

Delta Hedging under Stochastic Volatility

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Static Position in a European Put

- A static position in a put with strike K pays off $(K - S_T)^+$ at its maturity T :

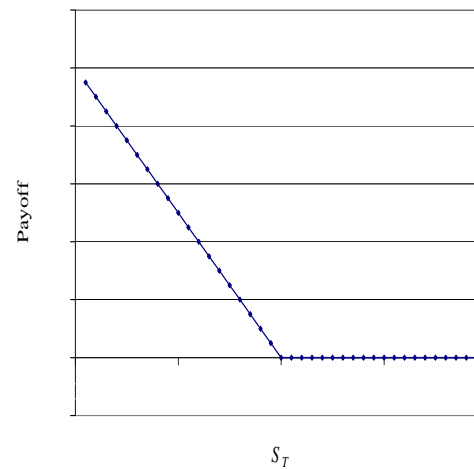


Figure 1: Put Payoff

- If the initial purchase price is borrowed, the terminal P&L is:

$$P\&L_T = (K - S_T)^+ - P_0(K, T)/B_0(T).$$

Static Position in a European Call

- A static position in a call with strike K pays off $(S_T - K)^+$ at its maturity T :

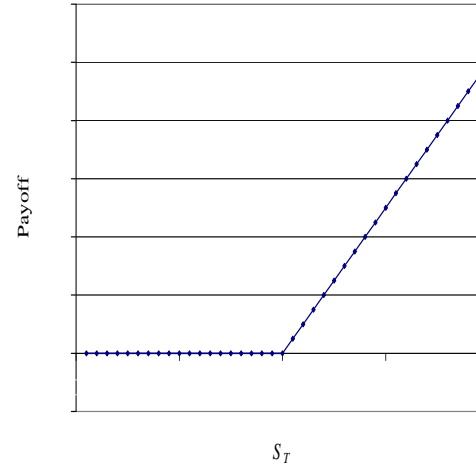


Figure 2: Call Payoff

- If the initial purchase price is borrowed, the terminal P&L is:

$$P\&L_T = (S_T - K)^+ - C_0(K, T)/B_0(T).$$

Transforming Option P&L into Local Variance Bets

- Assume frictionless markets, continuous trading opportunities, constant interest rates r_d and r_f , and a positive continuous spot FX process S (expressed in domestic units per foreign currency unit):

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t, \quad t \in [0, T],$$

where the coefficients μ and σ are arbitrary adapted processes. W is a \mathbb{P} standard Brownian motion.

- If one initially buys a call for its initial implied σ_{i0} , and is short $e^{-r_f(T-t)}N(d_1(S_t, t; \sigma_{i0}))$ units of the foreign currency at each $t \in [0, T]$, and uses the domestic riskless asset to finance, then the terminal P&L is:

$$P\&L_T = \int_0^T e^{r_d(T-t)} (\sigma_t^2 - \sigma_{i0}^2) \frac{S_t^2 e^{-r_f(T-t)} N'(d_1(S_t, t; \sigma_{i0}))}{2 S_t \sigma_{i0} \sqrt{T-t}} dt.$$

- The quantity multiplying the local variance bet payoff $\sigma_t^2 - \sigma_{i0}^2$ is half the Black Scholes dollar gamma evaluated at the initial implied.

From Local Bets to Global Bets

- Recall that the terminal P&L from delta hedging a long call at its initial implied is:

$$P\&L_T = \int_0^T e^{rd(T-t)} (\sigma_t^2 - \sigma_{i0}^2) \frac{S_t^2 e^{-rf(T-t)} N'(d_1(S_t, t; \sigma_{i0}))}{2 S_t \sigma_{i0} \sqrt{T-t}} dt.$$

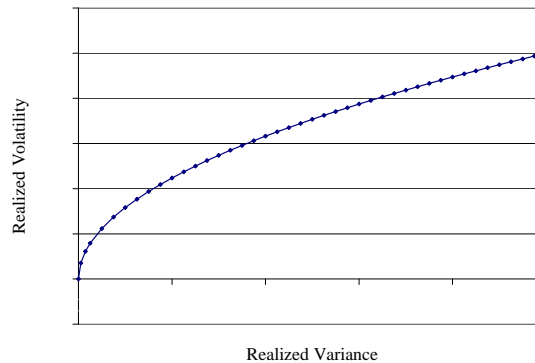
- To eliminate the dependence of the terminal P&L on the S path, Neuberger (90) showed that one can instead delta hedge a fictitious contract that pays $2 \ln(S_T/S_0)$ at T . Dupire (92) points out that this log contract can be replicated by a static position in puts and calls. Each option is delta hedged at the same vol σ_{v0} resulting in the simpler payoff:

$$P\&L_T = \int_0^T (\sigma_t^2 - \sigma_{v0}^2) dt.$$

- As with any zero cost self financing trading strategy, there exists a measure \mathbb{Q} under which this P&L has zero mean. The in principle ex ante observable quantity $\sigma_{v0}^2 T$ is thus the \mathbb{Q} mean of the subsequent realized (non-annualized) variance.
- Forward, corridor, and local variance swaps can also be created in this way.

The Love of Money is the Root of all Evil

- Volatility is defined as the square root of Variance:



- Traders, marketers, lawyers, and customers prefer to think in terms of volatility not variance. In fact, variance swaps are always quoted as annualized volatilities.
- It seems that only quants can be square.

Variance Swaps, Vol Swaps, & the Variance of Realized Vol

- Assuming only positivity, we can define $VOL \equiv \sqrt{\frac{1}{T} \int_0^T \left(\frac{dS_t}{S_{t-}}\right)^2}$ as the realized volatility.
- By definition, a vol swap pays the above realized vol, less a constant initially chosen so that the the contract has zero cost of entry.
- Under no arbitrage, there exists a probability measure \mathbb{Q} such that the initial vol swap rate σ_0^{vol} is the initial \mathbb{Q} mean of realized vol:

$$\sigma_0^{vol} = E_0^{\mathbb{Q}}VOL. \quad (1)$$

- Recall that the variance swap rate σ_{vo} is quoted as an annualized vol, so:

$$(\sigma_0^{var})^2 = E_0^{\mathbb{Q}}VOL^2. \quad (2)$$

- Subtracting the square of (1) from (2) implies:

$$(\sigma_0^{var})^2 - (\sigma_0^{vol})^2 = E_0^{\mathbb{Q}}VOL^2 - (E_0^{\mathbb{Q}}VOL)^2 \equiv \text{Var}_0^{\mathbb{Q}}VOL. \quad (3)$$

Getting to the Root of the Problem

- Suppose that your boss demands that you price and hedge a vol swap.
- Easy!, assume an SV model, eg Heston, fit it to options, and vega hedge the vol swap.
- Wait, we didn't need an SV model for variance swaps, so why do we need one for vol swaps?
- By Taylor expanding the realized variance VOL^2 about $(\sigma_0^{var})^2$, we get:

$$VOL \leq \sigma_0^{var} + \frac{1}{2\sigma_0^{var}}[VOL^2 - (\sigma_0^{var})^2],$$

so we can get another robust upper bound on a vol swap.

- How about an exact hedge and price?

Creating Arbitrary Functions of Realized Variance

- It is well known that butterfly spread levels reflect vol vol. Perhaps the whole smile reflects the risk-neutral distribution of realized variance?
- By further assuming that the instantaneous volatility evolves independently of the Brownian motion driving the FX rate, Carr and Lee (2005) show that one can replicate *any function of* realized variance by dynamic trading in options and their underlying asset.
- No assumption is placed on the instantaneous vol dynamics other than independent evolution. Furthermore, jumps in the FX rate are assumed which render consistency with observed risk reversal levels.
- Under independence, the **initial** vol swap rate is well approximated by the **initial** ATM forward implied. Otherwise, it depends.
- Other examples covered are options on realized variance, claims paying the Sharpe ratio, and barrier options knocking in or out on either price or realized variance.

Betting on Realized Variance without Options?

- The strategies thus far have required at least a static position in options.
- Can one replicate variance derivatives while never touching options?
- Perhaps surprisingly, the answer is a resounding maybe:
- For simplicity, again assume positive continuous price dynamics:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t, \quad t \geq 0.$$

- Let τ be the payoff time which can be any stopping time.
- Then payoffs $U \left(S_\tau, \int_0^\tau \left(\frac{dS_t}{S_t} \right)^2 \right)$ paid at τ are exactly replicated just by currency trading, so long as the payoff trades off “long vega” for “short gamma”, i.e.:

$$U_q(S, q) = -\frac{S^2}{2} U_{ss}(S, q), \quad S > 0, q \geq 0.$$

- An example is a possibly perpetual claim paying any desired function of the realized variance at the first exit time of a given corridor (L,H) containing S_0 .

Delta Hedging at the Running Implied

- Delta-hedging of options requires a volatility input.
- We have examined the P&L arising if the initial implied vol σ_{i0} is used throughout:

$$P\&L_T = \int_0^T e^{rd(T-t)} \frac{S_t^2}{2} \frac{\partial^2}{\partial S^2} BS(S_t, t; \sigma_{i0}) (\sigma_t^2 - \sigma_{i0}^2) dt,$$

where $BS(S, t; \sigma)$ is the Black Scholes model value of an option with spot FX rate S at time t and using vol σ .

- A common industry practice is to use the **running** implied σ_{it} of the option. How does the P&L of a delta hedged option change if this practice is used?
- Assuming only that the stock price S and implied vol σ_{it} are continuous processes:

$$P\&L_T = \int_0^T e^{rd(T-t)} \left\{ \frac{\partial^2}{\partial S^2} BS(S_t, t; \sigma_{it}) \frac{S_t^2}{2} (\sigma_t^2 dt + d[\sigma_{it}^2(T-t)]) \right. \\ \left. + \frac{\partial^2}{\partial S \partial \sigma} BS(S_t, t; \sigma_{it}) dS_t d\sigma_{it} + \frac{\partial^2}{\partial \sigma^2} BS(S_t, t; \sigma_{it}) \frac{(d\sigma_{it})^2}{2} \right\}.$$

- Now, gamma, vanna, and volga along the joint (S, σ_i) path dictate the P&L.

Modelling Implied Vol Dynamics

- Recall that the terminal P&L when delta hedging an option at its running implied is $P\&L_T = \int_0^T e^{rd(T-t)} dP\&L_t$, where $dP\&L_t =$

$$\frac{\partial^2 BS(S_t, t; \sigma_{it})}{\partial S^2} \frac{S_t^2}{2} (\sigma_t^2 dt + d[\sigma_{it}^2(T-t)]) + \frac{\partial^2 BS(S_t, t; \sigma_{it})}{\partial S \partial \sigma} dS_t d\sigma_{it} + \frac{\partial^2 BS(S_t, t; \sigma_{it})}{\partial \sigma^2} \frac{(d\sigma_{it})^2}{2}.$$

- We can rescale the delta hedged option position at each $t \in [0, T]$, so that dollar gamma is constant at 2: $P\&L_T = \int_0^T e^{rd(T-t)} dP\&L_t$, where $dP\&L_t$

$$\begin{aligned} &= \sigma_t^2 dt + d[\sigma_{it}^2(T-t)] + \frac{\frac{\partial^2 BS(S_t, t; \sigma_{it})}{\partial S \partial \sigma}}{\frac{\partial^2 BS(S_t, t; \sigma_{it})}{\partial S^2}} \frac{2dS_t d\sigma_{it}}{S_t^2} + \frac{\frac{\partial^2 BS(S_t, t; \sigma_{it})}{\partial \sigma^2}}{\frac{\partial^2 BS(S_t, t; \sigma_{it})}{\partial S^2}} \frac{(d\sigma_{it})^2}{S_t^2} \\ &= \sigma_t^2 dt + d[\sigma_{it}^2(T-t)] - d_2 \sqrt{T} 2 \frac{dS_t}{S_t} d\sigma_{it} + T d_1 d_2 (d\sigma_{it})^2 \end{aligned}$$

after massive simplification.

Risk-Neutral Drift of Remaining Implied Total Variance

- Recall that a constant dollar gamma strategy $\Rightarrow P\tilde{\&L}_T = \int_0^T e^{rd(T-t)} dP\tilde{\&L}_t$, where:

$$dP\tilde{\&L}_t = \sigma_t^2 dt + d[\sigma_{it}^2(T-t)] - d_2\sqrt{T}2\frac{dS_t}{S_t}d\sigma_{it} + Td_1d_2(d\sigma_{it})^2. \quad (4)$$

- No arbitrage \Rightarrow there exists an equivalent probability measure \mathbb{Q} , s.t. all increments in P&L from any zero cost/self-financed trading strategy has zero \mathbb{Q} mean:

$$E^{\mathbb{Q}}[dP\tilde{\&L}_t | \mathcal{F}_t] = 0, \quad t \in [0, T]. \quad (5)$$

- Let $I_t \equiv \sigma_{it}^2(T-t)$ be the remaining total implied variance at time t . Since σ_{it} is a continuous process under \mathbb{P} , I is some continuous process under \mathbb{Q} :

$$dI_t = m_t dt + \omega_t dB_t, \quad t \in [0, T], \text{ where } B \text{ is a } \mathbb{Q} \text{ standard Brownian motion.}$$

- A commonly advocated modelling approach is to specify ω , the volatility process of I , and then determine m , the risk-neutral drift of I , from (4) and (5).

It Sounded Like a Good Idea at the Time

- A necessary condition on the drift is thus that

$$E_t^{\mathbb{Q}} dI_t \equiv m_t dt = dP\tilde{\&}L_t = -\sigma_t^2 dt + d_2 \sqrt{T} 2 \frac{dS_t}{S_t} d\sigma_{it} - T d_1 d_2 (d\sigma_{it})^2.$$

- Unfortunately, there are 4 cross-sectional no arbitrage relations on term implieds that are not yet being respected:
 1. $\frac{\partial}{\partial K} C_t(K, T) < 0 \Rightarrow$ Upper Bound on Slope of Implied $I_t(T, K)$ in Strike K .
 2. $\frac{\partial}{\partial K} P_t(K, T) > 0 \Rightarrow$ Lower Bound on Slope of Implied $I_t(T, K)$ in Strike K .
 3. $\frac{\partial^2}{\partial K^2} [C/P]_t(K, T) > 0 \Rightarrow$ Lower Bound on Convexity of Implied $I_t(T, K)$ in Strike K .
 4. $\frac{\partial}{\partial T} [C/P]_t(K, T) > 0 \Rightarrow$ Lower Bound on Slope of Implied $I_t(T, K)$ in Maturity T .
- Incorporating these cross-sectional constraints at every future time $t > 0$ is difficult and probably dooms direct Implied Volatility modelling.

Delta Hedging at the Running Realized Volatility

- So far, we have examined the P&L arising from being long a call and delta-hedging:
 1. using the initial implied vol σ_{i0} at each $t \in [0, T]$
 2. using the running implied vol σ_{it} at each $t \in [0, T]$.
- We now consider a third trading strategy termed:
“Delta-hedging at the Running Realized Volatility”.
- To define the trading strategy, recall that delta-hedging a short call requires that we always be long $e^{-r_f(T-t)}N(d_1(S_t, t, \sigma_t))$ foreign currency units, where:

$$d_1(S, t, \sigma) \equiv \frac{\ln(S/K) + (r_d - r_f + \sigma^2/2)(T - t)}{\sqrt{\sigma^2(T - t)}}.$$

- The issue at hand is:

What Vol σ_t ?

- Notice that σ enters d_1 only through “Remaining Total Variance” $\sigma^2(T - t)$.

Delta Hedging at the Running Realized Volatility

- First, let $\langle \ln S \rangle_t \equiv \int_0^t \sigma_s^2 ds \equiv \int_0^t \left(\frac{dS_s}{S_s} \right)^2$ be the total variance realized between time 0 (when the call was sold) and the current time t .
- Next, let τ_H be the first time that $\langle \ln S \rangle$ reaches $\sigma_{i0}^2(T) \times T$, where recall $\sigma_{i0}(T)$ is the initial implied vol of the call that was sold.
- If $\tau_H \leq T$, then realized vol at T (weakly) exceeded the implied vol forecast $\sigma_{i0}(T)$, while if $\tau_H > T$, then realized vol at T fell short of the forecast.
- Let $\tau \equiv T \wedge \tau_H$ be the earlier of maturity and the first passage time.
- We define delta-hedging a short call at the running realized volatility to mean that the total variance defining the delta-hedging strategy at $t \in [0, \tau]$ is given by:

$$\sigma_t^2(T-t) \equiv \sigma_{i0}^2 T - \int_0^t \sigma_s^2 ds.$$

- If $\tau_H \leq T$, then for $t \in (\tau_H, T)$, we adopt a “stop-loss/start-gain strategy”, i.e. long one foreign currency unit and short K domestic units when $S_t > K$ and hold zero otherwise.

P&L when Delta Hedging at the Running Realized Volatility

- Carr & Jarrow (RFS 90) show that using the SLSG strategy over $[0, t]$ results in a loss given by the *local time* $L_t^F(K)$ of the forward FX rate $\{F_t\}$ at the strike K .
- Let $C(S, t; R) = Se^{-r_f(T-t)}N(d_1(S, t, R)) - Ke^{-r_d(T-t)}N(d_2(S, t, R))$ be the Black Scholes call value with remaining total variance R , where:

$$d_1(S, t, R) \equiv \frac{\ln(S/K) + (r_d - r_f)(T - t) + R/2}{\sqrt{R}} \quad d_2(S, t, R) \equiv d_1(S, t, R) - \sqrt{R}.$$

- The P&L from delta-hedging a short call at the running realized volatility is:

$$P\&L_T = -[L_T^F(K) - L_{\tau_H}^F(K)] < 0,$$

if $\tau_H < T$ (i.e. realized vol exceeded the forecast), and:

$$P\&L_T = C(S_T, T; \sigma_{i0}^2 T - \langle \ln S \rangle_T) - (S_T - K)^+ \geq 0,$$

if $\tau_H \geq T$ (i.e. realized vol fell short of the forecast).

Conclusions

- Under relatively weak assumptions, derivatives on realized variance and/or price can be manufactured by dynamic trading in options and/or their underlying.
- By restricting attention to the large class of payoffs which just depend on realized variance and/or price at a fixed time, it is not necessary to specify the dynamics of the (unobservable) instantaneous volatility beyond independence.
- An alternative approach explored in several papers is to specify the dynamics of the (term structure of) variance swap rates or ATM implieds. In either case, the “market price of vol risk” gets buried in the exogenous specification of the instantaneous volatility of these quantities.
- One can also try to model implied vol surfaces, but the rich information content of the initial surface makes it difficult to specify arbitrage-free dynamics.
- Further research on the robust linkages between volatility and credit derivatives is in progress.