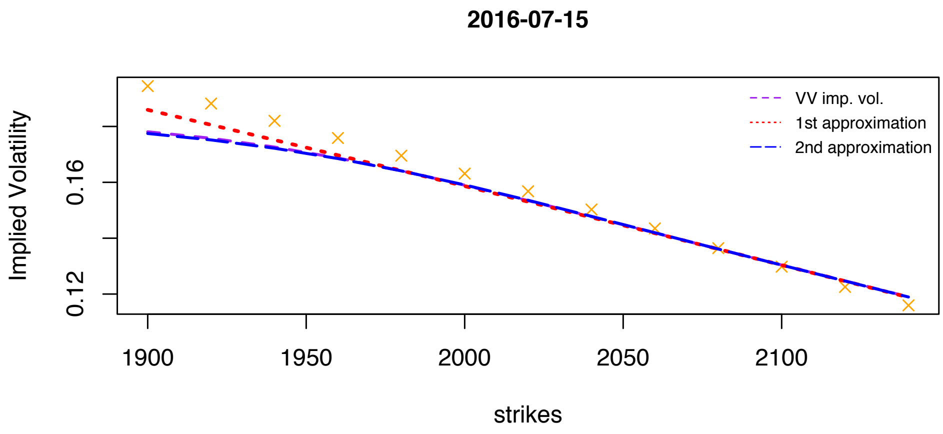

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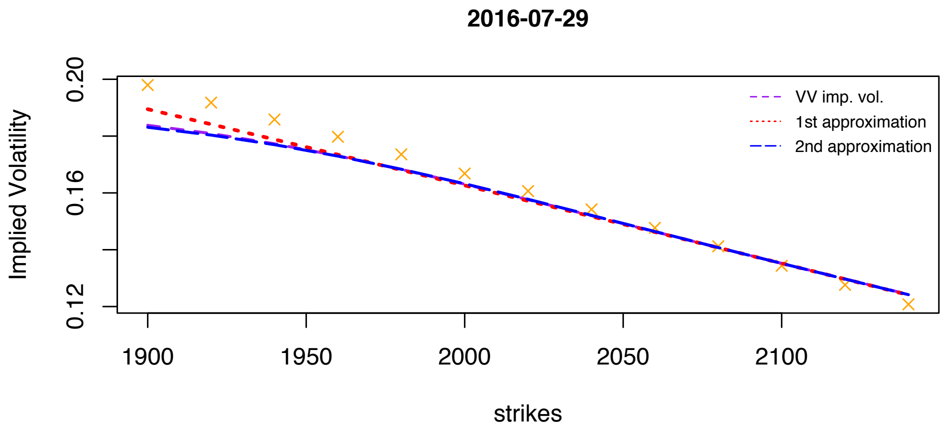
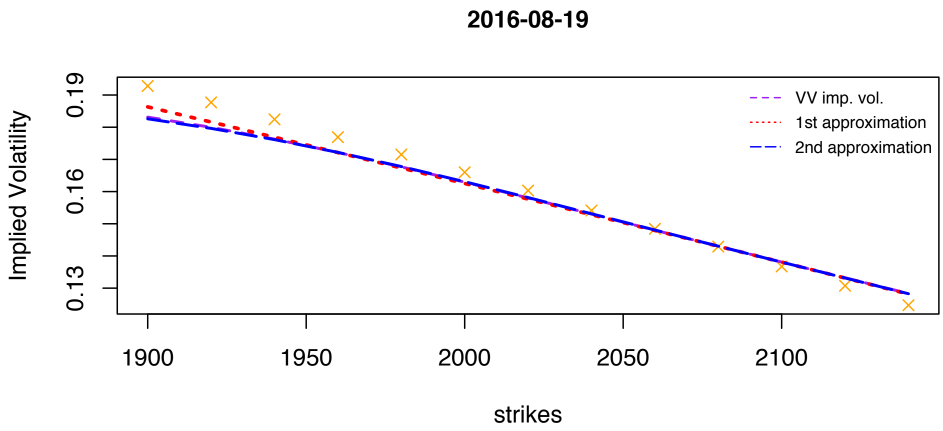
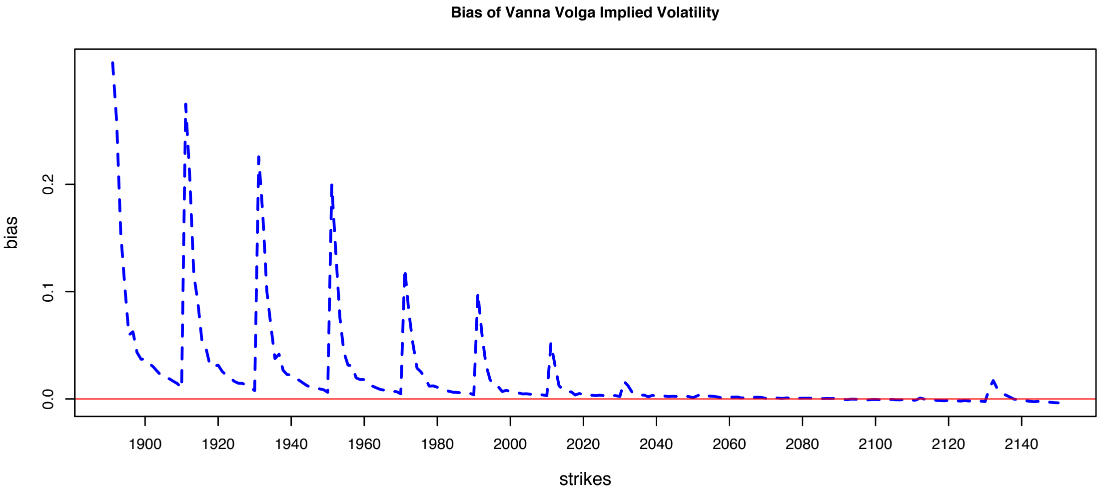
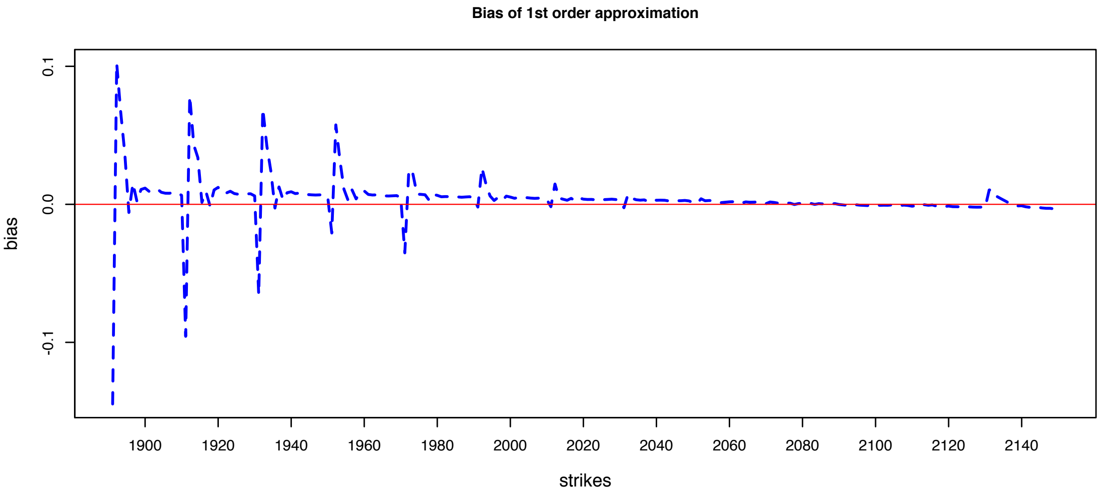


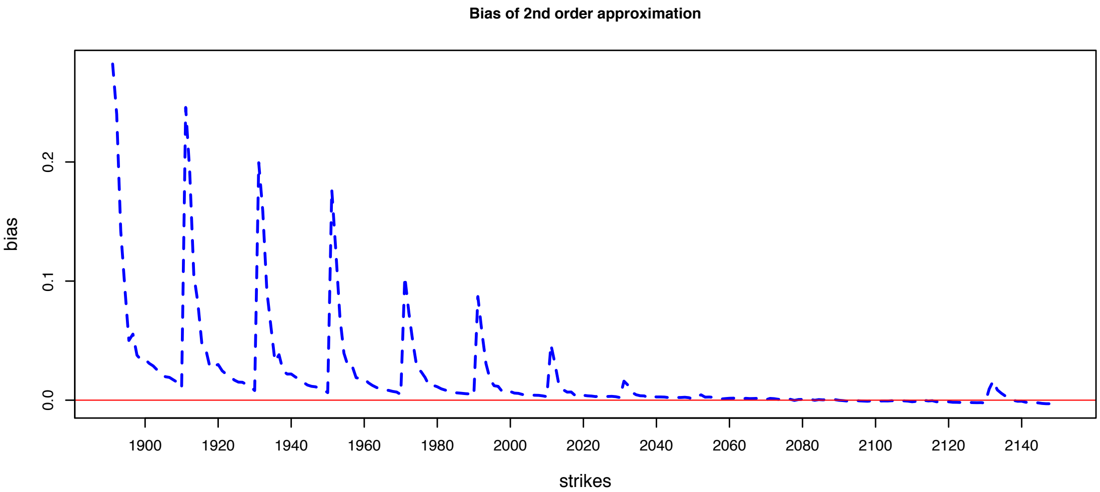
2016-08-19



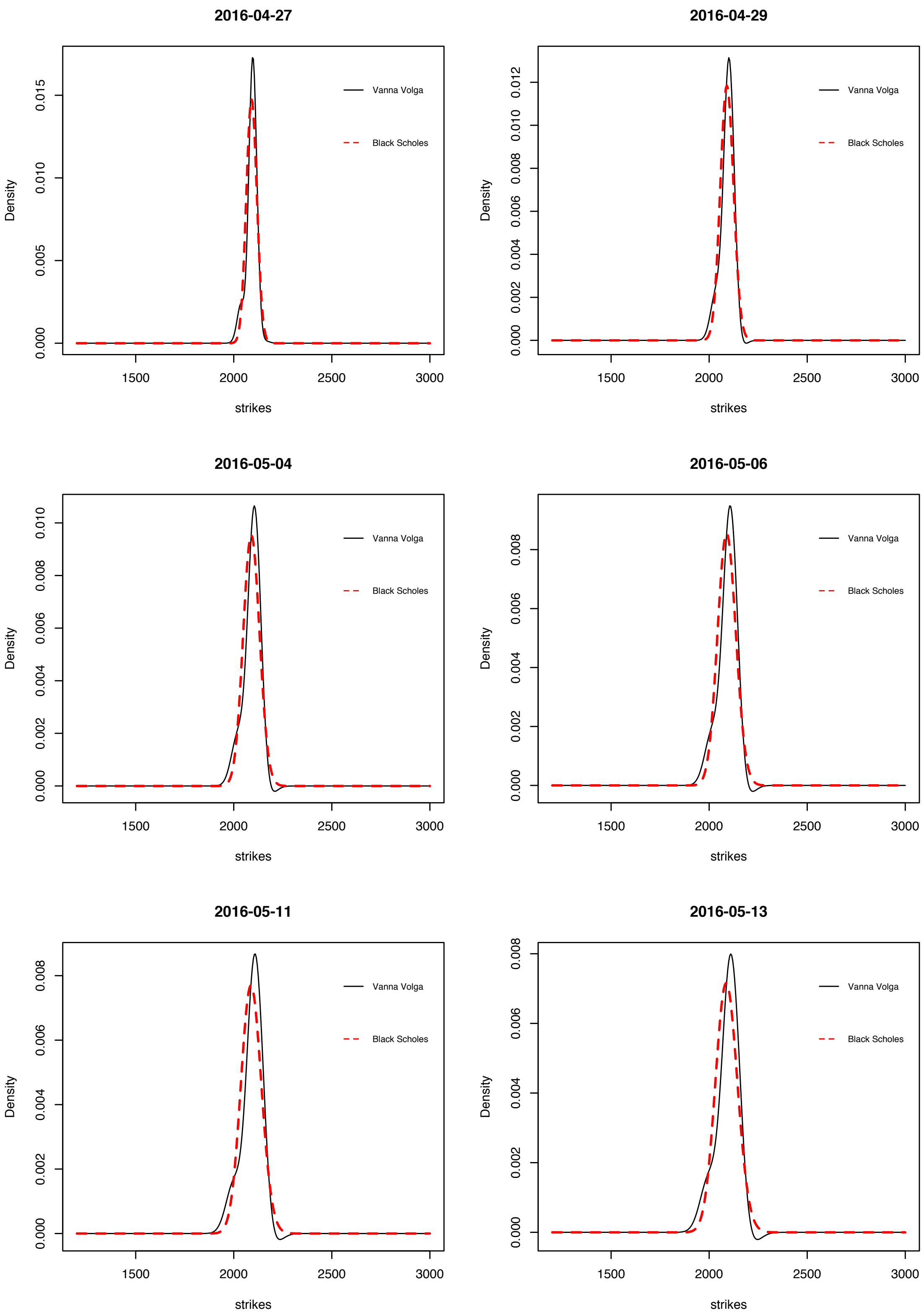
B Bias between Model Price and Market Price

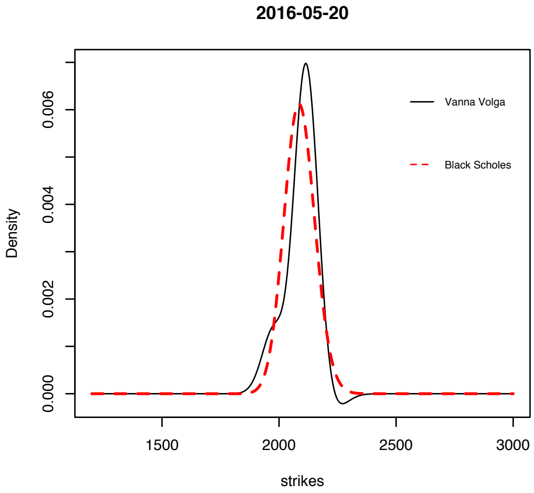






C Risk-neutral Density

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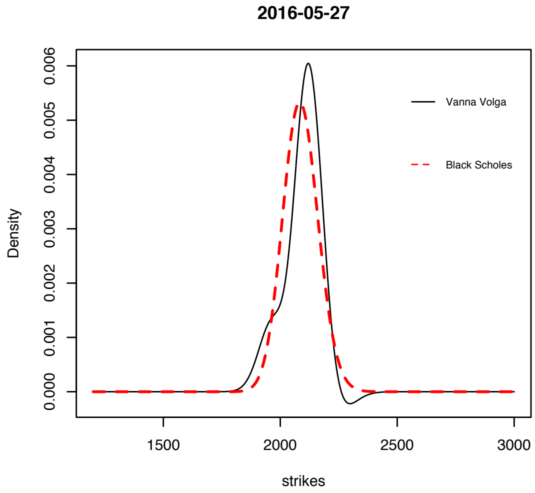
0.000
0.002
0.004
0.006
1500
2000
2500
3000
strikes
Vanna Volga
Black Scholes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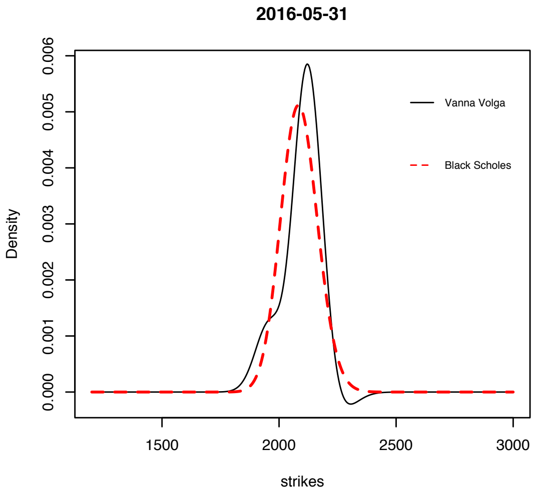
0.005
0.006
1500
2000
2500
3000
strikes
Vanna Volga
Black Scholes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Density
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0.002
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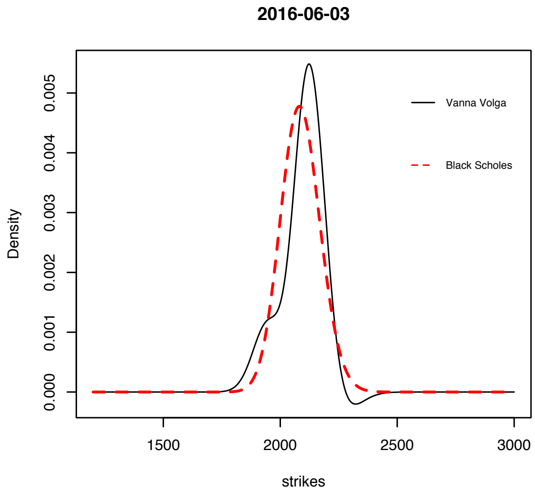
0.005
0.006
1500
2000
2500
3000
strikes
Vanna Volga
Black Scholes


2016-06-03
Density
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0.002
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0.004
0.005
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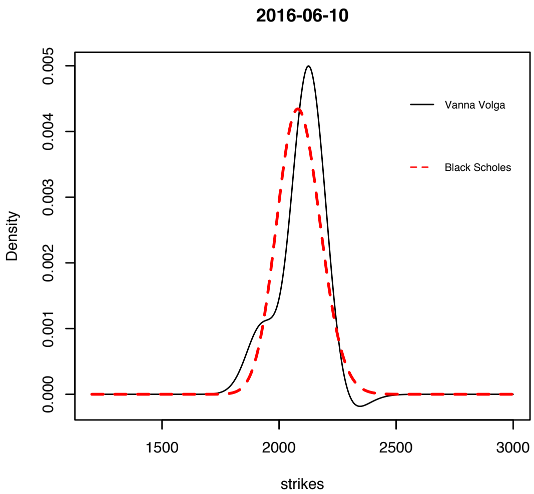
2000
2500
3000
strikes
Vanna Volga
Black Scholes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0.005
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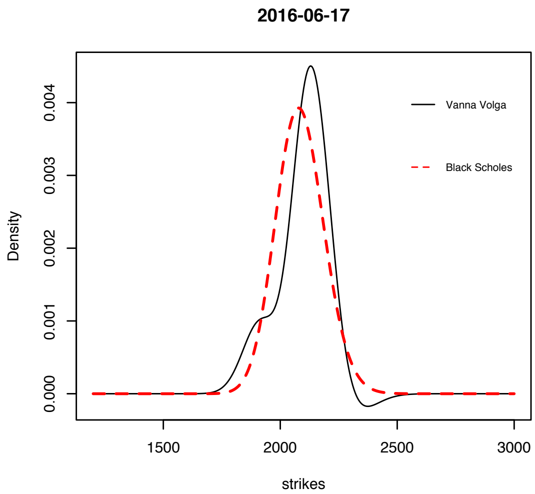
strikes
Vanna Volga
Black Scholes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2000
2500
3000
strikes
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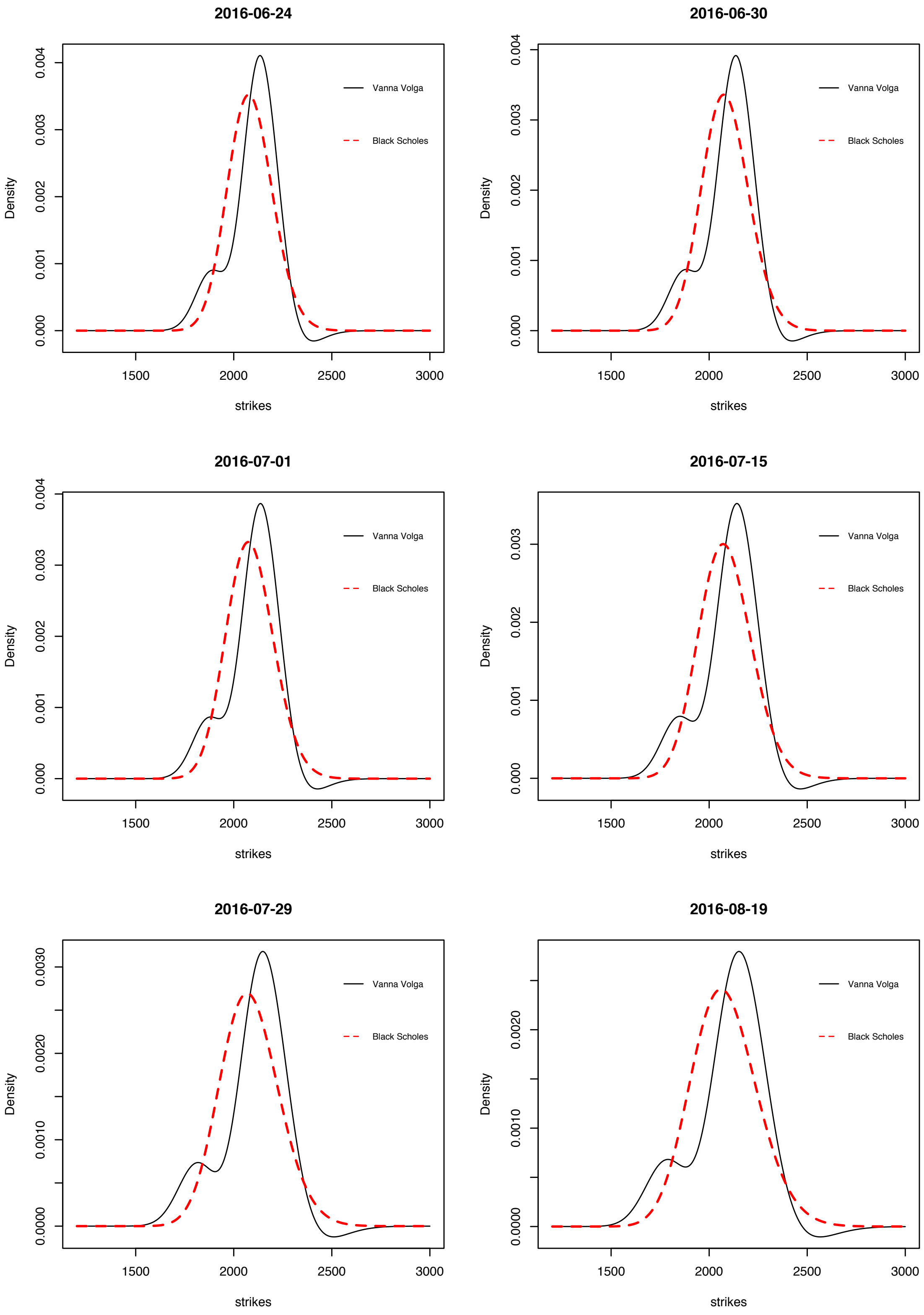
Black Scholes


2016-06-24
2016-06-30
2016-07-01
2016-07-15
2016-07-29
2016-08-19
Density
strikes
Vanna Volga
Black Scholes
1500
2000
2500
3000

D Compute Forward Price via Put-Call Parity

Pseudo Code

---

for each maturity T

% *get the following data*

- best-bid of call
- best-bid of put
- best-ask of call
- best-ask of put

% *unique the strikes of call and put*

- strikes

for each strike K

% *compute the mean of bid and ask price of call and put*

A = mean of best-bid of call option prices $C(K,T)$

B = mean of best-bid of put option prices $P(K,T)$

C = mean of best-ask of call option prices $C(K,T)$

D = mean of best-ask of put option prices $P(K,T)$

% *compute the mid-price*

mid-price = (A-B + C - D) / 2

optimize forward price by Put-Call parity C-P=PV·(F-K)

% *optimize only for near-the-money strikes*

near-the-money strikes

% *initial guess of forward price F and present value PV*

F.initial = mean (mid-price + strikes)

PV.initial = 1

% *objective function for optimization*

$$\text{error} = \Big(\text{PV} \cdot (\text{F} - \text{K}) - \text{mid-price}\Big) \cdot \text{near-the-money strike}$$

objective function = minimize $\sum(\text{errors})^2$

% run optimization with these boundaries

F_{min}= the minimum of strikes

F_{max}= the maximum of strikes

PV_{min}= 0.5

PV_{max}=2
