# Volatility has to be rough

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#### Abstract

First, we give an asymptotic expansion of short-dated at-the-money implied volatility that refines the preceding works and proves in particular that non-rough volatility models are inconsistent to a power law of volatility skew. Second, we show that given a power law of volatility skew in an option market, a continuous price dynamics of the underlying asset with non-rough volatility admits an arbitrage opportunity. The volatility therefore has to be rough in a viable market of the underlying asset of which the volatility skew obeys a power law.

#### 1 Introduction

It has been almost two decades since a power law of volatility skew in option markets was reported [7, 28, 12]. Denoting  $\sigma_{BS}(k, \theta)$  the Black-Scholes implied volatility with log moneyness *k* and time to maturity  $\theta > 0$ , the power law can be formulated as

$$\frac{\sigma_{\rm BS}(k,\theta) - \sigma_{\rm BS}(k',\theta)}{k - k'} \propto \theta^{H - 1/2}$$

for  $k \approx 0$  and  $k' \approx 0$ , with  $H \approx 0$ , when  $\theta \approx 0$ . It is now well known that classical local stochastic volatility models, where volatility is modeled as a diffusion, are not consistent to the power law, while some rough volatility models are so [1, 15, 3, 17, 19, 11, 23, 25, 9, 2, 4] as well as stable-type discontinuous price models [7, 13, 10]. The present article extends the preceding works and shows that there is an arbitrage opportunity if volatility is not rough given an option market with volatility skew obeying the power law, under the assumption that the asset price is a positive continuous Itô semimartingale.

In Section 2, we give an asymptotic expansion of short-dated at-the-money implied volatility, which is a refinement of the results in [17]. Both the result and proof are much simpler than in [17] thanks to choosing the square root of the variance swap fair strike, that is, VIX [30, 21], as the leading term of the expansion, as in [9]. Also, we adopt the forward variance framework [5, 3, 9] that justifies not to consider a time consistency issue treated in [17]. In Section 3, we construct the above mentioned arbitrage opportunity. Some concluding remarks are in Section 4. All the proofs are given in Appendix. Throughout the paper, interest rates are assumed to be zero for brevity.

### 2 An asymptotic expansion

**Theorem 2.1** Suppose that (the underlying asset price process) *S* is a positive continuous martingale under a measure Q with  $\langle \log S \rangle$  being absolutely continuous. Let

$$V_t = \frac{\mathrm{d}}{\mathrm{d}t} \langle \log S \rangle_t, \ v(t) = E[V_t],$$

where *E* is the expectation under *Q*, and assume that v(t) is positive and continuous at t = 0. If there exists  $H \in (0, 1/2]$  such that

$$\frac{1}{\theta^H} \left( \frac{V_\theta}{v(\theta)} - 1 \right)$$

is uniformly integrable and

$$\left(\frac{1}{\sqrt{\theta}}\left(\frac{S_{\theta}}{S_{0}}-1\right),\frac{1}{\theta^{H}}\left(\frac{V_{\theta}}{v(\theta)}-1\right)\right)$$

converges in law to a two dimensional random variable  $(\xi, \eta)$  as  $\theta \to 0$ , then

$$\sigma_{\rm BS}\left(z\,\sqrt{\theta},\theta\right) = \sqrt{\bar{v}(\theta)}(1+\alpha(z)\theta^H) + o(\theta^H), \ as \ \theta \to 0,$$

where  $\sigma_{BS}(k, \theta)$  is the Black-Scholes implied volatility as in Introduction at time t = 0,

$$\begin{split} \bar{v}(\theta) &= \frac{1}{\theta} \int_0^{\theta} v(t) \mathrm{d}t, \\ \alpha(z) &= \frac{1}{2} \int_0^1 \int_{-\infty}^{\infty} u^H E[\eta] \xi = z \sqrt{u} + \sqrt{v(0)(1-u)} w] \phi(w) \mathrm{d}w \mathrm{d}u, \end{split}$$

and  $\phi$  is the standard normal density.

**Remark 2.1**  $\sqrt{E[\eta]\xi = x]}$  is a renormalized limit of the Dupiré local volatility.

**Remark 2.2** The function *v* is called the forward variance curve.

**Remark 2.3** The leading term  $\sqrt{\overline{v}(\theta)}$  with  $\theta = 30$  days corresponds to the VIX.

**Corollary 2.1** Under the condition of Theorem 2.1, if  $(\xi, \eta) \sim \mathcal{N}(0, \Sigma)$  with covariance matrix  $\Sigma = [\Sigma_{ij}]$ , then  $\Sigma_{11} = v(0)$ ,  $E[\eta|\xi = x] = x\Sigma_{12}/\Sigma_{11}$ , and

$$\alpha(z) = \frac{\sum_{12}}{2v(0)} \frac{z}{H + 3/2}.$$

*In particular, a power law of volatility skew follows: for*  $\zeta \neq z$ *,* 

$$\frac{\sigma_{\rm BS}\left(z\,\sqrt{\theta},\theta\right)-\sigma_{\rm BS}\left(\zeta\,\sqrt{\theta},\theta\right)}{z\,\sqrt{\theta}-\zeta\,\sqrt{\theta}}\sim\frac{\Sigma_{12}}{\sqrt{v(0)}(2H+3)}\theta^{H-1/2}.$$

**Remark 2.4** When *V* is an integrable Itô semimartingale,  $v(\theta) = V_0 + O(\theta)$  and Theorem VIII.3.8 of [24] verifies the assumptions of Corollary 2.1 with H = 1/2 and

$$(\xi,\eta) \sim \mathcal{N}(0,\Sigma), \ \Sigma_{11} = \frac{\mathrm{d}}{\mathrm{d}t} \langle \log S \rangle_t \Big|_{t=0} = v(0), \ \Sigma_{12} = \frac{\mathrm{d}}{\mathrm{d}t} \langle \log S, \log V \rangle_t \Big|_{t=0}$$

In particular, for a local volatility model  $V_t = \sigma(S_t, t)^2$  with a smooth function  $\sigma$ , we have  $v(0) = \sigma(S_0, 0)^2$  and  $\Sigma_{12} = 2\sigma(S_0, 0)\partial_S\sigma(S_0, 0)$ . Thus we conclude the so-called 1/2 rule :

$$\frac{\sigma_{\rm BS}(z\,\sqrt{\theta},\theta) - \sigma_{\rm BS}(\zeta\,\sqrt{\theta},\theta)}{z\,\sqrt{\theta} - \zeta\,\sqrt{\theta}} \sim \frac{1}{2}\partial_S\sigma(S_0,0).$$

Remark 2.5 The martingale property of the rough Bergomi model

$$S_t = S_0 \exp\left(\int_0^t \sqrt{V_s} \left[\rho dW_s + \sqrt{1 - \rho^2} dW_s^{\perp}\right] - \frac{1}{2} \int_0^t V_s ds\right),$$
  
$$V_t = v(t) \exp\left(\int_0^t k(t, s) dW_s - \frac{1}{2} \int_0^t k(t, s)^2 ds\right)$$

with  $k(t,s) = \eta |t-s|^{H-1/2}$  was shown by [20] for  $\rho \in [-1,0]$ . Using that

$$\left(H+\frac{1}{2}\right)\theta^{-H-1/2}\int_0^\theta k(\theta,s)\mathrm{d}s \to \eta, \ 2H\theta^{-2H}\int_0^\theta k(\theta,s)^2\mathrm{d}s \to \eta^2 > 0$$

as  $\theta \to 0$ , the assumptions of Corollary 2.1 are verified with  $\Sigma_{12} = \sqrt{v(0)}\rho\eta/(H+1/2)$ , and therefore we have a power law of volatility skew

$$\frac{\sigma_{\rm BS}\left(z\,\sqrt{\theta},\theta\right)-\sigma_{\rm BS}\left(\zeta\,\sqrt{\theta},\theta\right)}{z\,\sqrt{\theta}-\zeta\,\sqrt{\theta}}\sim\frac{\rho\,\eta}{(H+1/2)(2H+3)}\theta^{H-1/2}.$$

**Remark 2.6** The model-free implied leverage is defined by [16] as the normalized difference of the gamma and variance swap fair strikes :

$$\lambda(\theta) = \frac{1}{E[\langle \log S \rangle_{\theta}]} E\left[\int_{0}^{\theta} \left(\frac{S_{t}}{S_{0}} - 1\right) d\langle \log S \rangle_{t}\right] = \frac{\sqrt{\theta}}{\bar{v}(\theta)} \int_{0}^{1} E[X_{u}^{\theta} V_{\theta u}] du,$$

where  $X^{\theta}$  is defined as (1) in Appendix. Under a slightly stronger assumption than in Corollary 2.1, namely,

$$E\left[\frac{1}{\theta^{H+1/2}}\left(\frac{S_{\theta}}{S_{0}}-1\right)\left(\frac{V_{\theta}}{v(\theta)}-1\right)\right]\to\Sigma_{12},$$

we have

$$\theta^{-H} E[X_u^{\theta} V_{\theta u}] = E\left[X_u^{\theta} \theta^{-H} \left(V_{\theta u} - v(\theta u)\right)\right] \to u^{H+1/2} v(0) \Sigma_{12}$$

uniformly in  $u \in [0, 1]$  and so, a model-free representation of the slope

$$\frac{\Sigma_{12}}{\sqrt{v(0)}(2H+3)}\theta^{H-1/2} \sim \frac{\lambda(\theta)}{2\theta\sqrt{\overline{v}(\theta)}}.$$

### 3 An arbitrage opportunity

In the previous section we considered the implied volatility at time t = 0 and varied the maturity  $\theta$ . Here, we fix a maturity T > 0 instead and consider the implied volatility at time  $\tau < T$ . The short-dated asymptotics corresponds to  $\tau \uparrow T$ . We start with a lemma that tells about the magnitude of the Black-Scholes delta hedging error for at-the-money options when volatility is Hölder continuous.

**Lemma 3.1** Suppose that *S* is a positive continuous semimartingale with  $\langle \log S \rangle$  being absolutely continuous. Let

$$V_t = \frac{\mathrm{d}}{\mathrm{d}t} \langle \log S \rangle_t$$

and assume that V is positive and  $H_0$ -Hölder continuous with  $H_0 \in (0, 1/2]$  a.s. on [0, T], that is,

$$\sup_{0 \le s < t \le T} \frac{|V_t - V_s|}{|t - s|^{H_0}} < \infty, \ a.s..$$

*Then, for any positive adapted process*  $K_{\tau}$ *, as*  $\tau \uparrow T$ *,* 

$$\left(S_T - \frac{S_{\tau}^2}{K_{\tau}}\right)_+ = c_{\rm BS}(S_{\tau}, T - \tau) + \int_{\tau}^T \frac{\partial c_{\rm BS}}{\partial S}(S_t, T - t) dS_t + O((T - \tau)^{H_0 + 1/2}), \ a.s., (K_{\tau} - S_T)_+ = p_{\rm BS}(S_{\tau}, T - \tau) + \int_{\tau}^T \frac{\partial p_{\rm BS}}{\partial S}(S_t, T - t) dS_t + O((T - \tau)^{H_0 + 1/2}), \ a.s.,$$

and

$$\begin{split} (K_{\tau} - S_T)_+ &- \frac{K_{\tau}}{S_{\tau}} \left( S_T - \frac{S_{\tau}^2}{K_{\tau}} \right)_+ \\ &= \int_{\tau}^T \left( \frac{\partial p_{\text{BS}}}{\partial S} (S_t, T - t) - \frac{K_{\tau}}{S_{\tau}} \frac{\partial c_{\text{BS}}}{\partial S} (S_t, T - t) \right) dS_t + O((T - \tau)^{H_0 + 1/2}), \ a.s., \end{split}$$

where  $c_{BS}(S, \theta)$  (resp.  $p_{BS}(S, \theta)$ ) is the Black-Scholes price of the call (resp. put) option with the underlying asset price S, time to maturity  $\theta$ , strike price  $S_{\tau}^2/K_{\tau}$  (resp.  $K_{\tau}$ ), and volatility parameter  $\sqrt{V_{\tau}}$ .

Now we assume a hypothetical option market where call and put options with the underlying asset *S* and maturity *T* are traded at any time  $\tau < T$  and for any strike price K > 0. Denote by  $\sigma_{BS,\tau}(K)$  for the market implied volatility for the strike price *K* at time  $\tau$ . For  $H \in (0, 1/2)$ , we say *the H-power law of negative volatility skew holds* if there exist adapted processes  $\sigma_{\tau}$  and  $\alpha_{\tau}$  such that

$$\liminf_{\tau\uparrow T} \sigma_{\tau} > 0, \ \limsup_{\tau\uparrow T} \sigma_{\tau} < \infty, \ \liminf_{\tau\uparrow T} \alpha_{\tau} > -\infty, \ \limsup_{\tau\uparrow T} \alpha_{\tau} < 0$$

and for any positive adapted process  $K_{\tau}$  with  $|K_{\tau}/S_{\tau} - 1| = O(\sqrt{T - \tau})$ ,

$$\sigma_{\mathrm{BS},\tau}(K_{\tau}) = \sigma_{\tau} + (T-\tau)^{H-1/2} \alpha_{\tau} \log \frac{K_{\tau}}{S_{\tau}} + o((T-\tau)^{H}) \text{ as } \tau \uparrow T.$$

Note that negative volatility skew is typically observed in equity option markets. What is essential in the following is that  $\alpha_{\tau}$  does not change its sign.

Now we construct building blocks of our arbitrage strategy. Let  $\tau_n = T - 1/n$  and choose  $K_{\tau_n}$  so that  $|K_{\tau_n}/S_{\tau_n} - 1| = O(n^{-1/2})$  and

$$\limsup_{n\to\infty} \sqrt{n}\log\frac{K_{\tau_n}}{S_{\tau_n}} < 0$$

Denote by  $\Pi^n$  the P&L of one unit short of the put option with strike  $K_{\tau_n}$  and  $K_{\tau_n}/S_{\tau_n}$  unit long of the call option with strike  $S^2_{\tau_n}/K_{\tau_n}$  with the Black-Scholes delta hedging :

$$\Pi^{n} = P_{\tau_{n}}(K_{\tau_{n}}) - \frac{K_{\tau_{n}}}{S_{\tau_{n}}}C_{\tau_{n}}\left(\frac{S_{\tau_{n}}^{2}}{K_{\tau_{n}}}\right) + \int_{\tau_{n}}^{T} \left(\frac{\partial p_{\mathrm{BS}}}{\partial S}(S_{t}, T-t) - \frac{K_{\tau_{n}}}{S_{\tau_{n}}}\frac{\partial c_{\mathrm{BS}}}{\partial S}(S_{t}, T-t)\right) \mathrm{d}S_{t}$$
$$- (K_{\tau_{n}} - S_{T})_{+} + \frac{K_{\tau_{n}}}{S_{\tau_{n}}}\left(S_{T} - \frac{S_{\tau_{n}}^{2}}{K_{\tau_{n}}}\right)_{+},$$

where  $C_{\tau}(K)$  and  $P_{\tau}(K)$  are respectively the market price of call and put options with strike *K* at time  $\tau$ , and  $c_{BS}$  and  $p_{BS}$  are as in Lemma 3.1 with  $\tau = \tau_n$ .

**Theorem 3.1** Suppose the *H*-power law of negative volatility skew holds. Under the condition of Lemma 3.1 with  $H_0 > H$ ,

$$\sum_{n=1}^{\infty} n^{H-1/2} \Pi^n = \infty, \ a.s..$$

The idea behind Theorem 3.1 is simple. If the volatility is  $H_0$ -Hölder continuous, the Black-Scholes delta hedging error of the specific option portfolio in the *n*th building block is only of  $O(n^{-H_0-1/2})$  a.s. by Lemma 3.1. The Black-Scholes price of the portfolio is zero due to the put-call symmetry [6] and the assumed power law of volatility skew implies the market price of the portfolio of  $O(n^{-H_-1/2})$ . That

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

while

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+H_0-H}} < \infty$$

enables us to make an almost sure infinite profit.

The implication of Theorem 3.1 is that in a viable market, the volatility cannot have a better Hölder regularity than *H*, that is, it has to be rough.

#### 4 Concluding remarks

**Remark 4.1** This paper concludes rough volatility as a consequence of the power law in option markets. The origin of the power law can be explained by a financial practice convention. In FX option markets the convention is to quote prices in terms of the implied volatility and tends to quote the same implied volatility for the same value of the Black-Scholes delta. Since the delta is approximately a function of  $k/\sqrt{\theta}$ , this convention makes  $\sigma_{BS}(z\sqrt{\theta},\theta)$  approximately independent of  $\theta$ , which is nothing but the *H*-power law with H = 0. The origin of this convention is not clear. Naively one may argue that this is due to the traditional financial engineering that perceives the risk of a position only via its delta.

Remark 4.2 The volatility is indeed statistically estimated to be rough; see [18].

Remark 4.3 A model-free bound of volatility skew

$$\left|\frac{\partial \sigma_{\rm BS}}{\partial k}(0,\theta)\right| \le \sqrt{\frac{\pi}{2\theta}}$$

is given in [14] and shown to be sharp in [29]. This extreme skew corresponds to the *H*-power law with H = 0. Therefore the *H*-power law with H < 0 violates no static arbitrage principle in option markets.

**Remark 4.4** Volatility with regularity H = 0 can be understood as a Gaussian multiplicative chaos. It is however an open question whether there exists a continuous-time model with both the regularity of H = 0 and nondegenerate conditional skewness that is necessary to recover the power law of volatility skew stably in time.

**Remark 4.5** Derivations of rough volatility as a scaling limit of Hawkes-type market micro structure models are given in [26, 8, 27]. In [26, 8], a heavy-tailed nearly unstable self-exciting kernel of order flow is the source of the rough volatility. In [27], such a heavy-tailed kernel is derived via Tauberian theorems by assuming the existing of a scaling limit of market impact functions.

**Remark 4.6** An inspection of the proof of Lemma 3.1 reveals that the Hölder regularity of volatility only around the maturity *T* does matter. Therefore a more precise statement of our finding is that the volatility has to be rough near the maturities of options. The volatility has to be rough everywhere under a hypothetical framework where vanilla options are traded for any strike prices around at-the-money and any maturities. Note also that our study does not apply any stock price or index whose options are not traded.

## A Proof of Theorem 2.1

Step 1 [An expansion of a rescaled put option price ]. Denote

$$X_{u}^{\theta} = \frac{1}{\sqrt{\theta}} \left( \frac{S_{\theta u}}{S_{0}} - 1 \right) \tag{1}$$

for  $u \in [0, 1]$ . Note that  $X^{\theta}$  is a martingale and with

$$d\langle X^{\theta}\rangle_{u} = \left(\frac{S_{\theta u}}{S_{0}}\right)^{2} V_{\theta u} du = (1 + \sqrt{\theta} X_{u}^{\theta})^{2} V_{\theta u} du.$$

A rescaled put option price can be expressed as

$$\frac{E[(S_0 e^{z\sqrt{\theta}} - S_{\theta})_+]}{S_0\sqrt{\theta}} = E[(\Delta - X_1^{\theta})_+], \quad \Delta = \frac{e^{z\sqrt{\theta}} - 1}{\sqrt{\theta}}.$$
(2)

Consider the Bachelier pricing equation with time-dependent variance

$$\frac{\partial p}{\partial u}(x,u) + \frac{1}{2}v(\theta u)\frac{\partial^2 p}{\partial x^2}(x,u) = 0, \ p(x,1) = (\Delta - x)_+.$$

The solution and its derivatives are given explicitly :

$$p(x,u) = (\Delta - x)\Phi\left(\frac{\Delta - x}{\sqrt{w(1) - w(u)}}\right) + \sqrt{w(1) - w(u)}\phi\left(\frac{\Delta - x}{\sqrt{w(1) - w(u)}}\right),$$
$$\frac{\partial p}{\partial x}(x,u) = -\Phi\left(\frac{\Delta - x}{\sqrt{w(1) - w(u)}}\right),$$
$$\frac{\partial^2 p}{\partial x^2}(x,u) = \frac{1}{\sqrt{w(1) - w(u)}}\phi\left(\frac{\Delta - x}{\sqrt{w(1) - w(u)}}\right),$$

where  $\Phi$  and  $\phi$  are respectively the standard normal distribution function and the density, and

$$w(u) = \frac{1}{\theta} \int_0^{\theta u} v(t) \mathrm{d}t.$$

Since the process  $X^{\theta}$  takes values on the interval  $[-\theta^{-1/2}, \infty)$  and the function p(x, u) is bounded on  $[-\theta^{-1/2}, \infty) \times [0, 1]$  for each  $\theta > 0$ , Itô's formula, with the aid of a localization argument, gives that

$$E[(\Delta - X_1^{\theta})_+] = E[p(X_1^{\theta}, 1)]$$
  
=  $p(0, 0) + \frac{1}{2}E[\int_0^1 \frac{\partial^2 p}{\partial x^2} (X_u^{\theta}, u)((1 + \sqrt{\theta}X_u^{\theta})^2 V_{\theta u} - v(\theta u))du].$  (3)

By the assumption,

$$\left(X_{u}^{\theta}, \theta^{-H}(V_{\theta u} - v(\theta u))\right) \to (X_{u}, Y_{u}) := (\sqrt{u}\xi, u^{H}v(0)\eta).$$

We have  $\xi \sim N(0, v(0))$  by the martingale central limit theorem. Since

$$\frac{\partial^2 p}{\partial x^2}(x,u) \to \frac{1}{\sqrt{v(0)(1-u)}} \phi\left(\frac{z-x}{\sqrt{v(0)(1-u)}}\right)$$

as  $\theta \rightarrow 0$ , we have

$$\frac{\partial^2 p}{\partial x^2}(X_u^\theta, u) \to \frac{1}{\sqrt{v(0)(1-u)}} \phi\left(\frac{z-X_u}{\sqrt{v(0)(1-u)}}\right)$$

in law for each  $u \in [0, 1)$ . For any polynomial q, there exists a constant C > 0 such that

$$\left|q(x)\frac{\partial^2 p}{\partial x^2}(x,u)\right| \le \frac{C}{\sqrt{1-u}}.$$
(4)

Therefore, the dominated convergence theorem gives that

$$\theta^{-H} \int_{0}^{1} E\left[\frac{\partial^{2} p}{\partial x^{2}}(X_{u}^{\theta}, u)(V_{\theta u} - v(\theta u))\right] du$$
  

$$\rightarrow \int_{0}^{1} E\left[\frac{1}{\sqrt{v(0)(1-u)}}\phi\left(\frac{z-X_{u}}{\sqrt{v(0)(1-u)}}\right)Y_{u}\right] du$$
  

$$= 2\alpha(z)\sqrt{v(0)}\phi\left(\frac{z}{\sqrt{v(0)}}\right)$$

and that

$$\begin{split} &\int_0^1 E\left[\frac{\partial^2 p}{\partial x^2}(X_u^\theta, u)X_u^\theta V_{\theta u}\right] \mathrm{d}u \\ &\to v(0) \int_0^1 E\left[\frac{1}{\sqrt{v(0)(1-u)}}\phi\left(\frac{z-X_u}{\sqrt{v(0)(1-u)}}\right)X_u\right] \mathrm{d}u \\ &= \frac{z\sqrt{v(0)}}{2}\phi\left(\frac{z}{\sqrt{v(0)}}\right). \end{split}$$

From (2) and (3), we have then that

$$\frac{E[(S_0e^{z\sqrt{\theta}} - S_{\theta})_+]}{S_0\sqrt{\theta}} = p(0,0) + \alpha(z)\sqrt{v(0)}\phi\left(\frac{z}{\sqrt{v(0)}}\right)\theta^H + \frac{z\sqrt{v(0)}}{2}\phi\left(\frac{z}{\sqrt{v(0)}}\right)\sqrt{\theta} + o(\theta^H) \quad (5)$$

$$= \Delta\Phi\left(\frac{\Delta}{\sqrt{\overline{v}(\theta)}}\right) + \sqrt{\overline{v}(\theta)}\phi\left(\frac{\Delta}{\sqrt{\overline{v}(\theta)}}\right)\left(1 + \alpha(z)\theta^H + \frac{z}{2}\sqrt{\theta}\right) + o(\theta^H).$$

**Step 2** [A comparison with the Black-Scholes model]. The Black-Scholes model  $\sqrt{V_{\theta}} \equiv \sigma$ , the volatility parameter, satisfies the assumption with H = 1/2 and  $\eta = 0$ . Therefore, (5) gives

$$\frac{P_{\rm BS}(S_0 e^{z\sqrt{\theta}}, \theta, \sigma)}{S_0\sqrt{\theta}} = \Delta \Phi\left(\frac{\Delta}{\sigma}\right) + \sigma \phi\left(\frac{\Delta}{\sigma}\right) \left(1 + \frac{z}{2}\sqrt{\theta}\right) + o(\theta^{1/2}),\tag{6}$$

where  $P_{BS}(K, \theta, \sigma)$  is the Black-Scholes price of put option with strike *K*, time to maturity  $\theta$  and volatility parameter  $\sigma$ . By the Taylor expansion,

$$\frac{P_{\text{BS}}(S_0 e^{z\sqrt{\theta}}, \theta, \sigma + a\theta^H)}{S_0\sqrt{\theta}} = \Delta \Phi\left(\frac{\Delta}{\sigma}\right) + \sigma \phi\left(\frac{\Delta}{\sigma}\right) \left(1 + \frac{z}{2}\sqrt{\theta} + \frac{a}{\sigma}\theta^H\right) + o(\theta^H).$$

We can equate this and (5) by setting

$$\sigma = \sqrt{\overline{v}(\theta)}, \ a = \sigma \alpha(z),$$

which implies the result.

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# B Proof of Lemma 3.1

Since the Black-Scholes prices  $c_{BS}$  and  $p_{BS}$  satisfy the Black-Scholes equation

$$\frac{\partial c_{\rm BS}}{\partial \theta} = \frac{1}{2} V_{\tau} S^2 \frac{\partial^2 c_{\rm BS}}{\partial S^2}, \quad \frac{\partial p_{\rm BS}}{\partial \theta} = \frac{1}{2} V_{\tau} S^2 \frac{\partial^2 p_{\rm BS}}{\partial S^2}$$

with

$$c_{\rm BS}(S,0) = \left(S - \frac{S_\tau^2}{K_\tau}\right)_+, \ p_{\rm BS}(S,0) = (K_\tau - S)_+\,,$$

Itô's formula gives

$$\begin{split} \left(S_T - \frac{S_\tau^2}{K_\tau}\right)_+ &= c_{\rm BS}(S_\tau, T - \tau) + \int_\tau^T \frac{\partial c_{\rm BS}}{\partial S}(S_t, T - t) \mathrm{d}S_t \\ &+ \frac{1}{2} \int_\tau^T (V_t - V_\tau) S_t^2 \frac{\partial^2 c_{\rm BS}}{\partial S^2}(S_t, T - t) \mathrm{d}t, \\ (K_\tau - S_T)_+ &= p_{\rm BS}(S_\tau, T - \tau) + \int_\tau^T \frac{\partial p_{\rm BS}}{\partial S}(S_t, T - t) \mathrm{d}S_t \\ &+ \frac{1}{2} \int_\tau^T (V_t - V_\tau) S_t^2 \frac{\partial^2 p_{\rm BS}}{\partial S^2}(S_t, T - t) \mathrm{d}t. \end{split}$$

Since  $|V_t - V_\tau| \le C|t - \tau|^{H_0}$  for some finite random variable *C* by the assumption and

$$\left|\frac{\partial^2 c_{\rm BS}}{\partial S^2}(S_t, T-t)\right| \vee \left|\frac{\partial^2 p_{\rm BS}}{\partial S^2}(S_t, T-t)\right| \leq \frac{1}{\sqrt{2\pi V_\tau(T-t)} \inf_{t \in [\tau,T]} S_t},$$

we obtain the first two equations. The last equation follows from the first two with aid of the put-call symmetry [6]:

$$p_{\rm BS}(S_\tau, T-\tau) = \frac{K_\tau}{S_\tau} c_{\rm BS}(S_\tau, T-\tau).$$

### C Proof of Theorem 3.1

Let

$$Z_n = \sqrt{n} \log \frac{K_{\tau_n}}{S_{\tau_n}}.$$

Then,  $\liminf_{n\to\infty} Z_n > -\infty$ ,  $\limsup_{n\to\infty} Z_n < 0$  and

$$\sigma_{\mathrm{BS},\tau_n}(K_{\tau_n}) = \sigma_{\tau_n} + n^{-H}\alpha_{\tau_n}Z_n + o(n^{-H}),$$
  
$$\sigma_{\mathrm{BS},\tau_n}(S_{\tau_n}^2/K_{\tau_n}) = \sigma_{\tau_n} - n^{-H}\alpha_{\tau_n}Z_n + o(n^{-H})$$

by the assumed power law. The Taylor expansion of the Black-Scholes price with respect to the volatility parameter gives

$$P_{\tau_n}(K_{\tau_n}) = p_{\rm BS} + \frac{\partial p_{\rm BS}}{\partial \sigma} n^{-H} \alpha_{\tau_n} Z_n + o(n^{-H})$$

and

$$C_{\tau_n}(S_{\tau_n}^2/K_{\tau_n}) = c_{\rm BS} - \frac{\partial c_{\rm BS}}{\partial \sigma} n^{-H} \alpha_{\tau_n} Z_n + o(n^{-H}),$$

where  $p_{BS}$  and  $c_{BS}$  are the Black-Scholes prices with volatility parameter  $\sigma_{\tau_n}$  of, respectively, put option with strike  $K_{\tau_n}$  and call option with strike  $S^2_{\tau_n}/K_{\tau_n}$ . By the put-call symmetry [6] of the Black-Scholes prices,

$$P_{\tau_n}(K_{\tau_n}) - \frac{K_{\tau_n}}{S_{\tau_n}} C_{\tau_n}(S^2_{\tau_n}/K_{\tau_n}) = \left(\frac{\partial p_{\mathrm{BS}}}{\partial \sigma} + \frac{K_{\tau_n}}{S_{\tau_n}}\frac{\partial c_{\mathrm{BS}}}{\partial \sigma}\right) n^{-H} \alpha_{\tau_n} Z_n + o(n^{-H}).$$

Note that  $\liminf_{n\to\infty} Z_n > -\infty$  ensures

$$\liminf_{n\to\infty} \sqrt{n} \left( \frac{\partial p_{\rm BS}}{\partial \sigma} + \frac{K_{\tau_n}}{S_{\tau_n}} \frac{\partial c_{\rm BS}}{\partial \sigma} \right) > 0.$$

Further, we have  $\liminf_{n\to\infty} \alpha_{\tau_n} Z_n > 0$  and so,

$$\sum_{n=1}^{\infty} n^{H-1/2} \left( P_{\tau_n}(K_{\tau_n}) - \frac{K_{\tau_n}}{S_{\tau_n}} C_{\tau_n}(S_{\tau_n}^2/K_{\tau_n}) \right) = \infty.$$

The result then follows from Lemma 3.1.

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